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General helices that lie on the sphere S^{2n} in Euclidean space E^{2n+1}

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Abstract

In this work, we give two methods to generate general helices that lie on the sphere S^{2n} in Euclidean (2n+1)-space E^{2n+1}.

1. Introduction

In Euclidean 3-space E^3, the condition for a curve to lie on a sphere (spherical curve) is usually given in the form

$$\frac{k_2}{k_1}+\left(\frac{1}{k_2}\left(\frac{1}{k_1}\right)'\right)'=0$$

where $k_1>0$ and $k_2\neq 0$ [8]. The integral form of the above equation was given in [2] as

$$\frac{1}{k_1}=A\cos\int k_2\,ds+B\sin\int k_2\,ds.$$

Besides, researchers gave different characterizations about spherical curves by using the equations above [9, 10].
In E^3, general helices are defined by the property that their tangent makes a constant angle with a fixed direction in every point. In this paper we use this definition for higher dimensions too. But, the general helix notion in $\mathbb{R}^3$ can be generalized to higher dimensions in many ways. In [7], the same definition is proposed but in $\mathbb{R}^n$. The definition of a general helix is more restrictive in [5]; the fixed direction makes a constant angle with all vectors of the Frenet frame. It is easy to check that this definition only works in odd dimensions [3]. Moreover, in the same paper, it is proven that this definition is equivalent to the fact that the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4}, \ldots, \frac{k_{n-4}}{k_{n-3}}, \frac{k_{n-2}}{k_{n-1}}$, where curvatures k_i are constants. This statement is related with the Lancret theorem for general helices in $\mathbb{R}^3$.
If a general helix lies on S^n, we call it spherical helix. This topic have become an active research area in recent years. In [6], Monterde studied constant curvature ratio curves (ccr-curves) for which all the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4}, \ldots$ are constant. He found explicit examples of spherical ccr-curves that lie on S^2 with non-constant curvatures. He showed that a ccr-curve on S^2 is a general helix. After that in [1], authors presented some necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space. They gave an example for a slant helix in E^5 whose tangent indicatrix is a spherical helix that lie on S^4.
In literature, there are studies about spherical helices in E^3 and there is only one example when $n\geq 4$ [1]. By means of the papers mentioned above, the goal of this paper is to find methods for generating spherical helices that lies on S^{2n} in E^{2n+1}.

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2. Basic concepts

The real vector space R^n with standard inner product and the standart orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ is given by

$$< X, Y >= \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, ..., x_n)$, $Y = (y_1, y_2, ..., y_n) \in R^n$. In particular, the norm of a vector $X \in R^n$ is given by $||X||^2 =< X, X >$.
Let $\alpha : I \subset R \to E^n$ be a regular curve in E^n and $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the curve α, where V_i $(i = 1, 2, ..., n)$ denote ith Frenet vector field. Then, the Frenet formulas are given by

$$\begin{cases} V_1'(t) = \nu(t) k_1(t)\, V_2\ (t) \\ V_i'(t) = \nu(t)(-k_{i-1}(t)\, V_{i-1}\ (t) + \, k_i(t)\, V_{i+1}\ (t)), \quad i = 2, 3, ..., n-1 \\ V_n'(t) = -\nu(t) k_{n-1}(t)\, V_{n-1}\ (t) \end{cases} \tag{2.1}$$

where $\nu(t) = ||d\alpha(t)/dt|| = ||\alpha'(t)||$ and k_i $(i = 1, 2, ..., n-1)$ denote the ith curvature function of the curve [4].

Definition 2.1. *The curve $\alpha : I \subset R \to E^n$ is called general helix if its tangent vector V_1 makes a constant angle with a fixed direction [7].*

A sphere of center $P = (p_1, p_2, ..., p_n) \in E^n$ and radius $R > 0$ is the surface

$$S^n(P, R) = \left\{ (x_1, x_2, ..., x_n) \in E^n | \quad (x_1 - p_1)^2 + \cdots + (x_n - p_n)^2 = R^2 \right\}.$$

When, P is the origin of E^n and $R = 1$, we denote this with S^n, that is,

$$S^n = \left\{ (x_1, x_2, ..., x_n) \in E^n | \quad {x_1}^2 + \cdots + {x_n}^2 = 1 \right\}.$$

3. Spherical helices in E^{2n+1}

Now, we give two theorems to generate general helices that lie on $S^{2n} \subset E^{2n+1}$. To reach our goal; First, we use W-curves, i.e. a curve which has constant Frenet curvatures [3].

Theorem 3.1. *Let,*

$$\gamma(s) = \left(\gamma_1(s), \gamma_2(s), \ldots, \gamma_{2n}(s), \sqrt{1 - R^2} \right) \subset S^{2n-1}(P, R) \subset S^{2n} \subset E^{2n+1}$$

be a unit speed W-curve with the Frenet vector fields $\{u_1, u_2, \ldots, u_{2n}\}$ and the curvatures $\{k_1, k_2, \ldots, k_{2n-1}\}$ where $P = \left(0, 0, \ldots, 0, \sqrt{1 - R^2}\right) \in E^{2n+1}$, $R = 1/a$, $a > 1$. Then, $\alpha(s) = \sin(s)\gamma(s) + \cos(s)u_1(s)$ with the Frenet vector fields $\{V_1, V_2, \ldots, V_{2n}\}$ is a general helix that lies on S^{2n}.

Proof. With straightforward calculations, it is clear that

$$\|\alpha\| = 1,$$

then α is a spherical curve which lies on S^{2n}.
We know $\langle \gamma, e_{2n+1} \rangle = \sqrt{1 - R^2}$. If we take derivatives of this equation with respect to s, we have

$$\langle u_i, e_{2n+1} \rangle = 0, \quad i = 1, 2, ..., 2n. \tag{3.1}$$

Since $\langle \gamma - P, \gamma - P \rangle = R^2$, for $i = 1, 2, ..., n$ we also have

$$\begin{aligned} \langle u_{2i-1}, \gamma \rangle &= 0, \\ \langle u_{2i}, \gamma \rangle &= \frac{-\prod_{j=0}^{i-1} k_{2i}}{\prod_{j=1}^{i} k_{2i-1}} \end{aligned} \tag{3.2}$$

where $k_0 = 1$. Then, by using Equations (3.1) and (3.2), we can write

$$\gamma = P + \lambda_1 u_1 + \lambda_2 u_3, \cdots + \lambda_n u_{2n-1}$$

So,

$$\langle V_1, e_{2n+1} \rangle = \sqrt{\frac{1 - R^2}{{k_1}^2 - 1}}.$$

This completes the proof.

Corollary 3.2. *If*

$$\gamma(s)=\frac{R}{\sqrt{n}}\left(\sum_{j=1}^{n}\sin\left(c_js\right)e_{2j-1}+\sum_{j=1}^{n}\cos\left(c_js\right)e_{2j}\right)+\sqrt{1-R^2}\,e_{2n+1} \tag{3.3}$$

where $R=\left(\frac{n}{\sum_{j=1}^{n}c_j^2}\right)^{1/2}$, $a=\left(\frac{\sum_{j=1}^{n}c_j^2}{n}\right)^{1/2}>1$ *and* $c_i\neq c_j$, $1\leq i<j\leq n$.
Then, $\alpha(s)=\sin(s)\gamma(s)+\cos(s)u_1(s)$ *is a general helix that lies on* S^{2n} *in* E^{2n+1}.

Example 3.3. *If we take* $c_1=2$, $c_2=4$, *and* $n=2$ *in Corollary 3.2 we have*

$$P=\left(0,0,0,0,\frac{1}{\sqrt{10}}\right),$$

$$R=\frac{3}{\sqrt{10}},$$

$$\gamma(s)=\left(\frac{\sin(2s)}{2\sqrt{5}},\frac{\cos(2s)}{2\sqrt{5}},\frac{\sin(4s)}{2\sqrt{5}},\frac{\cos(4s)}{2\sqrt{5}},\frac{3}{\sqrt{10}}\right)\subset S^3\left(P,R\right)\subset S^4,$$

$$v_1(s)=\left(\frac{\cos(2s)}{\sqrt{5}},-\frac{\sin(2s)}{\sqrt{5}},\frac{2\cos(4s)}{\sqrt{5}},-\frac{2\sin(4s)}{\sqrt{5}},0\right).$$

Then, γ *is a unit speed spherical W-curve with the curvatures*

$$k_1=2\sqrt{\frac{17}{5}},$$

$$k_2=\frac{12}{\sqrt{85}},$$

$$k_3=4\sqrt{\frac{5}{17}},$$

$$k_4=0.$$

Therefore, we get

$$\alpha(s)=\left(\frac{\cos^3(s)}{\sqrt{5}},-\frac{3\sin(s)+\sin(3s)}{4\sqrt{5}},\frac{2\cos^3(s)(3\cos(2s)-2)}{\sqrt{5}},-\frac{5\sin(3s)+3\sin(5s)}{4\sqrt{5}},\frac{3\sin(s)}{\sqrt{10}}\right)$$

with the tangent vector field

$$V_1(s)=\left(-\frac{\sin(s)\cos(s)}{\sqrt{7}},-\frac{(\cos(s)+\cos(3s))\sec(s)}{4\sqrt{7}},\frac{5(\sin(s)-\sin(3s))\cos(s)}{\sqrt{7}},-\frac{5(\cos(3s)+\cos(5s))\sec(s)}{4\sqrt{7}},\frac{1}{\sqrt{14}}\right)$$

where $\|\alpha\|=1$.

By means of Theorem 3.1 and Corollary 3.2 we can give a new theorem.

Theorem 3.4. *Let* $\alpha: I\subset R\to E^{2n+1}$

$$\alpha(t)=(\alpha_1(t),\alpha_2(t),\ldots,\alpha_{2n+1}(t))$$

be a regular curve given by

$$\alpha_{2i-1}(t)=\frac{1}{\left(\sum_{j=1}^{n}c_j^2\right)^{1/2}}\left(c_i\cos\left(t\right)\cos\left(c_it\right)+\sin\left(t\right)\sin\left(c_it\right)\right),$$

$$\alpha_{2i}(t)=\frac{1}{\left(\sum_{j=1}^{n}c_j^2\right)^{1/2}}\left(\cos\left(c_it\right)\sin\left(t\right)-c_i\cos\left(t\right)\sin\left(c_it\right)\right),$$

for $i=1,2,\ldots n$ *and*

$$\alpha_{2n+1}(t)=\left(1-\frac{n}{\sum_{j=1}^{n}c_j^2}\right)^{1/2}\sin\left(t\right)$$

where $c_1,c_2,\ldots,c_n>1$ *with* $c_i\neq c_j$, $1\leq i<j\leq n$. *Then,* α *is a general helix which lies on* S^{2n}.

Proof. With straightforward calculations, we easily have

$$||\alpha(t)|| = 1,$$

$$||\alpha'(t)|| = \left(\frac{\left(\sum_{i=1}^{n} c_i{}^2\right)\left(1-\frac{n}{\sum_{i=1}^{n} c_i{}^2}\right)}{\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)}\right)^{1/2}\cos(t),$$

and

$$V_1(t) = \left(\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)\right)^{-1/2}\left(\sum_{i=1}^{n}\left(1-c_i{}^2\right)\left(\sin(c_it)e_{2i-1}-\cos(c_it)e_{2i}\right)+\left(\sum_{i=1}^{n} c_i{}^2\left(1-\frac{n}{\sum_{i=1}^{n} c_i{}^2}\right)\right)^{1/2} e_{2n+1}\right).$$

Therefore, we have

$$\langle V_1(t), e_{2n+1}\rangle = \left(\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)\right)^{-1/2}\left(\sum_{i=1}^{n} c_i{}^2\left(1-\frac{n}{\sum_{i=1}^{n} c_i{}^2}\right)\right)^{1/2}.$$

This completes the proof. □

Example 3.5. *If we take $n=4$ and $c_1=\sqrt{2}$, $c_2=\sqrt{3}$, $c_3=2$ in Theorem 3.4 we have,*

$$\alpha(t) = \left(\frac{\sqrt{2}}{3}\cos(t)\cos\left(\sqrt{2}t\right)+\frac{1}{3}\sin(t)\sin\left(\sqrt{2}t\right),\right.$$

$$\frac{1}{3}\cos\left(\sqrt{2}t\right)\sin(t)-\frac{\sqrt{2}}{3}\cos(t)\sin\left(\sqrt{2}t\right),$$

$$\frac{1}{\sqrt{3}}\cos(t)\cos\left(\sqrt{3}t\right)+\frac{1}{3}\sin(t)\sin\left(\sqrt{3}t\right),$$

$$\frac{1}{3}\cos\left(\sqrt{3}t\right)\sin(t)-\frac{1}{\sqrt{3}}\cos(t)\sin\left(\sqrt{3}t\right),$$

$$\frac{2}{3}\cos(t)\cos(2t)+\frac{1}{3}\sin(t)\sin(2t),$$

$$\frac{1}{3}\cos(2t)\sin(t)-\frac{2}{3}\cos(t)\sin(2t),$$

$$\left.\sqrt{\frac{2}{3}}\sin(t)\right),$$

$$||\alpha(t)|| = 1,$$

$$||\alpha'(t)|| = \frac{2\sqrt{5}\cos(t)}{3},$$

$$V_1(t) = \left(-\frac{\sin\left(\sqrt{2}t\right)}{2\sqrt{5}}, -\frac{\cos\left(\sqrt{2}t\right)}{2\sqrt{5}}, -\frac{\sin\left(\sqrt{3}t\right)}{\sqrt{5}}, -\frac{\cos\left(\sqrt{3}t\right)}{\sqrt{5}}, -\frac{3\sin(2t)}{2\sqrt{5}}, -\frac{3\cos(2t)}{2\sqrt{5}}, \sqrt{\frac{3}{10}}\right)$$

and

$$\langle V_1(t), e_7\rangle = \sqrt{\frac{3}{10}}.$$

Therefore, from Definition 2.1, α is a spherical helix.

□

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