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ON A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS SATISFYING SUBORDINATE CONDITIONS

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Abstract. The purpose of our present paper is to introduce a new subclass of bi-univalent functions associated with pseudo-starlike function with Sakaguchi type functions and to determine the coefficient estimates $|a_2|$ and $|a_3|$ for functions in each of this newly-defined class. We also highlight some known consequences of our main results.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. Let S be the subclass of A consisting of functions which are analytic and univalent in $\mathbb{U}$.

Here, we recall some definitions and concepts of classes of analytic functions. Let $f \in A$. Then f is said to be in the class $S(\alpha, s, t)$ if it satisfies

$$\Re\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right) > \alpha,$$

for some $0 \leq \alpha < 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and for all $z \in \mathbb{U}$. The class $S(\alpha, s, t)$ was introduced by Frasin [10]. The class $S(\alpha, 1, t)$ was introduced and studied by Owa et al. [18], and by taking $t = -1$, the class $S(\alpha, 1, -1) \equiv S_s(\alpha)$ was introduced by Sakaguchi [19] and is called Sakaguchi function of order α, where as $S_s(0) = S_s$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. Also, we note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ which is the familiar class of starlike functions of order $\alpha\,(0 \leq \alpha < 1)$.

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With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0, \ |w(z)| < 1, \ z \in \mathbb{U}\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f \prec g \ \text{ or } \ f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Keobe One-Quarter Theorem [9] states that, the range of every function of the class S contains the disk $\{w : |w| < 1/4\}$. Therefore, every $f \in S$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f); r_0(f) \geq 1/4).$$

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots. \tag{2}$$

A function $f(z)$ univalent in a neighbourhood of the origin and its inverse satisfy the condition $f(f^{-1}(w)) = w$.
By using (2) yields

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots \tag{3}$$

and now by using (3) we get the following results

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

An analytic function f is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.The class of analytic and bi-univalent function in $\mathbb{U}$ is denoted by Σ. For a brief history and interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava *et al.* [21], which has apparently revived the study of bi-univalent functions in recent years. From the work of Srivastava et al. [21], we choose to recall the following examples of functions in the class Σ

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the bi-univalent function class Σ. Such other common examples of functions in S as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [21]).

Historically, Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [16] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^\star(\beta)$ and $K(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [16]). The classes $S^\star_\Sigma(\beta)$ and $K_\Sigma(\beta)$ of bi-starlike functions of order β in $\mathbb{U}$ and bi-convex functions of order β in $\mathbb{U}$, corresponding to the function classes $S^\star(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S^\star_\Sigma(\beta)$ and $K_\Sigma(\beta)$, they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. [21], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [1], [3], [12], [15], [17], [22], [23] and [24]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions (see, for example, [2], [4], [13],[25]). The coefficient estimate problem for each of the coefficients

$$|a_n| \quad (n \in \mathbb{N}\setminus\{1,2\},\ \mathbb{N} = \{1,2,3,\ldots\})$$

is still an open problem.

Babalola [6] defined the class $L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as below:

Definition 1. *Let $f \in A$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. Then $f(z) \in L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk if and only if*

$$\Re\left(\frac{z\,[f'(z)]^\lambda}{f(z)}\right) > \beta. \tag{5}$$

Babalola [6] proved that, all pseudo-starlike functions are Bazilevic of type $(1-\frac{1}{\lambda})$ order $\beta^{\frac{1}{\lambda}}$ and univalent in open unit disk $\mathbb{U}$.

Definition 2. *A function $f \in \Sigma$ is said to be in the class $\angle_{\Sigma}^{\lambda}(\phi, s, t)$, if the following subordinations hold*

$$\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)} \prec \phi(z)$$

and

$$\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-f(tw)} \prec \phi(w)$$

where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t, |\lambda| > 0, |t| \leq 1$.

We note that, for suitable choices λ, s and t, the class $\angle_{\Sigma}^{\lambda}(\phi, s, t)$ reduces to the following known classes.

Definition 3. *(see [1]) For $\lambda = 1$, a function $f \in \Sigma$ is said to be in the class*

$$S_{\Sigma}(\phi, s, t) \qquad (s, t \in \mathbb{C},\ s \neq t,\ |t| \leq 1)$$

if it satisfies the following conditions respectively:

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \prec \phi(z)$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} \prec \phi(w)$$

where $g = f^{-1}$.

Definition 4. *For $\lambda = s = 1$, a function $f \in \Sigma$ is said to be in the class*

$$S_{\Sigma}(\phi, t) \qquad (t \in \mathbb{C},\ |t| \leq 1)$$

if it satisfies the following conditions respectively:

$$\frac{(1-t)zf'(z)}{f(z)-f(tz)} \prec \phi(z)$$

and

$$\frac{(1-t)wg'(w)}{g(w)-g(tw)} \prec \phi(w)$$

where $g = f^{-1}$. This class studied by Goyal and Goswami [11].

Definition 5. *For $\lambda = s = 1$ and $t = -1$, a function $f \in \Sigma$ is said to be in the class $S_{\Sigma}(\phi)$ if it satisfies the following conditions respectively:*

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z)$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)} \prec \phi(w)$$

where $g = f^{-1}$. This class studied by Shamugham et al.[20].

Definition 6. *For $s = 1$ and $t = 0$, a function $f \in \Sigma$ is said to be in the class $\angle_{\Sigma}^{\lambda}(\phi)$ if it satisfies the following conditions respectively:*

$$\frac{z[f'(z)]^{\lambda}}{f(z)} \prec \phi(z)$$

and

$$\frac{w[g'(w)]^{\lambda}}{g(w)} \prec \phi(w)$$

where $g = f^{-1}$.

Definition 7. *(see[5]) For $\lambda = s = 1$ and $t = 0$, a function $f \in \Sigma$ is said to be in the class $S_{\Sigma}^{*}(\phi)$ if it satisfies the following conditions respectively:*

$$\frac{zf'(z)}{f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w)}{g(w)} \prec \phi(w)$$

where $g = f^{-1}$.

The purpose of our present paper is to introduce a new subclass of bi-univalent functions associated with pseudo-starlike function with Sakaguchi type functions and to determine the coefficient estimates $|a_2|$ and $|a_3|$ for functions in each of this newly-defined class. We also highlight some known consequences of our main results.

2. Coefficient Estimates

Let ϕ be an analytic function with positive real part in $\mathbb{U}$, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(\mathbb{U})$ be starlike with respect to $\phi(0) = 1$ and symmetric with respect to the axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0). \tag{6}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk $\mathbb{U}$ with $u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n (|z| < 1, |w| < 1). \tag{7}$$

It is well known that

$$|b_1| \leq 1, |b_2| \leq 1 - |b_1|^2, |c_1| \leq 1, |c_2| \leq 1 - |c_1|^2. \tag{8}$$

Next, the equation (6) and (7) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \cdots, \quad |z| < 1 \tag{9}$$

and

$$\phi(u(z)) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + \cdots, \quad |w| < 1. \tag{10}$$

For functions in the class $\angle_{\Sigma}^{\lambda}(\phi, s, t)$ the following estimates are obtained.

Theorem 1. *Let the function f given by (1) be in the class $\angle_{\Sigma}^{\lambda}(\phi, s, t)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3\lambda - 2\lambda(s+t-\lambda+1) + st)B_1^2 - (2\lambda - s - t)^2 B_2| + |2\lambda - s - t|^2 B_1}} \tag{11}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{|3\lambda - s^2 - t^2 - st|}, & B_1 \leq \frac{|2\lambda - s - t|^2}{|3\lambda - s^2 - t^{-} st|} \\ \frac{|(3\lambda - 2\lambda(s+t-\lambda+1)+st)B_1^2 - (2\lambda - s - t)^2 B_2|B_1 - |3\lambda - s^2 - t^2 - st|B_1^3}{|3\lambda - s^2 - t^2 - st|[|(3\lambda - 2\lambda(s+t-\lambda+1)+st)B_1^2 - (2\lambda - s - t)^2 B_2| + |2\lambda - s - t|^2 B_1]}, & B_1 > \frac{|2\lambda - s - t|^2}{|3\lambda - s^2 - t^2 - st|} \end{cases} \tag{12}$$

Proof. Let $f \in \angle_{\Sigma}^{\lambda}(\phi, s, t)$. Then, there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ given by (7) such that

$$\frac{(s-t)z\,[f'(z)]^{\lambda}}{f(sz) - f(tz)} = \phi(u(z)) \tag{13}$$

and

$$\frac{(s-t)w\,[g'(w)]^{\lambda}}{g(sw) - g(tw)} = \phi(v(w)). \tag{14}$$

Since

$$\frac{(s-t)z\,[f'(z)]^{\lambda}}{f(sz) - f(tz)} =$$

$$1 + (2\lambda - s - t)a_2 z + \left[(3\lambda - s^2 - t^2 - st)a_3 - (2\lambda(s+t-\lambda-1) - s^2 - t^2 - 2st)a_2^2\right] z^2 + \cdots$$

and

$$\frac{(s-t)w\,[g'(w)]^{\lambda}}{g(sw) - g(tw)} =$$

$$1 - (2\lambda - s - t)a_2 w + \left[(6\lambda - s^2 - t^2 - 2\lambda(s+t-\lambda+1))a_2^2 - (3\lambda - s^2 - t^2 - st)a_3\right] w^2 + \cdots,$$

it follows from (9), (10), (13) and (14) that

$$(2\lambda - s - t)a_2 = B_1 b_1 \tag{15}$$

$$(3\lambda - s^2 - t^2 - st)a_3 - (2\lambda(s+t-\lambda+1) - s^2 - t^2 - 2st)a_2^2 = B_1 b_2 + B_2 b_1^2 \tag{16}$$

$$-(2\lambda - s - t)a_2 = B_1 c_1 \tag{17}$$

$$(6\lambda - s^2 - t^2 - 2\lambda(s+t-\lambda+1))a_2^2 - (3\lambda - s^2 - t^2 - st)a_3 = B_1 c_2 + B_2 c_1^2. \tag{18}$$

See that, (15) and (17) together yields:

$$c_1 = -b_1. \tag{19}$$

By adding (18) to (16), further computation using (15) and (19) leads to

$$\left[(6\lambda - 4\lambda(s+t-\lambda+1) + 2st)B_1^2 - 2(2\lambda - s - t)^2 B_2\right] a_2^2 = B_1^3(b_2 + c_2). \tag{20}$$

By using (19) and (20), together with (8), we find that

$$|(3\lambda - 2\lambda(s+t-\lambda+1) + st)B_1^2 - (2\lambda - s - t)^2 B_2||a_2|^2 \leq B_1^3(1-|b_1|^2) \qquad (21)$$

which gives the desired estimate on $|a_2|$ as asserted in (11).

In order to find the bound on $|a_3|$, by subtracting (18) from (16), we readily obtain

$$2(3\lambda - s^2 - t^2 - st)a_3 - 2(3\lambda - s^2 - t^2 - st)a_2^2 = B_1(b_2 - c_2) + B_2(b_1^2 - c_1^2). \qquad (22)$$

Then, from (8) and (19), we have

$$|3\lambda - s^2 - t^2 - st|B_1|a_3| \leq \left[|3\lambda - s^2 - t^2 - st|B_1 - |2\lambda - s - t|^2\right]|a_2|^2 + B_1^2.$$

Using (11), we get the estimate on $|a_3|$ as asserted in (12). □

3. Corollaries and consequences

This section is devoted to the presentation of some special cases of the main results. These results are given in the form of corollaries.

Corollary 1. *If we let*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1),$$

then the inequalities (11) *and* (12) *becomes*

$$|a_2| \leq \frac{2\alpha}{\sqrt{|2(3\lambda - 2\lambda(s+t-\lambda+1) + st) - (2\lambda - s - t)^2|\alpha + |2\lambda - s - t|^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2\alpha}{|3\lambda - s^2 - t^2 - st|}, & \text{if } 0 < \alpha \leq \dfrac{|2\lambda - s - t|^2}{2|3\lambda - s^2 - t^2 - st|} \\ \dfrac{2[|2(3\lambda - 2\lambda(s+t-\lambda+1) + st) - (2\lambda - s - t)^2| + 2|3\lambda - s^2 - t^2 - st|]\alpha^2}{|3\lambda - s^2 - t^2 - st|[|2(3\lambda - 2\lambda(s+t-\lambda+1) + st) - (2\lambda - s - t)^2|\alpha + |2\lambda - s - t|^2]}, \\ \qquad \text{if } \dfrac{|2\lambda - s - t|^2}{2|3\lambda - s^2 - t^2 - st|} < \alpha \leq 1. \end{cases}$$

Corollary 2. *If we let*

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1),$$

then the inequalities (11) *and* (12) *becomes*

$$|a_2| \leq \frac{2(1-\beta)}{\sqrt{|2(3\lambda - 2\lambda(s+t-\lambda+1) + st)(1-\beta) - (2\lambda - s - t)^2| + |2\lambda - s - t|^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2(1-\beta)}{|3\lambda - s^2 - t^2 - st|}, & \text{if } \dfrac{2|3\lambda - s^2 - t^2 - st| - |2\lambda - s - t|^2}{2|3\lambda - s^2 - t^2 - st|} \leq \beta < 1 \\ \dfrac{2[|2(3\lambda - 2\lambda(s+t-\lambda+1) + st)(1-\beta) - (2\lambda - s - t)^2| + 2|3\lambda - s^2 - t^2 - st|(1-\beta)](1-\beta)}{|3\lambda - s^2 - t^2 - st|[|2(3\lambda - 2\lambda(s+t-\lambda+1) + st)(1-\beta) - (2\lambda - s - t)^2| + |2\lambda - s - t|^2]}, \\ \qquad \text{if } 0 \leq \beta < \dfrac{2|3\lambda - s^2 - t^2 - st| - |2\lambda - s - t|^2}{2|3\lambda - s^2 - t^2 - st|}. \end{cases}$$

Remark 1. *Taking $\lambda = 1$ in the above Theorem 1 and Corollaries 1, 2, we obtain the results of Altinkaya and Yalcin* [1].

Corollary 3. *Let $f \in S_\Sigma(\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + 2|B_1^2 - 2B_2|}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{2}, & \text{if } B_1 \leq 2 \\ \dfrac{|B_1^2 - 2B_2|B_1 + B_1^3}{|2B_1^2 - 4B_2| + 4B_1}, & \text{if } B_1 > 2. \end{cases}$$

Corollary 4. *Let $f \in S^*_\Sigma(\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 - B_2| + B_1}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{2}, & \text{if } B_1 \leq \dfrac{1}{2} \\ \dfrac{|B_1^2 - B_2|B_1 + 2B_1^3}{2[|B_1^2 - B_2| + B_1]}, & \text{if } B_1 > \dfrac{1}{2}. \end{cases}$$

Remark 2. *For $f \in S^*_\Sigma(\phi)$, the function ϕ is given by*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \qquad (0 < \alpha \leq 1)$$

and so $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence the Corollary 4 reduces to an improved results of Brannan and Taha [8]. *On the other hand when*

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \leq \beta < 1),$$

$B_1 = B_2 = 2(1-\beta)$ and thus the Corollary 4 reduces to the estimate in Brannan and Taha [8].

Corollary 5. *Let $f \in \angle^\lambda_\Sigma(\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(2\lambda - 1)[|\lambda B_1^2 - (2\lambda - 1)B_2| + |2\lambda - 1|B_1]}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{(3\lambda-1)}, & \text{if } B_1 \leq \dfrac{|2\lambda-1|^2}{|3\lambda-1|} \\ \dfrac{(2\lambda-1)|\lambda B_1^2-(2\lambda-1)B_2|B_1+|3\lambda-1|B_1^3}{(3\lambda-1)[(2\lambda-1)|\lambda B_1^2-(2\lambda-1)B_2|+|2\lambda-1|^2B_1]}, & \text{if } B_1 > \dfrac{|2\lambda-1|^2}{|3\lambda-1|}. \end{cases}.$$

Remark 3. *For $f \in \angle_\Sigma^\lambda(\phi)$, the function ϕ is given by*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \qquad (0 < \alpha \leq 1)$$

and so $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence the Corollary 5 reduces to an improved results of Joshi et al. [12] *On the other hand when*

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \leq \beta < 1),$$

$B_1 = B_2 = 2(1-\beta)$ *and thus the Corollary 5 reduces to the estimate in Joshi et al.* [12]

Corollary 6. *Let $f \in S_\Sigma(\phi, t)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1-t)[|B_1^2-(1-t)B_2|+|1-t|B_1]}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{(|2-t-t^2|)}, & \text{if } B_1 \leq \dfrac{|1-t|^2}{|2-t-t^2|} \\ \dfrac{|(1-t)B_1^2-(1-t)^2B_2|B_1+|2-t-t^2|B_1^3}{(2-t-t^2)[|(1-t)B_1^2-(1-t)^2B_2|+|1-t|^2B_1]}, & \text{if } B_1 \leq \dfrac{|1-t|^2}{|2-t-t^2|}. \end{cases}.$$

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