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Geometry of bracket-generating distributions of step 2 on graded manifolds

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Abstract

A Z_2-graded analogue of bracket-generating distribution is given. Let $\mathscr{D}$ be a distribution of rank (p,q) on an (m,n)-dimensional graded manifold $\mathscr{M}$, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if F is of constant rank $(m-p,n-q)$ on $\mathscr{M}$.

1. Introduction

A smooth distribution $D \subset TM$ is said to be bracket-generating if all iterated brackets among its sections generate the whole tangent space to the manifold M, [1, 8]. D is a bracket-generating distribution of step 2 if $D^2 = TM$, where $D^2 = D + [D,D]$. Bejancu showed that a distribution of rank $k < m = dimM$ is a bracket-generating distribution of step 2, if and only if, the curvature of D is of constant rank $m-k$ on M, [1].

In this paper, a Z_2-graded analogue of bracket-generating distribution of step 2 is given. Some differences arise in the graded case due to the presence of odd generators. Given a distribution $\mathscr{D}$ of rank (p,q) on an (m,n)-dimensional graded manifold $\mathscr{M}$, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if F is of constant rank $(m-p,n-q)$ on $\mathscr{M}$. In particular, if $rank\mathscr{D}(z) = (m-1,n)$, then for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_0 \neq 0$ and if $rank\mathscr{D}(z) = (m,n-1)$, then $F_1 \neq 0$ on $\mathscr{M}$.

2. Preliminaries

Let M be a topological space and let $\mathscr{O}_M$ be a sheaf of super $\mathbb{R}$-algebras with unity. A graded manifold of dimension (m,n) is a ringed space $\mathscr{M} = (M, \mathscr{O}_M)$ which is locally isomorphic to $\mathbb{R}^{m|n}$, (see [6]) .

Let $\mathscr{M}$ and $\mathscr{N}$ be graded manifolds. Let $\phi : M \mapsto N$ be a continuous map such that $\phi^* : \mathscr{O}_N \longrightarrow \mathscr{O}_M$ takes $\mathscr{O}_N(V)$ into $\mathscr{O}_M(\phi^{-1}(V))$ for each open set $V \subset N$, then we say that $\Phi = (\phi, \phi^*) : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism between $\mathscr{M}$ and $\mathscr{N}$.

Let A be a super $\mathbb{R}$-algebra, $\varphi \in End_{\mathbb{R}}A$ is called a derivation of A, if for all $a,b \in A$,

$$\varphi(ab) = \varphi(a).b + (-1)^{|\varphi||a|} a.\varphi(b), \tag{2.1}$$

where for a homogeneous element x of some graded object, $|x| \in \{0,1\}$ denotes the parity of x (see [6]).

A vector field on $\mathscr{M}$ is a derivation of the sheaf $\mathscr{O}_M$. Let $U \subset M$ be an open subset, the $\mathscr{O}_M(U)-$super module of derivations of $\mathscr{O}_M(U)$ is defined by

$$T\mathscr{M}(U) := Der(\mathscr{O}_M(U)).$$

The $\mathscr{O}_M-$module $T\mathscr{M}$ is locally free of dimension (m,n) and is called the tangent sheaf of $\mathscr{M}$. A vector field is a section of $T\mathscr{M}$.

If $\Omega^1(\mathscr{M}) := T^*\mathscr{M}$ be the dual of the tangent sheaf of a graded manifold $\mathscr{M}$, then it is the sheaf of super $\mathscr{O}_M$-modules and

$$\Omega^1(\mathscr{M}) := Hom(T\mathscr{M}, \mathscr{O}_M). \tag{2.2}$$

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It is called the cotangent sheaf of a graded manifold $\mathscr{M}$, and the sections of $\Omega^1(\mathscr{M})$ are called super differential 1-forms [2, 6].
Let $\mathscr{M}=(M,\mathscr{O}_M)$ be an (m,n)-dimensional graded manifold and $\mathscr{D}$ be a distribution of rank (p,q) $(p<m, q<n)$ on $\mathscr{M}$. Then for each point $x\in M$ there is an open subset U over which any set of generators $\{D_i, D\mu | 1\leq i\leq p, 1\leq \mu\leq q\}$ of the module $\mathscr{D}(U)$ can be enlarged to a set

$$\left\{C_a, D_i, D_\mu, C_\alpha \,\middle|\, {}^{1\leq i\leq p}_{p+1\leq a\leq m} \; and \; {}^{1\leq \mu\leq q}_{q+1\leq \alpha\leq n} \; {}^{|C_a|=0}_{|D_i|=0} \; and \; {}^{|D_\mu|=1}_{|C_\alpha|=1}\right\}$$

of free generators of $Der\mathscr{O}_M$, [3].
We attach to $\mathscr{D}$ a sequence of distributions defined by,

$$\mathscr{D}\subset\mathscr{D}^2\subset\ldots\subset\mathscr{D}^r\subset\ldots\subset Der\mathscr{O}_M,$$

with

$$\mathscr{D}^2=\mathscr{D}+[\mathscr{D},\mathscr{D}],\ldots,\mathscr{D}^{r+1}=\mathscr{D}^r+[\mathscr{D},\mathscr{D}^r],$$

where

$$[\mathscr{D},\mathscr{D}^r]=\operatorname{span}\{[X,Y]: X\in\mathscr{D}, Y\in\mathscr{D}^r\}.$$

As in the classical case, we say that $\mathscr{D}$ is a bracket-generating distribution, if there exists an $r\geq 2$ such that $\mathscr{D}^r=Der\mathscr{O}_M$. In this case r is called the step of the distribution $\mathscr{D}$.
Suppose that $X,Y\in\mathscr{D}$ and consider the linear map on $\mathscr{D}$ as follows:

$$F(X,Y)=-(-1)^{|X||Y|}[X,Y] \bmod \mathscr{D}. \tag{2.3}$$

With respect to the above local basis $\{D_i, C_a, D_\mu, C_\alpha\}$ of $Der\mathscr{O}_M$, if

$$\begin{aligned}
[D_i,D_j]&=D^k_{ij}D_k+D^d_{ij}C_d+\tilde{D}^\nu_{ij}D_\nu+\tilde{D}^\gamma_{ij}C_\gamma,\\
[D_i,D_\xi]&=D^k_{i\xi}D_k+D^d_{i\xi}C_d+\tilde{D}^\nu_{i\xi}D_\nu+\tilde{D}^\gamma_{i\xi}C_\gamma,\\
[D_\mu,D_j]&=D^k_{\mu j}D_k+D^d_{\mu j}C_d+\tilde{D}^\nu_{\mu j}D_\nu+\tilde{D}^\gamma_{\mu j}C_\gamma,\\
[D_\mu,D_\xi]&=D^k_{\mu\xi}D_k+D^d_{\mu\xi}C_d+\tilde{D}^\nu_{\mu\xi}D_\nu+\tilde{D}^\gamma_{\mu\xi}C_\gamma,
\end{aligned}$$

then, by using (2.3), we conclude that

$$\begin{aligned}
F(D_j,D_i)&=D^d_{ij}C_d+\tilde{D}^\gamma_{ij}C_\gamma \mod \mathscr{D},\\
F(D_\xi,D_i)&=D^d_{i\xi}C_d+\tilde{D}^\gamma_{i\xi}C_\gamma \mod \mathscr{D},\\
F(D_j,D_\mu)&=D^d_{\mu j}C_d+\tilde{D}^\gamma_{\mu j}C_\gamma \mod \mathscr{D},\\
F(D_\xi,D_\mu)&=D^d_{\mu\xi}C_d+\tilde{D}^\gamma_{\mu\xi}C_\gamma \mod \mathscr{D}.
\end{aligned} \tag{2.4}$$

Each component D^a_{bc} of F is a superfunction on U.
Let $\overline{U}$ be an open subset of M such that $U\cap\overline{U}\neq\emptyset$. If we change the basis of $Der\mathscr{O}_M(U\cap\overline{U})$ to $\{\overline{D}_i,\overline{C}_a,\overline{D}_\mu,\overline{C}_\alpha\}$ then we have

$$\begin{aligned}
\overline{D}_j&=f^i_jD_i+f^\mu_jD_\mu,\\
\overline{D}_\nu&=f^i_\nu D_i+f^\mu_\nu D_\mu,\\
\overline{C}_b&=f^i_bD_i+g^a_bC_a+f^\mu_bD_\mu+g^\alpha_bC_\alpha,\\
\overline{C}_\beta&=f^i_\beta D_i+g^a_\beta C_a+f^\mu_\beta D_\mu+g^\alpha_\beta C_\alpha,
\end{aligned}$$

where

$$\begin{bmatrix} f^i_j & f^\mu_j\\ f^i_\nu & f^\mu_\nu\end{bmatrix} \text{ and } \begin{bmatrix} g^a_b & g^\alpha_b\\ g^a_\beta & g^\alpha_\beta\end{bmatrix},$$

are nonsingular supermatrices of smooth functions on $U\cap\overline{U}$. Both of these matrices are even. With respect to the basis $\{\overline{D}_j,\overline{C}_b,\overline{D}_\nu,\overline{C}_\beta\}$ on $\overline{U}$, if $\{\overline{D}^b_{kh},\overline{D}^\beta_{kh},\ldots,\overline{D}^\beta_{\xi\rho}\}$ are the local components of F, then we have

$$\begin{bmatrix} \overline{D}^b_{kh} & \overline{D}^\beta_{kh}\\ \overline{D}^b_{\xi h} & \overline{D}^\beta_{\xi h}\\ \overline{D}^b_{k\rho} & \overline{D}^\beta_{k\rho}\\ \overline{D}^b_{\xi\rho} & \overline{D}^\beta_{\xi\rho}\end{bmatrix}\begin{bmatrix} g^a_b & g^\alpha_b\\ g^a_\beta & g^\alpha_\beta\end{bmatrix}=\begin{bmatrix} f^j_h & 0 & f^\mu_h & 0\\ 0 & f^\mu_\rho & 0 & f^j_\rho\\ f^j_\rho & 0 & f^\mu_\rho & 0\\ 0 & f^\mu_h & 0 & f^j_h\end{bmatrix}\begin{bmatrix} f^i_k & 0 & f^\nu_k & 0\\ 0 & f^\nu_\xi & 0 & -f^i_\xi\\ 0 & -f^\nu_k & 0 & f^i_k\\ f^i_\xi & 0 & f^\nu_\xi & 0\end{bmatrix}\begin{bmatrix} D^a_{ij} & D^\alpha_{ij}\\ D^a_{\nu\mu} & D^\alpha_{\nu\mu}\\ D^a_{\nu j} & D^\alpha_{\nu j}\\ D^a_{i\mu} & D^\alpha_{i\mu}\end{bmatrix} \tag{2.5}$$

Since $\begin{bmatrix} f^i_j & f^\mu_j \\ f^i_\nu & f^\mu_\nu \end{bmatrix}$ is invertible at $x \in U \cap \overline{U}$, we see that $\begin{bmatrix} f^i_j & 0 \\ 0 & f^\mu_\nu \end{bmatrix}$ is invertible and from (2.5) we conclude that if

$$D(x) = \begin{bmatrix} D^{p+q+1}_{1\,2} & D^{p+q+1}_{1\,3} & \dots & D^{p+q+1}_{1\,p+q} & D^{p+q+1}_{2\,3} & \dots & D^{p+q+1}_{2\,p+q} & \dots & D^{p+q+1}_{p+q-1\,p+q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ D^{m+n}_{1\,2} & D^{m+n}_{1\,3} & \dots & D^{m+n}_{1\,p+q} & D^{m+n}_{2\,3} & \dots & D^{m+n}_{2\,p+q} & \dots & D^{m+n}_{p+q-1\,p+q} \end{bmatrix}(x)$$

then rank $D(x) = \mathrm{rank}\overline{D}(x)$.
Now we can define the rank of F, which is related to its coefficients matrix. Before doing this, in view of (2.4), we note that the submatrices

$$\begin{bmatrix} D^a_{ij}(x) & D^a_{\mu\nu}(x) \\ \tilde{D}^\alpha_{ij}(x) & \tilde{D}^\alpha_{\mu\nu}(x) \end{bmatrix} \, and \begin{bmatrix} D^a_{i\mu}(x) & D^a_{\mu i}(x) \\ \tilde{D}^\alpha_{i\mu}(x) & \tilde{D}^\alpha_{\mu i}(x) \end{bmatrix}$$

are even and odd respectively. The rank of the first submatrix can be defined but for the second submatrix, since $D^a_{\mu i}(x)$ and $\tilde{D}^\alpha_{i\mu}(x)$ are even, we consider the matrix $\begin{bmatrix} D^a_{\mu i}(x) & D^a_{i\mu}(x) \\ \tilde{D}^\alpha_{\mu i}(x) & \tilde{D}^\alpha_{i\mu}(x) \end{bmatrix}$ to define its rank. Now set

$$r := rank \begin{bmatrix} D^a_{ij}(x) & D^a_{\mu\nu}(x) \\ \tilde{D}^\alpha_{ij}(x) & \tilde{D}^\alpha_{\mu\nu}(x) \end{bmatrix} \, and \quad s := rank \begin{bmatrix} D^a_{\mu i}(x) & D^a_{i\mu}(x) \\ \tilde{D}^\alpha_{\mu i}(x) & \tilde{D}^\alpha_{i\mu}(x) \end{bmatrix},$$

where $i, j = 1, ..., p, a = p+1, .., m$ and $\mu, \nu = 1, ..., q, \alpha = q+1, ..., n$. Thus we define

$$rankF(x) = (r, s).$$

If $(q_{\bar{a}}, \xi_{\bar{\mu}})$ are local supercoordinates on a coordinate neighborhood U of $x \in M$, $(\bar{a} = 1, ..., m, \bar{\mu} = 1, ..., n)$, then $\mathscr{D}$ is locally given by the graded 1-forms

$$\phi_{\bar{b}} = \phi^{\bar{a}}_{\bar{b}} dq_{\bar{a}} + \tilde{\phi}^{\bar{\mu}}_{\bar{b}} d\xi_{\bar{\mu}} = 0, \quad \bar{b} = 1, ..., p$$
$$\phi_{\bar{\alpha}} = \phi^{\bar{a}}_{\bar{\alpha}} dq_{\bar{a}} + \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} d\xi_{\bar{\mu}} = 0, \quad \bar{\alpha} = 1, ..., q.$$

Since $\mathscr{D}$ is a distribution of rank (p, q), we may assume that the submatrices $(\phi^{\bar{a}}_{\bar{b}}), 1 \le \bar{a}, \bar{b} \le p$, and $(\tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}}), 1 \le \bar{\alpha}, \bar{\mu} \le q$ are invertible. Let the matrix $\psi = (\psi^\bullet_\bullet)$ denotes the inverse of the matrix $\begin{pmatrix} \phi^{\bar{a}}_{\bar{b}} & \tilde{\phi}^{\bar{\mu}}_{\bar{b}} \\ \phi^{\bar{a}}_{\bar{\alpha}} & \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} \end{pmatrix}$, $1 \le \bar{a}, \bar{b} \le p, 1 \le \bar{\alpha}, \bar{\mu} \le q$ and suppose

$$\bar{\phi}_{\bar{a}} = \psi^{\bar{b}}_{\bar{a}} \phi_{\bar{b}} + \tilde{\phi}^{\bar{\mu}}_{\bar{a}} \phi_{\bar{\mu}}, \quad \bar{\phi}_{\bar{\alpha}} = \phi^{\bar{b}}_{\bar{\alpha}} \phi_{\bar{b}} + \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} \phi_{\bar{\mu}}.$$

Therefore, the new notation

$$y_a = q_a, x_i = q_i, i = 1, ..., p, \quad a = p+1, ..., m,$$
$$\zeta_\alpha = \xi_\alpha, \eta_\mu = \xi_\mu, \mu = 1, ..., q, \quad \alpha = q+1, ..., n,$$

for the coordinates, may be performed to bring the local basis of $\Omega^1(\mathscr{M})$ into the form $\{dx_i, d\eta_\mu, dy_a + r^a_i dx_i + r^a_\mu d\eta_\mu, d\zeta_\alpha + r^\alpha_i dx_i + r^\alpha_\mu d\eta_\mu\}$. It is easy to check that

$$\frac{\delta}{\delta x_i} := \frac{\partial}{\partial x_i} - r^a_i \frac{\partial}{\partial y_a} - r^\alpha_i \frac{\partial}{\partial \zeta_\alpha}, \; i = 1, \dots, p,$$
$$\frac{\delta}{\delta \eta_\mu} := \frac{\partial}{\partial \eta_\mu} + r^a_\mu \frac{\partial}{\partial y_a} - r^\alpha_\mu \frac{\partial}{\partial \zeta_\alpha}, \; \mu = 1, \dots, q, \tag{2.6}$$

are (respectively even and odd) generators of $\mathscr{D}$ on U and $\{\delta/\delta x_i, \delta/\delta \eta_\mu, \partial/\partial y_a, \partial/\partial \zeta_\alpha\}$ is a local basis for $Der(\mathscr{O}_M(U))$, (see also [4, 5]). With respect to this basis, if we put

$$F(\frac{\delta}{\delta x_j}, \frac{\delta}{\delta x_i}) = F_i{}^a_j \frac{\partial}{\partial y_a} + \tilde{F}_i{}^\alpha_j \frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},$$
$$F(\frac{\delta}{\delta x_j}, \frac{\delta}{\delta \eta_\nu}) = F_\nu{}^a_j \frac{\partial}{\partial y_a} + \tilde{F}_\nu{}^\alpha_j \frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},$$
$$F(\frac{\delta}{\delta \eta_\mu}, \frac{\delta}{\delta x_i}) = F_i{}^a_\mu \frac{\partial}{\partial y_a} + \tilde{F}_i{}^\alpha_\mu \frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},$$
$$F(\frac{\delta}{\delta \eta_\mu}, \frac{\delta}{\delta \eta_\nu}) = F_\nu{}^a_\mu \frac{\partial}{\partial y_a} + \tilde{F}_\nu{}^\alpha_\mu \frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D}, \tag{2.7}$$

then by using (2.3) and (2.6), we deduce that

$$\begin{aligned}
F_{i\,j}^{\,a}\frac{\partial}{\partial y_a}+\tilde{F}_{i\,j}^{\,\alpha}\frac{\partial}{\partial \zeta_\alpha} &= [\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}] = (\frac{\delta r_i^a}{\delta x_j}-\frac{\delta r_j^a}{\delta x_i})\frac{\partial}{\partial y_a}+(\frac{\delta r_i^\alpha}{\delta x_j}-\frac{\delta r_j^\alpha}{\delta x_i})\frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},\\
F_{\nu\,j}^{\,a}\frac{\partial}{\partial y_a}+\tilde{F}_{\nu\,j}^{\,\alpha}\frac{\partial}{\partial \zeta_\alpha} &= [\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta x_j}] = (-\frac{\delta r_\nu^a}{\delta x_j}-\frac{\delta r_j^a}{\delta \eta_\nu})\frac{\partial}{\partial y_a}+(\frac{\delta r_\nu^\alpha}{\delta x_j}-\frac{\delta r_j^\alpha}{\delta \eta_\nu})\frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},\\
F_{i\,\mu}^{\,a}\frac{\partial}{\partial y_a}+\tilde{F}_{i\,\mu}^{\,\alpha}\frac{\partial}{\partial \zeta_\alpha} &= [\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}] = (\frac{\delta r_i^a}{\delta \eta_\mu}+\frac{\delta r_\mu^a}{\delta x_i})\frac{\partial}{\partial y_a}+(\frac{\delta r_i^\alpha}{\delta \eta_\mu}-\frac{\delta r_\mu^\alpha}{\delta x_i})\frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D},\\
F_{\nu\,\mu}^{\,a}\frac{\partial}{\partial y_a}+\tilde{F}_{\nu\,\mu}^{\,\alpha}\frac{\partial}{\partial \zeta_\alpha} &= [\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta \eta_\mu}] = (\frac{\delta r_\nu^a}{\delta \eta_\mu}+\frac{\delta r_\mu^a}{\delta \eta_\nu})\frac{\partial}{\partial y_a}+(-\frac{\delta r_\nu^\alpha}{\delta \eta_\mu}-\frac{\delta r_\mu^\alpha}{\delta \eta_\nu})\frac{\partial}{\partial \zeta_\alpha} \mod \mathscr{D}.
\end{aligned} \tag{2.8}$$

Now let us consider a distribution $\mathscr{D}$ of corank one on $\mathscr{M}$. For each $z\in M$, there are two cases.

Case1. Let $rank\mathscr{D}(z)=(m-1,n)$. Then there exist a coordinate system $(x_i,t,\eta_\mu), i=1,...,m-1, \mu=1,...,n$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by

$$dt+r_i dx_i+r_\mu d\eta_\mu=0.$$

Case2. Let $rank\mathscr{D}(z)=(m,n-1)$. Then there exist a coordinate system $(x_j,\eta_\nu,\theta), j=1,...,m, \nu=1,...,n-1$ defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by

$$d\theta+r_j dx_j+r_\nu d\eta_\nu=0.$$

Note that in the first case, (2.8) becomes

$$\begin{aligned}
F_{ij}\frac{\partial}{\partial t} &= [\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}] = (\frac{\delta r_i}{\delta x_j}-\frac{\delta r_j}{\delta x_i})\frac{\partial}{\partial t} \mod \mathscr{D},\\
F_{\nu j}\frac{\partial}{\partial t} &= [\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta x_j}] = (-\frac{\delta r_j}{\delta \eta_\nu}-(-1)^{|t|}\frac{\delta r_\nu}{\delta x_j})\frac{\partial}{\partial t} \mod \mathscr{D},\\
F_{i\mu}\frac{\partial}{\partial t} &= [\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}] = (\frac{\delta r_i}{\delta \eta_\mu}+(-1)^{|t|}\frac{\delta r_\mu}{\delta x_i})\frac{\partial}{\partial t} \mod \mathscr{D},\\
F_{\nu\mu}\frac{\partial}{\partial t} &= [\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta \eta_\mu}] = ((-1)^{|t|}\frac{\delta r_\nu}{\delta \eta_\mu}+(-1)^{|t|}\frac{\delta r_\mu}{\delta \eta_\nu})\frac{\partial}{\partial t} \mod \mathscr{D},
\end{aligned} \tag{2.9}$$

where $F_{ij},F_{\nu j},F_{i\mu}$ and $F_{\nu\mu}$ are the local components of F with respect to the local basis $\{\delta/\delta x_i,\delta/\delta x_\mu,\partial/\partial t\}$.

3. Bracket-generating distribution of step 2

In this section, we want to find the conditions under which a distribution $\mathscr{D}$ is bracket-generating of step 2. As mentioned in the previous section, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. We will have several types of possibilities for the rank of F. Using this, we find conditions to describe the problem.

Theorem 3.1. *Let $\mathscr{D}$ be a distribution of rank (p,q) $(p<m,q<n)$ on an (m,n)-dimensional graded manifold $\mathscr{M}$ such that*

$$m-p\leq\frac{p(p-1)}{2}+\frac{q(q-1)}{2}, n-q\leq\frac{q(q-1)}{2}, \tag{3.1}$$

Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if, the linear map F associated to $\mathscr{D}$ is of constant rank $(m-p,n-q)$ on $\mathscr{M}$.

Proof. Let $x\in M$. Suppose $\mathscr{D}$ is a bracket-generating distribution of step 2 and let $\{\delta/\delta x_i,\delta/\delta\eta_\mu,\partial/\partial y_a,\partial/\partial\zeta_\alpha\}$ be a basis of $Der\mathscr{O}_M(U)$ in a coordinate neighborhood U of x. Then $rank[\mathscr{D},\mathscr{D}](x)=(m-p,n-q)$. This means that the number of linearly independent graded vector fields of the set $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}],[\frac{\delta}{\delta \eta_\mu},\frac{\delta}{\delta \eta_\nu}],1\leq i,j\leq p,1\leq\mu,\nu\leq q\}$, (respectively $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}],1\leq i\leq p,1\leq\mu\leq q\}$) is $m-p$ (respectively $n-q$). Therefore the coefficient matrix, the matrix consisting of the coefficients of the Lie brackets of graded vector fields $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}],[\frac{\delta}{\delta \eta_\mu},\frac{\delta}{\delta \eta_\nu}]\}$ at the point x, denoted by

$$\begin{bmatrix} D^a_{ij}(x) & D^a_{\mu\nu}(x)\\ \tilde{D}^\alpha_{ij}(x) & \tilde{D}^\alpha_{\mu\nu}(x)\end{bmatrix}_{\substack{a=1,...,m-p\\ \alpha=1,...,n-q}}, (mod\mathscr{D}),$$

having the rank $m-p$, is invertible. Similarly, the coefficient matrix

$$\begin{bmatrix} D^a_{i\mu}(x)\\ \tilde{D}^\alpha_{i\mu}(x)\end{bmatrix}_{\substack{a=1,...,m-p\\ \alpha=1,...,n-q}}, (\mod\mathscr{D}),$$

the matrix consisting of the coefficients of the Lie brackets of graded vector fields $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}]\}$ at the point x, has rank $n-q$, (i.e. $n-q=rank\begin{pmatrix}\tilde{D}^\alpha_{i\mu}(x)\\ D^a_{i\mu}(x)\end{pmatrix}$, and this matrix is even). Hence associated with F is the graded vector field, represented by the matrix $\begin{pmatrix}D^a_{bc}(x)\\ \tilde{D}^\alpha_{ef}(x)\end{pmatrix}, (mod\mathscr{D})$, relative to the above basis. It is clear that $rankF(x)=(m-p,n-q)$.

Conversely, suppose that $x \in M$ and $rankF(x) = (m-p, n-q)$ on $\mathscr{M}$. Let $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial y_a, \partial/\partial\zeta_\alpha\}$ be a basis of $Der\mathscr{O}_M(U)$ in a coordinate neighborhood U of x. Consider the coefficient matrix of the graded vector fields $F(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta x_j})$ and $F(\frac{\delta}{\delta\eta_\mu}, \frac{\delta}{\delta\eta_\nu})$, which is even and denoted by

$$\begin{bmatrix} F^a_{ij}(x) & F^a_{\mu\nu}(x) \\ \tilde{F}^\alpha_{ij}(x) & \tilde{F}^\alpha_{\mu\nu}(x) \end{bmatrix}. \tag{3.2}$$

Note that its rank is $m-p$, otherwise F would not be a map of the given rank. Thus there are two non-negative integers r and s such that $r+s = m-p$ and $rank(F^a_{ij}(x)) = r, rank(\tilde{F}^\alpha_{\mu\nu}(x)) = s$. Hence we may assume that the submatrices $G = (F^{a'}_{i'j'}(x)), 1 \leq a', j'-1 \leq r, i' < j'$ and $J = (\tilde{F}^{\alpha'}_{\mu'\nu'}(x)), 1 \leq \alpha', \nu'-1 \leq s, \mu' < \nu'$, are both invertible. Therefore, the submatrix,

$$\begin{bmatrix} G & H \\ I & J \end{bmatrix} = \begin{bmatrix} F^{a'}_{i'j'}(x) & F^{a'}_{\mu'\nu'}(x) \\ \tilde{F}^{\alpha'}_{i'j'}(x) & \tilde{F}^{\alpha'}_{\mu'\nu'}(x) \end{bmatrix}, \ {}_{\substack{1\leq a', j'-1\leq r \\ i'<j'}}, \ {}_{\substack{1\leq \alpha', \nu'-1\leq s \\ \mu'<\nu'}}, \tag{3.3}$$

is invertible.

Similarly, consider the coefficient matrix of the graded vector fields $F(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta\eta_\mu})$, which is odd and its rank is $n-q$. We denote it by

$$\begin{bmatrix} F^a_{i\mu}(x) \\ \tilde{F}^\alpha_{i\mu}(x) \end{bmatrix}, 1 \leq a \leq m-p, 1 \leq \alpha \leq n-q.$$

Since $rank(\tilde{F}^\alpha_{i\mu}(x)) = n-q$, we may assume that the submatrice $(\tilde{F}^\alpha_{i'\mu'}(x)), 1 \leq \mu'-1 \leq n-q, i' < \mu'$, is invertible. We thus consider

$$\begin{bmatrix} F^a_{i'\mu'}(x) \\ \tilde{F}^\alpha_{i'\mu'}(x) \end{bmatrix}, 1 \leq \mu'-1 \leq n-q, i' < \mu'. \tag{3.4}$$

Given the matrices (3.3) and (3.4), we may change the generators of $Der\mathscr{O}_{\mathscr{M}}$ to $\{\delta/\delta x_i, \delta/\delta\eta_\mu, Y_b, Z_\nu\}, b = 1, ..., m-p; \nu = 1, ..., n-q$, where $Y_b \in \{[\frac{\delta}{\delta x_{i'}}, \frac{\delta}{\delta x_{j'}}], [\frac{\delta}{\delta\eta_{\mu'}}, \frac{\delta}{\delta\eta_{\nu'}}]\}$, with local coefficients $(F^{a'}_{i'j'}(x) \ \ \tilde{F}^{\alpha'}_{i'j'}(x))$ or $(F^{a'}_{\mu'\nu'}(x) \ \ \tilde{F}^{\alpha'}_{\mu'\nu'}(x))$ of the matrix (3.3) and $Z_\nu \in \{[\frac{\delta}{\delta x_{i'}}, \frac{\delta}{\delta\eta_{\mu'}}]\}$, with local coefficients $(F^{a'}_{i'\mu'}(x) \ \ \tilde{F}^{\alpha'}_{i'\mu'}(x))$ of the matrix (3.3). Thus $\mathscr{D}$ is bracket-generating of step 2. □

By using Theorem (3.1) we can easily prove the following theorems.

Theorem 3.2. *Let $\mathscr{M}$ be an (m,n) dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(m-1,n)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_0 \neq 0$ on $\mathscr{M}$.*

Proof. Since $rank\mathscr{D}(z) = (m-1,n)$, there exist a coordinate system $(x_i, t, \eta_\mu), i = 1, ..., m-1, \mu = 1, ..., n$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by $\{\delta/\delta x_i, \delta/\delta\eta_\mu\}$ and $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial t\}$ is a local basis for $Der\mathscr{O}_M$. Therefore, according to the Theorem 3.1, the coefficient matrix,

$$\begin{bmatrix} D^1_{ij}(x) & D^1_{\mu\nu}(x) \end{bmatrix}, \quad (mod\mathscr{D}),$$

has the rank 1. Hence $F_0 \neq 0$. □

Theorem 3.3. *Let $\mathscr{M}$ be an $(m,n)-$dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(m,n-1)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_1 \neq 0$ on $\mathscr{M}$.*

Proof. Since $rank\mathscr{D}(z) = (m,n-1)$, there exist a coordinate system $(x_i, \eta_\mu, \theta), i = 1, ..., m, \ \mu = 1, ..., n-1$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by $\{\delta/\delta x_i, \delta/\delta\eta_\mu\}$ and $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial\theta\}$ is a local basis for $Der\mathscr{O}_M$. Therefore, according to the Theorem 3.1, the coefficient matrix,

$$\begin{bmatrix} \tilde{D}^1_{i\mu}(x) \end{bmatrix}, \quad (\ \ \mathrm{mod}\ \mathscr{D}),$$

has the rank n. Hence $F_1 \neq 0$. □

Theorem 3.4. *Let $\mathscr{M}$ be an (m,n) dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(0,n)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $rankF_0 = m$ on $\mathscr{M}$.*

Proof. The details are the same as those given in the proof of Theorem 3.1. □

Example 3.5. *Consider the graded manifold $\mathscr{M} = R^{3|1}$. Let $(x_i, t, \eta), i = 1, ..., 2$ be local supercoordinates on a coordinate neighborhood U of $x \in R^3$. Suppose that $\mathscr{D}$ is the distribution spanned by $\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}$ and $\frac{\delta}{\delta\eta}$ where*

$$\frac{\delta}{\delta\eta} = \frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial t}, \quad \frac{\delta}{\delta x_1} = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial t}, \quad \frac{\delta}{\delta x_2} = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial t}.$$

A simple calculation shows that $[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}] = \frac{\partial}{\partial t}$ and $\{\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}, \frac{\partial}{\partial t}, \frac{\delta}{\delta\eta}\}$ is a basis of $Der\mathscr{O}_{R^3}(U)$. Thus $\mathscr{D}$ is bracket-generating of step 2.

Example 3.6. *Consider the graded manifold $\mathscr{M} = R^{4|4}$. Let $(x_i, \eta_\mu), i, \mu = 1, \dots, 4$ be local supercoordinates on a coordinate neighborhood U of $x \in R^4$. Suppose that $\mathscr{D}$ is the distribution (see [7]) spanned by $\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}$ and $\frac{\delta}{\delta\eta_4}$, where*

$$\frac{\delta}{\delta\eta_1} = \frac{\partial}{\partial\eta_1} - i\eta_3\frac{\partial}{\partial x_1} - i\eta_4\frac{\partial}{\partial x_2} - \eta_4\frac{\partial}{\partial x_3} - i\eta_3\frac{\partial}{\partial x_4},$$
$$\frac{\delta}{\delta\eta_2} = \frac{\partial}{\partial\eta_2} - i\eta_4\frac{\partial}{\partial x_1} - i\eta_3\frac{\partial}{\partial x_2} + \eta_3\frac{\partial}{\partial x_3} + i\eta_4\frac{\partial}{\partial x_4},$$
$$\frac{\delta}{\delta\eta_3} = \frac{\partial}{\partial\eta_3} - i\eta_1\frac{\partial}{\partial x_1} - i\eta_2\frac{\partial}{\partial x_2} + \eta_2\frac{\partial}{\partial x_3} - i\eta_1\frac{\partial}{\partial x_4},$$
$$\frac{\delta}{\delta\eta_4} = \frac{\partial}{\partial\eta_4} - i\eta_2\frac{\partial}{\partial x_1} - i\eta_1\frac{\partial}{\partial x_2} - \eta_1\frac{\partial}{\partial x_3} + i\eta_2\frac{\partial}{\partial x_4}.$$

Here $i = \sqrt{-1}$. Thus the vector fields $[\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}], [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_2}], [\frac{\delta}{\delta\eta_3}, \frac{\delta}{\delta\eta_3}]$, and $[\frac{\delta}{\delta\eta_3}, \frac{\delta}{\delta\eta_4}]$ are zero and

$$[\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}] = -2i\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_4}, \quad [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}] = -2i\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial x_3},$$
$$[\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}] = -2i\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}, \quad [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}] = -2i\frac{\partial}{\partial x_1} + 2i\frac{\partial}{\partial x_4}.$$

In the notation used in Theorem 3.1, all of the entries $D^a_{ij}, \tilde{D}^\alpha_{ij}, \tilde{D}^\alpha_{\mu\nu}, D^a_{i\mu}, \tilde{D}^\alpha_{i\mu}$ of the coefficient matrix except $D^a_{\mu\nu}$ are zero and

$$[D^a_{\mu\nu}] = \begin{bmatrix} 0 & -2i & 0 & 0 & -2i & 0 \\ 0 & 0 & -2i & -2i & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & -2i & 0 & 0 & +2i & 0 \end{bmatrix}. \tag{3.5}$$

So we have rank$(D^a_{\mu\nu}) = 4$, and we conclude from Corollary 3.4, that $\mathscr{D}$ is a bracket-generating distribution of step 2. By calculation we have

$$\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}]) + (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}]) - 2(\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}])) = \frac{\partial}{\partial x_1},$$
$$\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}]) + (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}]) - 2(\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}])) = \frac{\partial}{\partial x_2},$$
$$\frac{1}{4}((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}]) - (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}])) = \frac{\partial}{\partial x_3},$$
$$\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}]) - (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}])) = \frac{\partial}{\partial x_4}.$$

Example 3.7. *Let $\mathscr{M} = R^{3|1}$ equiped with local supercoordinates (x_1, x_2, x_3, η) and $\mathscr{D}$ be the distribution spanned by $\{\frac{\delta}{\delta x_1} = \frac{\partial}{\partial x_1}, \frac{\delta}{\delta x_2} = \frac{\partial}{\partial x_2} + (x_1)^2\frac{\partial}{\partial x_3}, \frac{\delta}{\delta\eta} = \frac{\partial}{\partial\eta}\}$. In this case we have*

$$[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta\eta}] = [\frac{\delta}{\delta x_2}, \frac{\delta}{\delta\eta}] = 0,$$
$$[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}] = 2x_1\frac{\partial}{\partial x_3},$$
$$[\frac{\delta}{\delta x_1}, [\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}]] = 2\frac{\partial}{\partial x_3}.$$

We conclude from Corollary 3.2 that $\mathscr{D}$ is not bracket-generating of step 2 on the whole $R^{3|1}$. It is bracket-generating of step 3.

Example 3.8. *Let $\mathscr{M} = R^{1|2}$ equiped with local supercoordinates (x, η_1, η_2) and $\mathscr{D}$ be the distribution spanned by $\{\frac{\delta}{\delta x} = \frac{\partial}{\partial x}, \frac{\delta}{\delta\eta_1} = \frac{\partial}{\partial\eta_1} + x\frac{\partial}{\partial\eta_2}\}$. Then $[\frac{\partial}{\partial x}, \frac{\delta}{\delta\eta_1}] = \frac{\delta}{\delta\eta_2}$ and from Corollary 3.3, we see that $\mathscr{D}$ is bracket-generating of step 2 on $R^{1|2}$.*

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