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On the paranormed binomial sequence spaces

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Abstract

In this paper the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ which are the generalization of the classical Maddox's paranormed sequence spaces have been introduced and proved that the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are linearly isomorphic to spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively. Besides this, the $\alpha-, \beta-$ and $\gamma-$duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, and $b^{r,s}(p)$ have been computed, their bases have been constructed and some topological properties of these spaces have been studied. Finally, the classes of matrices $(b_0^{r,s}(p):\mu)$, $(b_c^{r,s}(p):\mu)$ and $(b^{r,s}(p):\mu)$ have been characterized, where μ is one of the sequence spaces ℓ_∞, c and c_0 and derives the other characterizations for the special cases of μ.

1. Introduction

We shall denote the space of all real-valued sequences by w as a classical notation. Any vector subspace of w is called a sequence space. The spaces ℓ_∞, c and c_0 are the most common and frequently used spaces which are all bounded, convergent and null sequences, respectively. Also bs, cs, ℓ_1 and ℓ_p notations are used for the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1<p<\infty$.
First, we point out the concept of a paranorm. A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta)=0, g(x)=g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n-\alpha| \to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n - \alpha x)\to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.
Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L=\max\{1,H\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [19] (see also Simons [21] and Nakano [20]) as follows:

$$\ell_\infty(p)=\{x=(x_k)\in w: \sup_{k\in\mathbb{N}}|x_k|^{p_k}<\infty\},$$

$$c(p)=\{x=(x_k)\in w: \lim_{k\to\infty}|x_k-l|^{p_k}=0 \text{ for some } l\in\mathbb{R}\},$$

$$c_0(p)=\{x=(x_k)\in w: \lim_{k\to\infty}|x_k|^{p_k}=0\}$$

$$\ell(p)=\left\{x=(x_k)\in w: \sum_k |x_k|^{p_k}<\infty\right\},$$

which are the complete spaces paranormed by

$$g_1(x)=\sup_{k\in\mathbb{N}}|x_k|^{p_k/L} \Longleftrightarrow \inf p_k>0 \text{ and } g_2(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. For convenience in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathscr{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n\in\mathbb{N}$ such that $n\geq k$, respectively. We shall assume throughout that $p_k^{-1}+(p_k')^{-1}=1$ provided $1<\inf p_k\leq H<\infty$.

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Let λ,μ be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A:\lambda\to\mu$, if for every sequence $x=(x_k)\in\lambda$, the sequence $Ax=\{(Ax)_n\}$, the $A-$transform of x, is in μ, where

$$(Ax)_n=\sum_k a_{nk}x_k,\ (n\in\mathbb{N}). \tag{1.1}$$

By $(\lambda:\mu)$, we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if the series on the right-hand side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\mu$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

2. The sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we define the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$, and prove that $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are the complete paranormed linear spaces.
For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A=\{x=(x_k)\in w: Ax\in\lambda\}. \tag{2.1}$$

In [7], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S-$transforms are in $\ell_{(p)}$, where $S=(s_{nk})$ is defined by

$$s_{nk}=\begin{cases}1 & , \quad 0\le k\le n,\\ 0 & , \quad k>n.\end{cases}$$

Başar and Altay [3] have studied the space $bs(p)$ which is formerly defined by Başar in [4] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Altay and Başar have studied the sequence spaces $r^t(p), r^t_\infty(p)$ in [1] and $r^t_c(p), r^t_0(p)$ in [2] which are derived by the Riesz means from the sequence spaces $\ell(p), \ell_\infty(p), c(p)$ and $c_0(p)$ of Maddox, respectively.
With the notation of (2.1), the spaces $\overline{\ell(p)}, bs(p), r^t(p), r^t_\infty(p), r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)}=[\ell(p)]_S, bs(p)=[\ell_\infty(p)]_S, r^t(p)=[\ell(p)]^t_R$$

$$r^t_\infty(p)=[\ell_\infty(p)]^t_R, r^t_c(p)=[c(p)]^t_R, r^t_0(p)=[c_0(p)]^t_R.$$

In [8], Demiriz and Çakan have defined the sequence spaces $e^r_0(u,p)$ and $e^r_c(u,p)$ which consists of all sequences such that $E^{r,u}$- transforms are in $c_0(p)$ and $c(p)$, respectively $E^{r,u}=\{e^r_{nk}(u)\}$ is defined by

$$e^r_{nk}(u)=\begin{cases}\binom{n}{k}(1-r)^{n-k}r^k u_k & , \quad (0\le k\le n),\\ 0 & , \quad (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$ and $0<r<1$.
In [5] and [6], the Binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$, which are the matrix domains of Binomial mean $B^{r,s}$ in the sequence spaces c_0, c, ℓ_∞ and ℓ_p, respectively, are introduced, some inclusion relations and Schauder basis for the spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ are given, and the $\alpha-,\beta-$ and $\gamma-$ duals of those spaces are determined. For more papers related to sequence spaces and matrix domains of different infinite matrices one can see [13, 12] and references therein. The main purpose of this paper is to introduce the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ which are the set of all sequences whose $B^{r,s}-$transforms are in the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively; where $B^{r,s}$ denotes the matrix $B^{r,s}=\{b^{r,s}_{nk}\}$ defined by

$$b^{r,s}_{nk}=\begin{cases}\frac{1}{(s+r)^n}\binom{k}{n}s^{n-k}r^k & , \quad 0\le k\le n,\\ 0 & , \quad k>n,\end{cases}$$

where $sr>0$. Also, we have constructed the basis and computed the $\alpha-,\beta-$ and $\gamma-$duals and investigated some topological properties of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$.
Following Choudhary and Mishra [7], Başar and Altay [3], Altay and Başar [1, 2], Demiriz [8], Kirişçi [14, 15], Candan and Güneş [16] and Ellidokuzoğlu and Demiriz [9], we define the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$, as the sets of all sequences such that $B^{r,s}-$transforms of them are in the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively, that is,

$$b_0^{r,s}(p)=\left\{x=(x_k)\in w: \lim_{n\to\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}=0\right\},$$

$$b_c^{r,s}(p)=\left\{x=(x_k)\in w: \exists l\in\mathbb{C}\ni \lim_{n\to\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k-l\right|^{p_n}=0\right\},$$

$$b_\infty^{r,s}(p)=\left\{x=(x_k)\in w: \sup_{n\in\mathbb{N}}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}<\infty\right\},$$

$$b^{r,s}(p)=\left\{x=(x_k)\in w: \sum_n\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}<\infty\right\}.$$

In the case $(p_n) = e = (1,1,1,...)$, the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are, respectively, reduced to the sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ which are introduced by Bişgin [5, 6]. With the notation of (2.1), we may redefine the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ as follows:

$$b_0^{r,s}(p) = [c_0(p)]_{B^{r,s}},\ b_c^{r,s}(p) = [c(p)]_{B^{r,s}}, b_\infty^{r,s}(p) = [\ell_\infty(p)]_{B^{r,s}} \text{ and } b^{r,s}(p) = [\ell(p)]_{B^{r,s}}.$$

Define the sequence $y = \{y_n(r,s)\}$, which will be frequently used, as the $B^{r,s}-$transform of a sequence $x = (x_k)$, i.e.,

$$y_n(r,s) := \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k; \text{ for all } k \in \mathbb{N}. \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *$b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$ are the complete linear metric space paranormed by g, defined by*

$$g(x) = \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^{p_n/L}. \tag{2.3}$$

In addition, $b^{r,s}(p)$ is the complete linear metric space paranormed by h, defined by

$$h(x) = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^{p_n} \right)^{1/M}. \tag{2.4}$$

Proof. First, we give the proof for $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$. Since the proof is similar for $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$, we give the proof only for the space $b_0^{r,s}(p)$. The linearity of $b_0^{r,s}(p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x,z \in b_0^{r,s}(p)$ (see Maddox [18, p.30])

$$\left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (x_k+z_k) \right|^{p_n/L} \leq \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^{p_n/L} + \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k z_k \right|^{p_n/L} \tag{2.5}$$

and for any $\alpha \in \mathbb{R}$ (see [21])

$$|\alpha|^{p_n} \leq \max\{1, |\alpha|^L\} = K. \tag{2.6}$$

Using (2.6) inequality, we get

$$\begin{aligned} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\alpha x_k) \right|^{p_n/L} &= |\alpha|^{p_n/L} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^{p_n/L} \\ &\leq K^{1/L} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^{p_n/L} \end{aligned}$$

for $x \in b_0^{r,s}(p)$. This shows the space $b_0^{r,s}(p)$ is a linear space.
Now we will see that g is a paranorm on $b_0^{r,s}(p)$. It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in b_0^{r,s}(p)$.
Let $\{x^n\}$ be any sequence of the points $x^n \in b_0^{r,s}(p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_n x^n - \alpha x) &= \sup_{k\in\mathbb{N}} \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j (\alpha_n x_j^n - \alpha x_j) \right|^{p_k/L} \\ &\leq |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x) \end{aligned} \tag{2.7}$$

which tends to zero as $n \to \infty$. This means that the scalar multiplication is continuous. Hence, g is a paranorm on the space $b_0^{r,s}(p)$.

It remains to prove the completeness of the space $b_0^{r,s}(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $b_0^{r,s}(p)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \frac{\varepsilon}{2}$$

for all $i, j > n_0(\varepsilon)$. Using the definition of g we obtain for each fixed $k \in \mathbb{N}$ that

$$|(B^{r,s}x^i)_k - (B^{r,s}x^j)_k|^{p_k/L} \leq \sup_{k\in\mathbb{N}} |(B^{r,s}x^i)_k - (B^{r,s}x^j)_k|^{p_k/L} < \frac{\varepsilon}{2} \tag{2.8}$$

for every $i,j > n_0(\varepsilon)$ which leads to the fact that $\{(B^{r,s}x^0)_k, (B^{r,s}x^1)_k, (B^{r,s}x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(B^{r,s}x^i)_k \to (B^{r,s}x)_k$ as $i \to \infty$. Using these infinitely many limits $(B^{r,s}x)_0, (B^{r,s}x)_1, \ldots$, we define the sequence $\{(B^{r,s}x)_0, (B^{r,s}x)_1, \ldots\}$. From (2.8) with $j \to \infty$, we have

$$|(B^{r,s}x^i)_k - (B^{r,s}x)_k|^{p_k/L} \leq \frac{\varepsilon}{2} \ (i,j > n_0(\varepsilon)) \tag{2.9}$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in b_0^{r,s}(p)$ for each $i \in \mathbb{N}$, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that

$$|(B^{r,s}x^i)_k|^{p_k/L} < \frac{\varepsilon}{2} \tag{2.10}$$

for every $k \geq k_0(\varepsilon)$ and for each fixed $i \in \mathbb{N}$. Therefore, taking a fixed $i > n_0(\varepsilon)$ we obtain by (2.9) and (2.10) that

$$|(B^{r,s}x)_k|^{p_k/L} \leq |(B^{r,s}x)_k - (B^{r,s}x^i)_k|^{p_k/L} + |(B^{r,s}x^i)_k|^{p_k/L} < \frac{\varepsilon}{2}$$

for every $k > k_0(\varepsilon)$. This shows that $x \in b_0^{r,s}(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $b_0^{r,s}(p)$ is complete and this concludes the proof.
Now lets show that, $b^{r,s}(p)$ is the complete linear metric space paranormed by h defined by (2.4). It is easy to see that the space $b^{r,s}(p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm h defined by (2.4).

It is clear that $h(\theta) = 0$ where $\theta = (0,0,0,\ldots)$ and $h(x) = h(-x)$ for all $x \in b^{r,s}(p)$.

Let $x,y \in b^{r,s}(p)$; then by Minkowski's inequality we have

$$\begin{aligned} h(x+y) &= \left(\sum_{k=0}^{\infty} \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j (x_j + y_j) \right|^{p_k} \right)^{1/M} \\ &= \left(\sum_{k=0}^{\infty} \left[\left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j (x_j + y_j) \right|^{p_k/M} \right]^M \right)^{1/M} \\ &\leq \left(\sum_{k=0}^{\infty} \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \right|^{p_k} \right)^{1/M} + \left(\sum_{k=0}^{\infty} \left[\frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j y_j \right|^{p_k} \right)^{1/M} \\ &= h(x) + h(y) \end{aligned} \tag{2.11}$$

and for any $\alpha \in \mathbb{R}$ we immediately see that

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{2.12}$$

Let $\{x^n\}$ be any sequence of the points $x^n \in b^{r,s}(p)$ such that $h(x^n - x) \to 0$ and (λ_n) also be any sequence of scalars such that $\lambda_n \to \lambda$. We observe that

$$h(\lambda_n x^n - \lambda x) \leq h[(\lambda_n - \lambda)(x^n - x)] + h[\lambda(x^n - x)] + h[(\lambda_n - \lambda)x]. \tag{2.13}$$

It follows from $\lambda_n \to \lambda \, (n \to \infty)$ that $|\lambda_n - \lambda| < 1$ for all sufficiently large n; hence

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)(x^n - x)] \leq \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.14}$$

Furthermore, we have

$$\lim_{n\to\infty} h[\lambda(x^n - x)] \leq \max\{1, |\lambda|^M\} \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.15}$$

Also, we have

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)x)] \leq \lim_{n\to\infty} |\lambda_n - \lambda| h(x) = 0. \tag{2.16}$$

Then, we obtain from (2.13), (2.14), (2.15) and (2.16) that $h(\lambda_n x^n - \lambda x) \to 0$, as $n \to \infty$. This shows that h is a paranorm on $b^{r,s}(p)$.
Now, we show that $b^{r,s}(p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in the space $b^{r,s}(p)$, where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $h(x^n - x^m) < \varepsilon$ for all $n,m > n_0(\varepsilon)$. Since for each fixed $k \in \mathbb{N}$ that

$$|(B^{r,s}x^n)_k - (B^{r,s}x^m)_k| \leq \left[\sum_k |(B^{r,s}x^n)_k - (B^{r,s}x^m)_k|^{p_k} \right]^{\frac{1}{M}} = h(x^n - x^m) < \varepsilon \tag{2.17}$$

for every $n,m > n_0(\varepsilon)$, $\{(B^{r,s}x^0)_k, (B^{r,s}x^1)_k, (B^{r,s}x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(B^{r,s}x^n)_k \to (B^{r,s}x)_k$ as $n \to \infty$. Using these infinitely many limits $(B^{r,s}x)_0, (B^{r,s}x)_1, \ldots$, we define the sequence $\{(B^{r,s}x)_0, (B^{r,s}x)_1, \ldots\}$. For each $K \in \mathbb{N}$ and $n,m > n_0(\varepsilon)$

$$\left[\sum_{k=0}^{K} |(B^{r,s}x^n)_k - (B^{r,s}x^m)_k|^{p_k} \right]^{\frac{1}{M}} \leq h(x^n - x^m) < \varepsilon. \tag{2.18}$$

By letting $m,K \to \infty$, we have for $n > n_0(\varepsilon)$ that

$$h(x^n - x) = \left[\sum_k |(B^{r,s}x^n)_k - (B^{r,s}x)_k|^{p_k}\right]^{\frac{1}{M}} < \varepsilon. \tag{2.19}$$

This shows that $x^n - x \in b^{r,s}(p)$. Since $b^{r,s}(p)$ is a linear space, we conclude that $x \in b^{r,s}(p)$; it follows that $x^n \to x$, as $n \to \infty$ in $b^{r,s}(p)$, thus we have shown that $b^{r,s}(p)$ is complete. □

Note that the absolute property does not hold on the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$, since there exists at least one sequence in the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ and such that $g(x) \neq g(|x|)$, where $|x| = (|x_k|)$. This says that $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ are the sequence spaces of non-absolute type.

Theorem 2.2. *The sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are linearly isomorphic to the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. To avoid repetition of similar statements, we give the proof only for $b_0^{r,s}(p)$. We should show the existence of a linear bijection between the spaces $b_0^{r,s}(p)$ and $c_0(p)$. With the notation of (2.2), define the transformation T from $b_0^{r,s}(p)$ to $c_0(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Furthermore, it is obvious that $x = \theta$ whenever $Tx = \theta$, and hence T is injective.
Let $y \in c_0(p)$ and define the sequence

$$x_k = \frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^j y_j; \quad (k \in \mathbb{N}).$$

Then, we have

$$\begin{aligned}
(B^{r,s}x)_n &= \frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k \\
&= \frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^j y_j \\
&= \frac{1}{(s+r)^n}\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}s^{n-k}(-s)^{k-j}(s+r)^j\right)y_j \\
&= \frac{1}{(s+r)^n}\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}s^{n-j}(s+r)^j\right)y_j \\
&= \frac{1}{(s+r)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}(s+r)^j\left(\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j}\right)y_j \\
&= \frac{1}{(s+r)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}(s+r)^j\delta_{nk}y_j \\
&= \frac{1}{(s+r)^n}\binom{n}{n}s^{n-n}(s+r)^n 1 y_n \\
&= y_n.
\end{aligned}$$

Thus, we have that $x \in b_0^{r,s}(p)$ and consequently T is surjective. Hence, T is a linear bijection and this says that the spaces $b_0^{r,s}(p)$ and $c_0(p)$ are linearly isomorphic, as was desired.

□

3. The basis for the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

Let (λ, g) be a paranormed space. Recall that a sequence (β_k) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$g\left(x - \sum_{k=0}^{n}\alpha_k\beta_k\right) \to 0 \text{ as } n \to \infty.$$

The series $\sum \alpha_k\beta_k$ which has the sum x is then called the expansion of x with respect to (β_n), and written as $x = \sum \alpha_k\beta_k$. Since it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [11, Remark 2.4]), we have the following. Because of the isomorphism T is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of those spaces $c_0(p)$, $c(p)$ and $\ell(p)$ are the basis of the new spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$, respectively. Therefore, we have the following:

Theorem 3.1. *Let $\lambda_k = (B^{r,s}x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)} = \{b^{(k)}\}_{k\in\mathbb{N}}$ of the elements of the space $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ by*

$$b_n^{(k)} = \begin{cases} \frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(s+r)^k &, \quad n \geq k \\ 0 &, \quad 0 \leq k < n \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then

(a) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $b_0^{r,s}(p)$*, and any* $x\in b_0^{r,s}(p)$ *has a unique representation of the form*

$$x=\sum_k \lambda_k b^{(k)}.$$

(b) *The set* $\{e,b^{(1)}(r),b^{(2)}(r),\ldots\}$ *is a basis for the space* $b_c^{r,s}(p)$*, and any* $x\in b_c^{r,s}(p)$ *has a unique representation of the form*

$$x=le+\sum_k[\lambda_k-l]b^{(k)},$$

where $l=\lim_{k\to\infty}(B^{r,s}x)_k$.

(c) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $b^{r,s}(p)$*, and any* $x\in b^{r,s}(p)$ *has a unique representation of the form*

$$x=\sum_k \lambda_k b^{(k)}.$$

4. The $\alpha-,\beta-$ and $\gamma-$duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we state and prove the theorems determining the $\alpha-,\beta-$ and $\gamma-$duals of the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ of non-absolute type.

We shall firstly give the definition of $\alpha-,\beta-$ and $\gamma-$duals of sequence spaces and after quoting the lemmas which are needed in proving the theorems given in Section 4.
The set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_kz_k)\in\mu \text{ for all } x=(x_k)\in\lambda\} \tag{4.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda\supset\nu\supset\mu$ that the inclusions

$$S(\lambda,\mu)\subset S(\nu,\mu) \text{ and } S(\lambda,\mu)\subset S(\lambda,\nu)$$

hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha=S(\lambda,\ell_1),\lambda^\beta=S(\lambda,cs) \text{ and } \lambda^\gamma=S(\lambda,bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as Köthe- Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively.

For to give the alpha-, beta- and gamma-duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ of non-absolute type, we need the following lemma:

Lemma 4.1. *[10,* $q_n=1$*] Let* $A=(a_{nk})$ *be an infinite matrix. Then, the following statements hold*

(i) $A\in(c_o(p):\ell(q))$ *if and only if*

$$\sup_{K\in\mathscr{F}}\sum_n\left|\sum_{k\in K}a_{nk}M^{-1/p_k}\right|<\infty,\ \exists M\in\mathbb{N}_2. \tag{4.2}$$

(ii) $A\in(c(p):\ell(q))$ *if and only if (4.2) holds and*

$$\sum_n\left|\sum_k a_{nk}\right|<\infty. \tag{4.3}$$

(iii) $A\in(c_0(p):c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|M^{-1/p_k}<\infty,\ \exists M\in\mathbb{N}_2, \tag{4.4}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\lim_{n\to\infty}|a_{nk}-\alpha_k|=0 \textit{ for all } k\in\mathbb{N}, \tag{4.5}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\sup_{n\in\mathbb{N}}\sum_k|a_{nk}-\alpha_k|M^{-1/p_k}<\infty,\ \exists M\in\mathbb{N}_2. \tag{4.6}$$

(iv) $A\in(c(p):c(q))$ *if and only if (4.4), (4.5), (4.6) hold and*

$$\exists\alpha\in\mathbb{R}\ni\lim_{n\to\infty}\left|\sum_k a_{nk}-\alpha\right|=0. \tag{4.7}$$

(v) *$A \in (c_o(p) : \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.8}$$

(vi) *$A \in (c(p) : \ell_\infty(q))$ if and only if (4.8) holds and*

$$\sup_n \left| \sum_k a_{nk} \right| < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.9}$$

(vii) *$A \in (\ell(p) : \ell_1)$ if and only if*

(a) *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then*

$$\sup_{N\in\mathscr{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} a_{nk} \right|^{p_k} < \infty. \tag{4.10}$$

(b) *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{N\in\mathscr{F}} \sum_k \left| \sum_{n\in N} a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.11}$$

Lemma 4.2. *[17] Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i) *$A \in (\ell(p) : \ell_\infty)$ if and only if*

(a) *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then,*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.12}$$

(b) *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.13}$$

(ii) *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.12) and (4.13) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \ \forall k \in \mathbb{N}. \tag{4.14}$$

Theorem 4.3. *Let $K \in \mathscr{F}$ and $K^* = \{k \in \mathbb{N} : n \ge k\} \cap K$ for $K \in \mathscr{F}$. Define the sets $T_1^r(p)$, T_2^r, $T_3(p)$ and $T_4(p)$ as follows:*

$$T_1(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{K\in\mathscr{F}} \sum_n \left| \sum_{k\in K^*} c_{nk} M^{-1/p_k} \right| < \infty \right\},$$

$$T_2 = \left\{ a = (a_k) \in w : \sum_n \left| \sum_{k=0}^n c_{nk} \right| \text{ exists for each } n \in \mathbb{N} \right\},$$

$$T_3(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathscr{F}} \sum_k \left| \sum_{n\in N} c_{nk} M^{-1} \right|^{p_k'} < \infty, \right\},$$

$$T_4(p) = \left\{ a = (a_k) \in w : \sup_{N\in\mathscr{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} c_{nk} \right|^{p_k} < \infty \right\},$$

where the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{1}{r^n} \sum_{k=0}^n \binom{n}{k} (-s)^{n-k} (s+r)^k a_n & , \quad 0 \le k \le n, \\ 0 & , \quad k \ge n. \end{cases} \tag{4.15}$$

Then, $[b_0^{r,s}(p)]^\alpha = T_1(p)$, $[b_c^{r,s}(p)]^\alpha = T_1(p) \cap T_2$ and

$$[b^{r,s}(p)]^\alpha = \begin{cases} T_3(p) & 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & 0 < p_k \le 1, \forall k \in \mathbb{N}. \end{cases} \tag{4.16}$$

Proof. We chose the sequence $a = (a_k) \in w$. We can easily derive that with the (2.2) that

$$a_n x_n = \frac{1}{r^n} \sum_{k=0}^n \binom{n}{k} (-s)^{n-k} (s+r)^k a_n y_k = (Cy)_n, \quad (n \in \mathbb{N}) \tag{4.17}$$

for all $k, n \in \mathbb{N}$, where $C = (c_{nk})$ defined by (4.15). It follows from (4.17) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in b_0^{r,s}(p)$ if and only if $Cy \in \ell_1$ whenever $y \in c_0(p)$. This means that $a = (a_n) \in [b_0^{r,s}(p)]^\alpha$ if and only if $C \in (c_0(p) : \ell_1)$. Then, we derive by (4.2) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[b_0^{r,s}(p)]^\alpha = T_1^r(p)$.
Using the (4.3) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.17), the proof of the $[b_c^{r,s}(p)]^\alpha = T_1^r(p) \cap T_2$ can also be obtained in a similar way. Also, using the (4.10),(4.11) and (4.17), the proof of the

$$[b^{r,s}(p)]^\alpha = \begin{cases} T_3(p) & 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & 0 < p_k \le 1, \forall k \in \mathbb{N}, \end{cases}$$

can also be obtained in a similar way. □

Theorem 4.4. *The matrix $D=(d_{nk})$ is defined by*

$$d_{nk}=\begin{cases}\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^{k}a_j & , \quad (0\leq k\leq n)\\ 0 & , \quad (k>n)\end{cases} \tag{4.18}$$

for all $k,n\in\mathbb{N}$. Define the sets $T_5(p)$, T_6, $T_7(p)$, T_8, $T_9(p)$, T_{10} and $T_{11}(p)$ as follows:

$$T_5(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}|M^{-1/p_k}<\infty\right\},$$

$$T_6=\left\{a=(a_k)\in w:\lim_{n\to\infty}|d_{nk}| \text{ exists for each } k\in\mathbb{N}\right\},$$

$$T_7(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\exists(\alpha_k)\subset\mathbb{R}\ni\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}-\alpha_k|M^{-1/p_k}<\infty\right\},$$

$$T_8=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}|d_{nk}| \text{ exists}\right\},$$

$$T_9(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k}\left|d_{nk}M^{-1}\right|^{p_k'}<\infty\right\},$$

$$T_{10}=\left\{a=(a_k)\in w:\lim_{n\to\infty}d_{nk} \text{ exists for each } k\in\mathbb{N}\right\},$$

$$T_{11}(p)=\left\{a=(a_k)\in w:\sup_{n,k\in\mathbb{N}}|d_{nk}|^{p_k}<\infty\right\}.$$

Then, $[b_0^{r,s}(p)]^{\beta}=T_5(p)\cap T_6\cap T_7(p)$, $[b_c^{r,s}(p)]^{\beta}=[b_0^{r,s}(p)]^{\beta}\cap T_8$ and

$$[b^{r,s}(p)]^{\beta}=\begin{cases}T_9(p)\cap T_{10} & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}\cap T_{11}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}.\end{cases} \tag{4.19}$$

Proof. We give the proof again only for the space $b_0^{r,s}(p)$. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n}a_kx_k&=\sum_{k=0}^{n}\left[\frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^{j}y_j\right]a_k\\&=\sum_{k=0}^{n}\left[\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^{k}a_j\right]y_k=(Dy)_n,\end{aligned} \tag{4.20}$$

where $D=(d_{nk})$ defined by (4.18). Thus, we deduce from (4.20) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in b_0^{r,s}(p)$ if and only if $Dy\in c$ whenever $y=(y_k)\in c_0(p)$. That is to say that $a=(a_k)\in[b_0^{r,s}(p)]^{\beta}$ if and only if $D\in(c_0(p):c)$. Therefore, we derive from (4.4),(4.5) and (4.6) with $q_n=1$ for all $n\in\mathbb{N}$ that $[b_0^{r,s}(p)]^{\beta}=T_5(p)\cap T_6\cap T_7(p)$.
Using the (4.4),(4.5), (4.6) and (4.7) with $q_n=1$ for all $n\in\mathbb{N}$ and (4.20), the proofs of the $[b_c^{r,s}(p)]^{\beta}=[b_0^{r,s}(p)]^{\beta}\cap T_8$ can also be obtained in a similar way. Also, using the (4.12),(4.13), (4.14) and (4.20), the proofs of the

$$[b^{r,s}(p)]^{\beta}=\begin{cases}T_9(p)\cap T_{10} & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}\cap T_{11}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}\end{cases}$$

can also be obtained in a similar way. □

Theorem 4.5. *Define the set T_{12} by*

$$T_{12}=\left\{a=(a_k)\in w:\sup_{n}\left|\sum_{k}a_{nk}\right|<\infty\right\}.$$

Then, $[b_0^{r,s}(p)]^{\gamma}=T_5(p)$, $[b_c^{r,s}(p)]^{\gamma}=[b_0^{r,s}(p)]^{\gamma}\cap T_{12}$ and

$$[b^{r,s}(p)]^{\gamma}=\begin{cases}T_8(p) & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}.\end{cases}$$

Proof. This is obtained in the similar way used in the proof of Theorem 4.4. □

5. Certain matrix mappings on the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we characterize some matrix mappings on the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$.
We known that, if $b_0^{r,s}(p) \cong c_0(p)$, $b_c^{r,s}(p) \cong c(p)$ and $b^{r,s}(p) \cong \ell(p)$, we can say: The equivalence "$x \in b_0^{r,s}(p), b_c^{r,s}(p)$ or $b^{r,s}(p)$ if and only if $y \in c_0(p), c(p)$ or $\ell(p)$" holds.
In what follows, for brevity, we write,

$$\tilde{a}_{nk} := \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj}$$

for all $k, n \in \mathbb{N}$.

Theorem 5.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \tag{5.1}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then,

(i) *$A \in (b_0^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$.*
(ii) *$A \in (b_c^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c(p) : \mu)$.*
(iii) *$A \in (b^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell(p) : \mu)$.*

Proof. We prove only part of (i). Let μ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $b_0^{r,s}(p)$ and $c_0(p)$ are linearly isomorphic.
Let $A \in (b_0^{r,s}(p) : \mu)$ and take any $y = (y_k) \in c_0(p)$. Then $EB^{r,s}$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in T_5(p) \cap T_6$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in c_0(p)$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.
We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0(p) : \mu)$.
Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$ hold, and take any $x = (x_k) \in b_0^{r,s}(p)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in (b_0^{r,s}(p) : \mu)$. This completes the proof of part of (i). □

Theorem 5.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} := \frac{1}{(s+r)^n} \sum_{j=0}^{n} \binom{n}{j} s^{n-j} r^j a_{jk} \text{ for all } k, n \in \mathbb{N}. \tag{5.2}$$

Let μ be any given sequence space. Then,

(i) *$A \in (\mu : b_0^{r,s}(p))$ if and only if $B \in (\mu : c_0(p))$.*
(ii) *$A \in (\mu : b_c^{r,s}(p))$ if and only if $B \in (\mu : c(p))$.*
(iii) *$A \in (\mu : b^{r,s}(p))$ if and only if $B \in (\mu : \ell(p))$.*

Proof. We prove only part of (i). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{j-n} \left(\sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{B^{r,s}(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in b_0^{r,s}(p)$ whenever $z \in \mu$ if and only if $Bz \in c_0(p)$ whenever $z \in \mu$. This completes the proof of part of (i). □

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space μ. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(b_0^{r,s}(p) : \mu)$, $(\mu : b_0^{r,s}(p))$, $(b_c^{r,s}(p) : \mu)$, $(\mu : b_c^{r,s}(p))$ and $(b^{r,s}(p) : \mu)$, $(\mu : b^{r,s}(p))$ may be derived by replacing the entries of C and A by those of the entries of $E = C\{B^{r,s}\}^{-1}$ and $B = B^{r,s}A$, respectively; where the necessary and sufficient conditions on the matrices E and B are read from the concerning results in the existing literature.
The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann [10]. Let N and K denote the finite subset of $\mathbb{N}$, L and M also denote the natural numbers. Prior to giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_n |a_{nk}|^{q_n} = 0, \text{ for all } k. \tag{5.3}$$

$$\forall L, \exists M \ni \sup_n L^{1/q_n} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \tag{5.4}$$

$$\lim_n |\sum_k a_{nk}|^{q_n} = 0, \tag{5.5}$$

$$\forall L, \sup_n \sup_{k \in K_1} \left| a_{nk} L^{1/q_n} \right|^{p_k} < \infty, \tag{5.6}$$

$$\forall L, \exists M \ni \sup_n \sum_{k \in K_2} |a_{nk} L^{1/q_n} M^{-1}|^{p'_k} < \infty, \tag{5.7}$$

$$\forall M, \lim_n \left(\sum_k |a_{nk}| M^{1/p_k} \right)^{q_n} = 0, \tag{5.8}$$

$$\forall M, \sup_n \sum_k |a_{nk}| M^{1/p_k} < \infty, \tag{5.9}$$

$$\forall M, \sup_K \sum_n \left| \sum_{k \in K} a_{nk} M^{1/p_k} \right|^{q_n} < \infty. \tag{5.10}$$

Lemma 5.3. *Let $A = (a_{nk})$ be an infinite matrix. Then*

(i) $A = (a_{nk}) \in (c_0(p) : \ell_\infty(q))$ *if and only if (4.8) holds.*
(ii) $A = (a_{nk}) \in (c(p) : \ell_\infty(q))$ *if and only if (4.8) and (4.9) hold.*
(iii) $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ *if and only if (4.12) and (4.13) hold.*
(iv) $A = (a_{nk}) \in (c_0(p) : c(q))$ *if and only if (4.4), (4.5) and (4.6) hold.*
(v) $A = (a_{nk}) \in (c(p) : c(q))$ *if and only if (4.4), (4.5), (4.6) and (4.7) hold.*
(vi) $A = (a_{nk}) \in (\ell(p) : c)$ *if and only if (4.12), (4.13) and (4.14) hold.*
(vii) $A = (a_{nk}) \in (c_0(p) : c_0(q))$ *if and only if (5.3) and (5.4) hold.*
(viii) $A = (a_{nk}) \in (c(p) : c_0(q))$ *if and only if (5.3), (5.4) and (5.5) hold.*
(ix) $A = (a_{nk}) \in (\ell(p) : c_0(q))$ *if and only if (5.3), (5.6) and (5.7) hold.*
(x) $A = (a_{nk}) \in (\ell_\infty(p) : c_0(q))$ *if and only if (5.8) holds.*
(xi) $A = (a_{nk}) \in (\ell_\infty(p) : c(q))$ *if and only if (5.9) holds.*
(xii) $A = (a_{nk}) \in (\ell_\infty(p) : \ell(q))$ *if and only if (5.10) holds.*
(xiii) $A = (a_{nk}) \in (c_0(p) : \ell(q))$ *if and only if (4.2) holds.*
(xiv) $A = (a_{nk}) \in (c(p) : \ell(q))$ *if and only if (4.2) and (4.4) hold.*

Corollary 5.4. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b_0^{r,s}(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (b_0^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3) and (5.4) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b_0^{r,s}(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5) and (4.6) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.5. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b_c^{r,s}(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) and (4.9) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (b_c^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.4) and (5.5) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b_c^{r,s}(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5), (4.6) and (4.7) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.6. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b^{r,s}(p) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.12) and (4.13) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
(ii) $A \in (b^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.6) and (5.7) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b^{r,s}(p) : c)$ *if and only if* $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.12), (4.13) and (4.14) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 5.7. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b_0^{r,s}(p))$ *if and only if (5.8) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b_0^{r,s}(p))$ *if and only if (5.3) and (5.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b_0^{r,s}(p))$ *if and only if (5.3), (5.4) and (5.5) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.8. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b_c^{r,s}(p))$ *if and only if (5.9) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b_c^{r,s}(p))$ *if and only if (4.4), (4.5) and (4.6) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b_c^{r,s}(p))$ *if and only if (4.4), (4.5), (4.6) and (4.7) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.9. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b^{r,s}(p))$ *if and only if (5.10) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b^{r,s}(p))$ *if and only if (4.2) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b^{r,s}(p))$ *if and only if (4.2) and (4.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

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