Hacettepe Journal of Mathematics and Statistics
Volume 46 (1) (2017), 103 – 110

Weakly discontinuous and resolvable functions between topological spaces

Dedicated to the memory of Prof. Dr. L. Michael Brown

Taras Banakh * and Bogdan Bokalo†

Abstract

We prove that a function $f : X \to Y$ from a first-countable (more generally, Preiss-Simon) space X to a regular space Y is weakly discontinuous (which means that every subspace $A \subset X$ contains an open dense subset $U \subset A$ such that $f|U$ is continuous) if and only if f is open-resolvable (in the sense that for every open subset $U \subset Y$ the preimage $f^{-1}(U)$ is a resolvable subset of X) if and only if f is resolvable (in the sense that for every resolvable subset $R \subset Y$ the preimage $f^{-1}(R)$ is a resolvable subset of X). For functions on metrizable spaces this characterization was announced (without proof) by Vinokurov in 1985.

Keywords: Weakly discontinuous function, resolvable function, Preiss-Simon space

2000 AMS Classification: 54C08

1. Introduction and Main Result

In this paper we present a proof of a characterization of weakly discontinuous functions announced (without proof) by Vinokurov in [14].

A function $f : X \to Y$ between topological spaces is called *weakly discontinuous* if every subspace $A \subset X$ contains a dense open subset $U \subset A$ such that the restriction $f|U$ is continuous. It is well-known that for weakly discontinuous maps $f : X \to Y$ and $g : Y \to Z$ the composition $g \circ f : X \to Z$ is weakly discontinuous. Weakly discontinuous functions were introduced by Vinokurov [14]. Many properties of functions, equivalent to the weak discontinuity were discovered in

*Instytut Matematyki, Jan Kochanowski University in Kielce (Poland)
Email : t.o.banakh@gmail.com
†Department of Mathematics, Ivan Franko National University of Lviv (Ukraine)
Email : b.m.bokalo@gmail.com

Doi : 10.15672/HJMS.2016.399

[1, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14]. By [14, Theorem 8], a function $f : X \to Y$ from a metrizable space X to a regular space Y is weakly discontinuous if and only if for every open set $U \subset Y$ the preimage $f^{-1}(U)$ is a resolvable subset of X. We recall [10, I.12] that a subset A of a topological space X is *resolvable* if for every closed subset $F \subset X$ the set $\overline{F \cap A} \cap \overline{F \setminus A}$ is nowhere dense in F. Observe that a subset $A \subset X$ is resolvable if and only if its characteristic function $\chi_A : X \to \{0,1\}$ is weakly discontinuous. It is known [10, I.12] that the family resolvable subsets of a topological space X is closed under intersections, unions, and complements.

A function $f : X \to Y$ between topological spaces is called (*open-*)*resolvable* if for every (open) resolvable subset $R \subset Y$ the preimage $f^{-1}(R)$ is a resolvable subset of X. It is clear that each resolvable function is open-resolvable.

1.1. Proposition. *If a function $f : X \to Y$ between topological spaces is weakly discontinuous, then f is resolvable.*

Proof. If a subset $A \subset Y$ is resolvable, then its characteristic function $\chi_A : Y \to \{0,1\}$ is weakly discontinuous. Since the weak discontinuity is preserved by compositions (see, e.g., [2, 4.1]), the composition $g = \chi_A \circ f : X \to \{0,1\}$ is weakly discontinuous, which implies that the set $g^{-1}(1) = f^{-1}(A)$ is resolvable in X. □

By [14, Theorem 8], for functions between metrizable spaces, Proposition 1.1 can be reversed. However the paper [14] contains no proof of this important fact. In this paper we present a proof of this Vinokurov's characterization in a more general case of functions defined on Preiss-Simon spaces.

We define a topological space X to be *Preiss-Simon* at a point $x \in X$ if for any subset $A \subset X$ with $x \in \overline{A}$ there is a sequence $(U_n)_{n\in\omega}$ of non-empty open subsets of A that converges to x in the sense that each neighborhood of x contains all but finitely many sets U_n. By $PS(X)$ we denote the set of points $x \in X$ at which X is Preiss-Simon. A topological space X is called a *Preiss-Simon* space if $PS(X) = X$ (that is X is Preiss-Simon at each point $x \in X$).

It is clear that each first-countable space is Preiss-Simon and each Preiss-Simon space is Fréchet-Urysohn. A less trivial fact due to Preiss and Simon [12] asserts that each Eberlein compact space is Preiss-Simon.

A base $\mathcal{B}$ of the topology of a space X will be called *countably additive* if the union $\cup\mathcal{C}$ of any countable subfamily $\mathcal{C} \subset \mathcal{B}$ belongs to $\mathcal{B}$.

A function $f : X \to Y$ between topological spaces will be called *base-resolvable* if there exists a countably additive base $\mathcal{B}$ of the topology of Y such that for every set $B \subset Y$ the preimage $f^{-1}(B)$ is a resolvable subset of X.

It is clear that for any function $f : X \to Y$ we have the implications:

weakly discontinuous ⇒ resolvable ⇒ open-resolvable ⇒ base-resolvable.

For functions on Preiss-Simon spaces these implications can be reversed, which is proved in the following characterization. For functions on metrizable spaces it was announced (without written proof) by Vinokurov in [14, Theorem 8].

1.2. Theorem. *For a functions $f : X \to Y$ from a Preiss-Simon space X to a regular space Y the following conditions are equivalent:*

1. *f is weakly discontinuous;*
2. *f is resolvable;*

(3) *f is open-resolvable;*
(4) *f is base-resolvable.*

This theorem will be proved in Section 3 after some preliminary work made in Section 2.

By Theorem 1.2, any open-resolvable map $f : X \to Y$ from a Preiss-Simon space X to a regular space Y is resolvable. We do not know if this implication still holds for any function between regular spaces. The authors are grateful to Sergey Medvedev for turning their attention to this intriguing question.

1.3. Problem (Medvedev)**.** Is each open-resolvable function $f : X \to Y$ between regular spaces resolvable?

The following example indicates that Problem 1.3 can be difficult and shows that the countable additivity of the base $\mathcal{B}$ cannot be removed from the definition of a base-resolvable function.

1.4. Example. Let $\mathbb{R}_{\mathbb{Q}}$ be the real line endowed with the metrizable topology generated by the countable base $\mathcal{B} = \big\{(a,b) : a < b,\ a,b \in \mathbb{Q}\big\} \cup \big\{\{q\} : q \in \mathbb{Q}\big\}$. The identity map $\mathrm{id} : \mathbb{R} \to \mathbb{R}_{\mathbb{Q}}$ is not (open-)resolvable as the preimage $\mathbb{Q} = \mathrm{id}^{-1}(\mathbb{Q})$ of the open set $\mathbb{Q} \subset \mathbb{R}_{\mathbb{Q}}$ is not resolvable in $\mathbb{R}$. Yet, for every basic set $B \in \mathcal{B}$ the preimage $\mathrm{id}^{-1}(B)$ is a resolvable set in $\mathbb{R}$.

2. Five Lemmas

In this section we shall prove some auxiliary results, which will be used in the proof of Theorem 1.2. For a subset A of a topological space by $\bar{A}$, A°, and $\overline{A}^\circ$ we denote the closure, the interior, and the interior of the closure of A in X, respectively. A family $\mathcal{B}$ of non-empty open subsets of a topological space X is called a *π-base* if each non-empty open set $U \subset X$ contains some set $B \in \mathcal{B}$.

Following [2], we define a function $f : X \to Y$ between topological spaces to be *scatteredly continuous* if for every non-empty subset $A \subset X$ the restriction $f|A$ has a continuity point. It is easy to see that each weakly discontinuous function is scatteredly continuous. For maps into regular spaces the converse implication is also true (see [1], [4] or [2, 4.4]):

2.1. Lemma. *A function $f : X \to Y$ from a topological space X to a regular space Y is weakly discontinuous if and only if f is scatteredly continuous.*

We recall that for a topological space X its *tightness* $t(X)$ is the smallest cardinal κ such that for every subset $A \subset X$ and point $a \in \bar{A}$ there exists a subset $B \subset A$ of cardinality $|B| \le \kappa$ such that $a \in \bar{B}$. The following lemma was proved in [2, 2.3].

2.2. Lemma. *A function $f : X \to Y$ between topological spaces is scatteredly continuous if and only if for any non-empty subset $A \subset X$ of cardinality $|A| \le t(X)$ the restriction $f|A$ has a continuity point.*

2.3. Lemma. *If A, B are disjoint resolvable subsets of a topological space X, then $\bar{A} \cap \bar{B}$ is nowhere dense in X.*

Proof. To derive a contradiction, assume that the set $F = \bar{A} \cap \bar{B}$ has a non-empty interior U in X. Then $U \cap A$ and $U \cap B$ are two dense disjoint sets in U. By the resolvability of A, the dense subset $A \cap \bar{U}$ of $\bar{U}$ has nowhere dense boundary in $\bar{U}$. Consequently, the interior U_A of the set $A \cap \bar{U}$ is dense in $\bar{U}$. By the same reason, the interior U_B of the set $B \cap \bar{U}$ is dense in $\bar{U}$. Then the non-empty space $\bar{U}$ contains two disjoint dense open sets U_A and U_B, which is not possible. □

A function $f : X \to Y$ between topological spaces is defined to be *almost continuous* (*weakly continuous*) at a point $x \in X$ if for any neighborhood $Oy \subset Y$ of the point $y = f(x)$ the (interior of the) set $f^{-1}(Oy)$ is dense in some neighborhood of the point x in X. By $AC(f)$ (resp. $WC(f)$) we shall denote the set of point of almost (resp. weak-) continuity of f.

2.4. Lemma. *Let $f : X \to Y$ be a base-resolvable map from a topological space X to a Hausdorff space Y. Then*

(1) $AC(f) = WC(f)$.
(2) *If D is dense in X, Y is regular, and $f|D$ has no continuity point, then $D \setminus AC(f)$ also is dense in X.*
(3) *If X has a countable π-base, then for any countable dense set $D \subset X$ there is a point $y \in f(D)$ such that for every neighborhood Oy of y the preimage $f^{-1}(Oy)$ has non-empty interior in X.*
(4) *The family $\{\overline{f^{-1}(y)}^{\circ} : y \in Y\}$ is disjoint.*

Proof. Since f is basic-resolvable, there exists a countably additive base $\mathcal{B}$ of the topology of Y such that for every $U \in \mathcal{B}$ the preimage $f^{-1}(U)$ is resolvable in X.

1. The inclusion $WC(f) \subset AC(f)$ is trivial. To prove that $AC(f) \subset WC(f)$, take any point $x \in AC(f)$. To show that $x \in WC(f)$, take any neighborhood $Oy \in \mathcal{B}$ of the point $y = f(x)$ and consider the preimage $f^{-1}(Oy)$. Since $x \in AC(f)$, the closure $F = \overline{f^{-1}(Oy)}$ is a neighborhood of x. Since the set $f^{-1}(Oy)$ is resolvable, the boundary $\overline{F \cap f^{-1}(Oy)} \cap \overline{F \setminus f^{-1}(Oy)}$ is nowhere dense in F. Consequently, the interior of the set $F \cap f^{-1}(O_y)$ in F is dense in F and $x \in WC(f)$.

2. Assume that $D \subset X$ is dense, Y is regular, and $f|D$ has no continuity point. Given a point $x \in D$, and a neighborhood $O_x \subset X$ of x we should find a point $x' \in O_x \cap D \setminus AC(f)$. If $x \notin AC(f)$, then we can take $x' = x$. So we assume that $x \in AC(f)$ and hence $x \in WC(f)$ by the preceding item. Since x is a discontinuity point of $f|D$, there is a neighborhood $O_{f(x)}$ of $f(x)$ such that $f(D \cap U_x) \not\subset O_{f(x)}$ for every neighborhood U_x of x. Using the regularity of Y choose a neighborhood $U_{f(x)} \subset Y$ of $f(x)$ with $\overline{U}_{f(x)} \subset O_{f(x)}$. Since f is weakly continuous at x, the closure of the interior of the preimage $f^{-1}(U_{f(x)})$ contains some open neighborhood W_x of x. By the choice of $O_{f(x)}$, we can find a point $x' \in D \cap O_x \cap W_x$ with $f(x') \notin O_{f(x)}$. Consider the neighborhood $O_{f(x')} = Y \setminus \overline{U}_{f(x)}$ of $f(x')$ and observe that $W_x \cap f^{-1}(O_{f(x')})$ is a nowhere dense subset of Ox (because it misses the interior of $f^{-1}(U_{f(x)})$ which is dense in W_x). This witnesses that $x' \notin AC(f)$.

3. Assume that X has countable π-base $\{W_n\}_{n\in\omega}$. We lose no generality assuming that the subfamilies $\{W_{2n}\}_{n\in\omega}$ and $\{W_{2n+1}\}_{n\in\omega}$ are countable π-bases in X. Given a countable dense subset $D \subset X$, we should find a point $y \in f(D)$ such that for every neighborhood $Oy \subset Y$ the preimage $f^{-1}(Oy)$ has non-empty

interior in X. Assume conversely that each point $y \in f(D)$ has a neighborhood $O_y \in \mathcal{B}$ such that the preimage $f^{-1}(O_y)$ has empty interior in X. The resolvability of $f^{-1}(O_y)$ implies that this set is nowhere dense in X. We shall inductively construct a sequence $(x_n)_{n\in\omega}$ of points of D and a sequence $(U_n)_{n\in\omega}$ of open sets in Y such that

(a) $f(x_n) \in U_n \in \mathcal{B}$ and the set $f^{-1}(U_n)$ is nowhere dense in X;
(b) $x_n \in D \cap W_n \setminus \bigcup_{k<n} f^{-1}(U_n)$;
(c) $U_n \cap \{f(x_k)\}_{k<n} = \emptyset$.

Taking into account that $\{W_{2n}\}_{n\in\omega}$ and $\{W_{2n+1}\}_{n\in\omega}$ are π-bases in X, we conclude that the disjoint sets $\{x_{2n}\}_{n\in\omega}$ and $\{x_{2n+1}\}_{n\in\omega}$ are dense in X. The countable additivity of the base $\mathcal{B}$ guarantees that the open sets $U_e = \bigcup_{n\in\omega} U_{2n}$ and $U_o = \bigcup_{n\in\omega} U_{2n+1}$ belong to $\mathcal{B}$. Then their preimages $f^{-1}(U_e) \supset \{x_{2n}\}_{n\in\omega}$ and $f^{-1}(U_e) \supset \{x_{2n+1}\}_{n\in\omega}$ are disjoint dense resolvable sets in X. But this contradicts Lemma 2.3.

4. Assuming that the family $\{\overline{f^{-1}(y)}^{\circ} : y \in Y\}$ is not disjoint, find two distinct points $y, z \in Y$ such that the intersection

$$W = \overline{f^{-1}(y)}^{\circ} \cap \overline{f^{-1}(z)}^{\circ}$$

is not empty. Observe that the sets $W \cap f^{-1}(y)$ and $W \cap f^{-1}(z)$ both are dense in W.

By the Hausdorff property of Y the points y, z have disjoint open neighborhoods $Oy, Oz \in \mathcal{B}$. The choice of $\mathcal{B}$ guarantees that the sets $f^{-1}(Oy)$ and $f^{-1}(Oz)$ are resolvable. By Lemma 2.3, the intersection $\overline{f^{-1}(Oy)} \cap \overline{f^{-1}(Oz)}$ is nowhere dense in X, which is not possible as this intersection contains the non-empty open set W. □

2.5. Lemma. *Let $f : X \to Y$ be a base-resolvable map from a topological space X to a regular space Y and D be a countable dense subset of X such that $f|D$ has no continuity point.*

(1) *For any finite subset $F \subset Y$ there is a dense subset $Q \subset D \setminus f^{-1}(F)$ in X such that $f|Q$ has no continuity point.*
(2) *If X has a countable π-base, then for any sequence $(U_n)_{n=1}^{\infty}$ of non-empty open subsets of X there are an infinite subset $I \subset \mathbb{N}$ and sequences $(V_n)_{n\in I}$ and $(W_n)_{n\in I}$ of pairwise disjoint non-empty open sets in X and Y, respectively, such that $V_n \subset U_n \cap f^{-1}(W_n)$ for all $n \in I$.*
(3) *If $D \subset PS(X)$, then there is a countable first countable subspace $Q \subset D$ such that Q contains no finite non-empty open subsets and the restriction $f|Q$ is a bijective map whose image $f(Q)$ is a discrete subspace of Y.*

Proof. Fix a countably additive base $\mathcal{B}$ of the topology of Y such that for every $U \in \mathcal{B}$ the preimage $f^{-1}(U)$ is resolvable in X.

1. The first statement will be proved by induction on the cardinality $|F|$ of the set F. If $|F| = 0$, then we can put $Q = D$ and finish the proof. Assume that for some $n > 0$ the first statement is proved for all sets $F \subset Y$ of cardinality $|F| < n$. Take any finite subset $F \subset Y$ of cardinality $|F| = n$. Choose any point $y \in F$. By the inductive hypothesis, for the set $F \setminus \{y\}$ there exists a dense subset $E \subset D \setminus f^{-1}(F \setminus \{y\})$ such that the function $f|E$ has no continuity points. We

claim that the set $E\setminus f^{-1}(y)$ is dense in X. In the opposite case, there exists a non-empty open set $U \subset X$ such that $E \cap U \subset f^{-1}(y)$. It follows that $E \cap U \subset AC(f)$. By Lemma 2.4(2), the set $E \setminus AC(f) \subset E \setminus U$ is dense in X, which is a desired contradiction showing that the set $E \setminus f^{-1}(y)$ is dense in X and so is the set $Q := (E \setminus f^{-1}(y)) \setminus \big(\overline{E \cap f^{-1}(y)} \setminus \overline{E \cap f^{-1}(y)}^{\circ}\big) \subset D \setminus f^{-1}(F)$.

It remains to check that the restriction $f|Q$ has no continuity points. To derive a contradiction, assume that some point $x_0 \in Q$ is a continuity point of the restriction $f|Q$. If $x_0 \notin \overline{E \cap f^{-1}(y)}$, then the discontinuity of the map $f|E$ at x_0 implies the discontinuity of $f|Q$ at x_0. So, x_0 belongs to the interior $\overline{E \cap f^{-1}(y)}^{\circ}$ of $\overline{E \cap f^{-1}(y)}$.

Let $y_0 = f(x_0)$ and observe that $y_0 \neq y$ (because $x_0 \notin f^{-1}(y)$). By the Hausdorff property of Y the points y_0 and y have disjoint open neighborhoods $Oy_0, Oy \in \mathcal{B}$. By the continuity of $f|Q$ at x_0, there is an open neighborhood $Ox_0 \subset \overline{E \cap f^{-1}(y)}^{\circ}$ of x_0 such that $f(Ox_0 \cap Q) \subset Oy_0$. It follows that the preimages $f^{-1}(Oy_0)$ and $f^{-1}(Oy)$ are disjoint resolvable subsets of X. The density of the set Q in X implies the density of the set $Ox_0 \cap f^{-1}(Oy_0) \supset Ox_0 \cap Q$ in Ox_0. On the other hand, the intersection $f^{-1}(y) \cap \overline{E \cap f^{-1}(y)}^{\circ}$ is dense in $\overline{E \cap f^{-1}(y)}^{\circ}$ and hence $Ox_0 \cap f^{-1}(Oy) \supset f^{-1}(y) \cap Ox_0$ is dense in Ox_0. Therefore, $\overline{f^{-1}(Oy_0)} \cap \overline{f^{-1}(Oy)}$ contains the non-empty open set Ox_0. But this contradicts Lemma 2.3.

2. Assume that the space X has a countable π-base and let $(U_n)_{n=1}^{\infty}$ be a sequence of non-empty open subsets of X. Applying Lemma 2.4(3) to the map $f|U_1$ and the dense subset $D \cap U_1$, find a point $y_0 \in f(D \cap U_1)$ such that for each neighborhood Oy_0 the preimage $U_1 \cap f^{-1}(Oy_0)$ has non-empty interior. By induction, for every $n \in \mathbb{N}$ we shall find a point $y_n \in f(D) \setminus \{y_i : i < n\}$ such that for every neighborhood Oy_n the set $U_n \cap f^{-1}(Oy_n)$ has non-empty interior.

Assuming that for some n the points $y_0, \dots, y_{n-1}$ have being chosen, we shall find a point y_n. It follows that the intersection $D \cap U_n$ is a countable dense subset of U_n such that $f|D \cap U_n$ has no continuity point. Applying Lemma 2.5(1), we can find a dense subset $Q \subset D \cap U_n \setminus f^{-1}(\{y_0, \dots, y_{n-1}\})$ in U_n such that the restriction $f|Q$ has no continuity point. Applying Lemma 2.4(3) to the map $f|U_n$ and the dense subset $Q \cap U_n$ of U_n, find a point $y_n \in f(Q) \subset f(D) \setminus \{y_i : i < n\}$ such that for each neighborhood Oy_n the preimage $U_n \cap f^{-1}(Oy_n)$ has non-empty interior. This completes the inductive construction.

The space $\{y_n : n \in \mathbb{N}\}$, being infinite and regular, contains an infinite discrete subspace $\{y_n : n \in I\}$. By induction, we can select pairwise disjoint open neighborhoods $W_n \subset Y$, $n \in I$, of the points y_n. For every $n \in I$, the choice of the point y_n guarantees that the set $U_n \cap f^{-1}(W_n)$ contains a non-empty open set V_n. Then the set $I \subset \mathbb{N}$ and sequences $(V_n)_{n\in I}$, $(W_n)_{n\in I}$ satisfy our requirements.

3. Assume that $D \subset PS(X)$. Using the density of the countable set D and the inclusion $D \subset PS(X)$, we can show that the space X has a countable π-base. Applying Lemma 2.4(2), we get that $D \setminus AC(f)$ is dense in X.

By induction on the tree $\omega^{<\omega}$ we shall construct sequences $(x_s)_{s\in\omega^{<\omega}}$ of points of the set $D \setminus AC(f)$, and sequences $(V_s)_{s\in\omega^{<\omega}}$ and $(U_s)_{s\in\omega^{<\omega}}$, $(W_s)_{s\in\omega^{<\omega}}$ of sets so that the following conditions hold for every finite number sequence $s \in \omega^{<\omega}$:

(a) V_s is an open neighborhood of the point x_s in X;
(b) $W_s \subset U_s$ are open neighborhoods of $f(x_s)$ in Y;
(c) $f(V_s) \subset U_s$;
(d) $V_{s\hat{}n} \subset V_s$ and $U_{s\hat{}n} \subset U_s$ for all $n \in \omega$;
(e) the sequence $(V_{s\hat{}n})_{n\in\mathbb{N}}$ converges to x_s;
(f) $W_s \cap U_{s\hat{}n} = \emptyset = U_{s\hat{}n} \cap U_{s\hat{}m}$ for all $n \neq m$ in ω.

We start the induction letting $V_\emptyset = X$, $U_\emptyset = Y$ and $x_\emptyset$ be any point of $D \setminus AC(f)$.

Assume that for a finite sequence $s \in \omega^{<\omega}$ the point $x_s \in D \setminus AC(f)$ and open sets $V_s \subset X$ and $U_s \subset Y$ with $x_s \in V_s$ and $f(V_s) \subset U_s$ have been constructed. Since $f|V_s$ fails to be almost continuous at x_s, there is a neighborhood $W_s \subset U_s$ of $f(x_s)$ such that the closure of the preimage $f^{-1}(\overline{W_s})$ is not a neighborhood of x_s in X. This fact and the Preiss-Simon property of X at x_s allows us to construct a sequence $(V'_k)_{k\in\omega}$ of open subsets of $V_s \setminus \mathrm{cl}_X\big(f^{-1}(\overline{W_s})\big)$ that converges to x_s in the sense that each neighborhood of x contains all but finitely many sets V'_k. Applying Lemma 2.5(2) to the map $f|V_s : V_s \to U_s$, we can find an infinite subset $N \subset \omega$ and a sequence $(U'_k)_{k\in N}$ of pairwise disjoint open sets of U_s such that each set $f^{-1}(U'_k) \cap V'_k$, $k \in N$, has non-empty interior in X. Let $N = \{k_n : n \in \omega\}$ be the increasing enumeration of the set N.

For every $n \in \omega$ let $U_{s\hat{}n} = U'_{k_n} \setminus \overline{W_s}$, $V_{s\hat{}n}$ be a non-empty open subset in $f^{-1}(U_{k_n}) \cap V'_{k_n}$ and $x_{s\hat{}n} \in V_{s\hat{}n} \cap D \setminus AC(f)$ be any point (such a point exists because of the density of $D \setminus AC(f)$ in X). One can check that the points $x_{s\hat{}n}$, $n \in \omega$ and sets W_s, $V_{s\hat{}n}$, $U_{s\hat{}n}$, $n \in \omega$ satisfy the requirements of the inductive construction.

After completing the inductive construction, consider the set $Q = \{x_s : s \in \omega^{<\omega}\}$ and note that it is first countable, contains no non-empty finite open subsets, $f|Q$ is bijective and $f(Q)$ is a discrete subspace of Y. □

3. Proof of Theorem 1.2

Given a function $f : X \to Y$ from a Preiss-Simon space X to a regular space Y we need to prove the equivalence of the following conditions:

(1) f is weakly discontinuous;
(2) f is resolvable;
(3) f is open-resolvable.
(4) f is base-resolvable.

The implication (1) ⇒ (2) follows from Proposition 1.1 and (2) ⇒ (3) ⇒ (4) are trivial. To prove that (4) ⇒ (1), assume that the function f is base-resolvable but not weakly discontinuous. By Lemma 2.1, f is not scatteredly continuous. The space X, being Preiss-Simon, has countable tightness. Then Lemma 2.2 yields a non-empty countable set $D \subset X$ such that the restriction $f|D$ has no continuity points. Applying Lemma 2.5(3) to the restriction $f|D$, we can find a countable first-countable subset $Q \subset D$ without finite open sets such that $f|Q$ is bijective and $f(Q)$ is a discrete subspace of Y. It is clear that $f|Q$ has no continuity point. The space X being second countable and without finite open sets, can be written as the union $Q = Q_1 \cup Q_2$ of two disjoint dense subsets of Q.

Let $\mathcal{B}$ be a countably additive base for Y such that for every $B \in \mathcal{B}$ the preimage $f^{-1}(B)$ is resolvable in X. Since the set $f(Q)$ is countable and discrete, for every $x \in Q$ we can select a neighborhood $O_{f(x)} \in \mathcal{B}$ of $f(x)$ so small that the family $\{O_{f(x)} : x \in Q\}$ is disjoint. The countable additivity of the base $\mathcal{B}$ implies that for every $i \in \{1,2\}$ the set $W_i = \bigcup_{x \in Q_i} O_{f(x)}$ belongs to $\mathcal{B}$. Consequently, the preimage $f^{-1}(W_i)$ is a resolvable subset of X and hence $\bar{Q} \cap f^{-1}(W_i) \supset Q_i$ is a dense resolvable subset of $\bar{Q}$. So, the space $\bar{Q}$ contains two disjoint dense resolvable subsets $\bar{Q} \cap f^{-1}(W_1)$ and $\bar{Q} \cap f^{-1}(W_2)$, which contradicts Lemma 2.3. This contradiction completes the proof of the implication $(4) \Rightarrow (1)$.

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