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$U(k)$- and $L(k)$-homotopic properties of digitizations of nD Hausdorff spaces

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Abstract

For $X \subset \mathbf{R}^n$ let (X, E^n_X) be the usual topological space induced by the nD Euclidean topological space $(\mathbf{R}^n, E^n)$. Based on the upper limit (U-, for short) topology (*resp.* the lower limit (L-, for brevity) topology), after proceeding with a digitization of (X, E^n_X), we obtain a U- (*resp.* an L-) digitized space denoted by $D_U(X)$ (*resp.* $D_L(X)$) in $\mathbf{Z}^n$ [16]. Further considering $D_U(X)$ (*resp.* $D_L(X)$) with a digital k-connectivity, we obtain a digital image from the viewpoint of digital topology in a graph-theoretical approach, *i.e.* Rosenfeld model [25], denoted by $D_{U(k)}(X)$ (*resp.* $D_{L(k)}(X)$) in the present paper. Since a Euclidean topological homotopy has some limitations of studying a digitization of (X, E^n_X), the present paper establishes the so called $U(k)$-homotopy (*resp.* $L(k)$-homotopy) which can be used to study homotopic properties of both (X, E^n_X) and $D_{U(k)}(X)$ (*resp.* both (X, E^n_X) and $D_{L(k)}(X)$). The goal of the paper is to study some relationships among an ordinary homotopy equivalence, a $U(k)$-homotopy equivalence, an $L(k)$-homotopy equivalence and a k-homotopy equivalence. Finally, we classify (X, E^n_X) in terms of a $U(k)$-homotopy equivalence and an $L(k)$-homotopy equivalence. This approach can be used to study applied topology, approximation theory and digital geometry.

Keywords: $U(k)$-digitization, $L(k)$-digitization, U- and L-localized neighborhood, $U(k)$- and $L(k)$-homotopy.

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1. Introduction

In relation to the digitizations of nD Euclidean spaces [3, 5, 14], the present paper uses two kinds of local rules associated with the *upper limit* (U-, for short) and the *lower limit* (L-, for brevity) topology [23]. These local rules are used to U- and L-digitize Euclidean nD subspace so that we obtain digital images from the viewpoint of digital topology in the graph-theoretical approach proposed in [25].

Let $\mathbf{Z}$ (*resp.* $\mathbf{N}$) represent the set of integers (*resp.* natural numbers), and $\mathbf{Z}^n$ the set of points in the Euclidean nD space with integer coordinates. In digital topology there are several approaches [1, 18, 25, 28] and so forth. Since the paper uses both digital graph theory on $\mathbf{Z}^n$ and topology on the nD Euclidean space, we need to recall the graph-theoretical approach to digital topology. Rosenfeld [25] introduced a digital image $X \subset \mathbf{Z}^n$ with k-adjacency, denoted by (X, k), and a (k_0, k_1)-continuous map $f : (X, k_0) \to (Y, k_1)$ of which f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1). We denote by DTC the category of digital images (X, k) as $Ob(DTC)$ and (k_0, k_1)-continuous maps between every pair of digital images (X, k_0) and (Y, k_1) in $Ob(DTC)$ as $Mor(DTC)$ [7, 9].

Let $(\mathbf{R}^n, E^n)$ be the nD real space with Euclidean topology [23]. For $X \subset \mathbf{R}^n$ we consider the subspace (X, E^n_X) induced by $(\mathbf{R}^n, E^n)$. In this paper we denoted by ETC the category of Euclidean topological spaces [27] consisting of the following two sets:
- the set of spaces (X, E^n_X) as objects, denoted by $Ob(ETC)$;
- for every ordered pair of objects (X, E^n_X) and (Y, E^n_Y), the set of (Euclidean topologically) continuous maps as morphisms denoted by $Mor(ETC)$.

To digitize (X, E^n_X) into a space in $\mathbf{Z}^n$ in a certain digital topological approach, we have often used graph theory and locally finite topological structures and so forth [1, 4, 5, 11, 16, 20, 21, 22, 24, 28]. Hereafter, based on the U-topology and the L-topology, after proceeding with a digitization of (X, E^n_X) [16], we obtain a U- (*resp.* an L-) digitized space denoted by $D_U(X)$ (*resp.* $D_L(X)$) in $\mathbf{Z}^n$ [16]. Further considering $D_U(X)$ (*resp.* $D_L(X)$) with a k-adjacency, we obtain a digital image denoted by $D_{U(k)}(X) := (D_U(X), k)$ (*resp.* $D_{L(k)}(X) := (D_L(X), k)$) in the present paper.

Since we have some difficulty in digitizing an ordinary map $f \in Mor(ETC)$ (see Lemma 6.1 in the present paper), the present paper develops both a $U(k)$-map and an $L(k)$-map and (see Definitions 11 and 12). The present paper proves that each of these maps is stronger than an ordinary map in ETC (see Lemma 6.1) but suitable for digitizing nD Euclidean spaces based on the graph-theoretical approach (see Theorem 6.5). Besides, we establish a category, denoted by UDC (*resp.* LDC), consisting of the sets of subspaces (X, E^n_X) and $U(k)$-maps (*resp.* $L(k)$-maps) (see Section 5).

Let $f : (X, E^n_X) \to (Y, E^n_Y)$ be a map in $Mor(ETC)$. Let $D_{U(k)}(f) : D_{U(k)}(X) \to D_{U(k)}(Y)$ be a k-continuous map induced by the map f (see Definition 11) and let $D_{L(k)}(f) : D_{L(k)}(X) \to D_{L(k)}(Y)$ be a k-continuous map induced by the map f (see Definition 12).

To study some homotopic properties of among (X, E^n_X) in $Ob(ETC)$, $D_{U(k)}(X)$ and $D_{L(k)}(X)$ in $Ob(DTC)$, the present paper develops a $U(k)$-homotopy in UDC (see Definition 15) and an $L(k)$-homotopy in LDC (see Definition 16). In relation to these homotopies, we may pose the following queries:
Assume two Euclidean topological spaces (X, E^n_X) and (Y, E^n_Y). Let $D_{U(k)}(X)$ and $D_{U(k)}(Y)$ in $Ob(DTC)$ be their $U(k)$-digitized spaces (or $U(k)$-spaces for short) and let $D_{L(k)}(X)$ and $D_{L(k)}(Y)$ in $Ob(DTC)$ be their $L(k)$-digitized spaces (or $L(k)$-spaces for brevity).

Assume that $f, g : (X, E^n_X) \to (Y, E^n_Y)$ are homotopic in ETC. Then we have the following queries (Q1)-(Q2) (see also the properties (4.1) and (4.2) and Definitions 13, 15, and 16):

(Q1) Are $D_{U(k)}(f)$ and $D_{U(k)}(g)$ k-homotopic ?

(Q2) Are $D_{L(k)}(f)$ and $D_{L(k)}(g)$ k-homotopic ?

Let us investigate homotopic properties of maps in $Mor(UDC)$ and $Mor(LDC)$.
(Q3) In case $f, g : (X, E^n_X) \to (Y, E^n_Y)$ are $U(k)$-homotopic in UDC, are $D_{U(k)}(f)$ and $D_{U(k)}(g)$ k-homotopic ?

(Q4) In case $f, g : (X, E^n_X) \to (Y, E^n_Y)$ are $L(k)$-homotopic in LDC, are $D_{L(k)}(f)$ and $D_{L(k)}(g)$ k-homotopic ?

More generally, we have the following:
(Q5) What are relationships among an ordinary homotopy equivalence in ETC, a $U(k)$-homotopy equivalence in UDC and an $L(k)$-homotopy equivalence in LDC?

The present paper shall address these issues in Sections 4-7. Roughly saying, both the first and the second question can be answered negatively and both the third and the fourth question can be answered affirmatively.

The rest of the paper proceeds as follows: Section 2 provides some basic notions on digital topology and various notions in UDC and LDC. Section 3 investigates some properties of a U- and an L-local rule of (X, E^n_X) to establish a local neighborhood. Section 4 proposes a $U(k)$- and an $L(k)$-digitization of (X, E^n_X). Section 5 develops two maps such as a $U(k)$-map and an $L(k)$-map and proves that these maps are not compatible with a map in $Mor(ETC)$ but suitable for studying a digitization of a map $f \in Mor(ETC)$. Section 6 develops a $U(k)$-homotopy and an $L(k)$-homotopy and investigates their properties. Section 7 investigates some relationships among a homotopy equivalence in ETC, a $U(k)$-homotopy equivalence in UDC and an $L(k)$-homotopy equivalence in LDC. Section 8 concludes the paper with a remark.

2. Preliminaries

This section recalls basic notions of the graph-theoretical approach to digital topology. A digital picture is usually represented as a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, a black points set $X \subset \mathbf{Z}^n$ is the set of points we regard as belonging to the image depicted, k represents as an adjacency relation for X and $\bar{k}$ represents an adjacency relation for white points set $\mathbf{Z}^n \setminus X$ [25]. We say that the pair (X, k) is a digital image in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$ [25]. Thus, motivated by 4- and 8-adjacencies of a 2D digital image and, 6-, 18-, and 26-adjacencies of a 3D digital image [20, 25], the k-adjacency relations of $\mathbf{Z}^n$ can be established to study a multi-dimensional digital image. Indeed, these are induced by the following operator [6] (see also [7]): for a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, ..., p_n), q = (q_1, q_2, ..., q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-(for brevity, k- or k_m-) adjacent if

$$\text{at most } m \text{ of their coordinates differ by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

The number k of the $k(m, n)$-adjacency is the number of points q which are k-adjacent to a given point p according to the number m in (2.1) [7] (see also [9], for more details, see [10]). Concretely, the k-adjacency relations of $\mathbf{Z}^n$ can be represented, as follows:

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!i!}. \tag{2.2}$$

For instance, $(n, m, k) \in \{(4, 1, 8), (4, 2, 32), (4, 3, 64), (4, 4, 80); (5, 1, 10), (5, 2, 50), (5, 3, 130), (5, 4, 210), (5, 5, 242)\}$ [6, 7, 9].
Owing to the digital k-connectivity paradox of a digital image (X, k) [20], we remind the reader that $k \neq \bar{k}$ except for the case $(\mathbf{Z}, 2, 2, X)$. For $\{a, b\} \subset \mathbf{Z}$ with $a < b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b \,|\, n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z}, 2, 2, [a, b]_{\mathbf{Z}})$ [20]. However, the present paper is not concerned with the $\bar{k}$-adjacency of $\mathbf{Z}^n \setminus X$. To follow the graph-theoretical approach to the study of nD digital images [26, 7], we use the k-adjacency relations of $\mathbf{Z}^n$ (see the property (2.2)), a digital k-neighborhood and so forth [25].

$$N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\}.$$

Furthermore, we often use the notation [20]

$$N_k^*(p) := N_k(p) \cup \{p\}.$$

We say that two subsets (A, k) and (B, k) of (X, k) are k-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent to each other [20]. We say that a set $X \subset \mathbf{Z}^n$ is k-connected if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [20].

For a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j \,| = 1$ [20]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [20, 6] (see Fig.2(a),(b)), is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i - j| = 1 (mod\, l)$ [20].

For a digital image (X, k), we define the digital k-neighborhood of $x_0 \in X$ with radius ε to be the following set [6]: $N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}$, where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$. Concretely, for $X \subset \mathbf{Z}^n$ we obtain [11]

$$N_k(x, 1) = N_k(x) \cap X. \tag{2.3}$$

The paper [25] established the notion of digital continuity. Motivated by this continuity, we can represent the digital continuity of maps between digital images, as follows:

2.1. Proposition. [6, 7, 10] *Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 2.1 in case $n_0 = n_1$ and $k_0 = k_1$, we call it k_0-continuous. Besides, the digital continuity of Proposition 2.1 has the transitive property.

Since the digital image (X, k) is considered to be a set $X \subset \mathbf{Z}^n$ with one of the adjacency relations of (2.2), we use the terminology a "(k_0, k_1)-isomorphism" as used in [8] rather than a "(k_0, k_1)-homeomorphism" as proposed in [2].

2.2. Definition. [2] (see also [8]) For two digital images (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.

In Definition 1, in case $n_0 = n_1$ and $k_0 = k_1$, we call it a k_0-isomorphism.

3. Some properties of a U and an L-local rule

When digitizing a space (X, E_X^n) into a digital image, it is required that the connectedness of the given spaces is preserved (see Lemma 6.4 in the present paper). To do this work, this section uses two types of local rules which are used to formulate special kinds of neighborhoods of the given point $p \in \mathbf{Z}^n$. And the structures of the neighborhoods depend on the digital topological structures related to the local rules. The U-topology on $\mathbf{R}$, denoted by $(\mathbf{R}, E_U)$, is induced by the set $\{(a, b] \,|\, a, b \in \mathbf{R} \text{ and } a < b\}$ as a base [23]. Then we obtain the product topology on $\mathbf{R}^n$, denoted by $(\mathbf{R}^n, E_U^n)$, induced by $(\mathbf{R}, E_U)$. Based on $(\mathbf{R}^n, E_U^n)$, we use a U-local rule [16] which is used to digitize $(\mathbf{R}^n, E_U^n)$ into $(\mathbf{Z}^n, D^n)$, where $(\mathbf{Z}^n, D^n)$ is the discrete topology on $\mathbf{Z}^n$.

3.1. Definition. [16] Under $(\mathbf{R}^n, E_U^n)$, for a point $p := (p_i)_{i \in [1,n]_\mathbf{Z}} \in \mathbf{Z}^n$ we define $N_U(p) := \{(x_i)_{i \in [1,n]_\mathbf{Z}} \,|\, x_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2}]\}$ and we call $N_U(p)$ the U-localized neighborhood of p associated with $(\mathbf{R}^n, E_U^n)$.

For instance, we see $N_U(p)$ in Fig.1(b) for a 2D case.

In relation to the digitization of $(\mathbf{R}^n, E_U^n)$, let us consider the following relation.

3.2. Definition. [16] For two points $x, y \in \mathbf{R}^n$, x is related to y if $x, y \in N_U(p)$ for some point $p \in \mathbf{Z}^n$, denoted by '$x \sim_U y$'. Then we say that $(\mathbf{R}^n, \sim_U)$ is a relation set associated with $(\mathbf{R}^n, E_U^n)$.

3.3. Lemma. [16] *The relation '$\sim_U$' of Definition 3 is an equivalence relation.*

3.4. Remark. [16] Since $\mathbf{R}^n = \bigcup_{p\in\mathbf{Z}^n} N_U(p)$ and further, for two points p, q in $\mathbf{Z}^n$ with $p \neq q$, $N_U(p) \cap N_U(q) = \emptyset$, we conclude that the set $\{N_U(p) \,|\, p \in \mathbf{Z}^n\}$ is a partition of $\mathbf{R}^n$.

By Lemma 3.1, we conclude that $\mathbf{Z}^n$ is the space obtained by identifying the points of $\mathbf{R}^n$ which belong to the same equivalence class of p. Namely, we may conclude $N_U(p) = [p]$, where $[p]$ is the equivalence class of the point p.

Concretely, based on $(\mathbf{R}^n, E^n_U)$ associated with the U-topology, we can digitize $\mathbf{R}^n$ according to the U-topology in such a way

$$(\mathbf{R}^n, E^n_U) \to (\mathbf{Z}^n, D^n) \text{ given by } N_U(p) \to p. \tag{3.1}$$

It is obvious that the process (3.1) is continuous.

Meanwhile, we may proceed the process of (3.1) in such a way:

$$(\mathbf{R}^n, E^n) \to (\mathbf{Z}^n, D^n) \text{ given by } N_U(p) \to p.$$

Then this process cannot be continuous in topological sense. This approach will be used in Section 4.

Let us now recall the L-local rule in [16]. The L-topology on $\mathbf{R}$, denoted by $(\mathbf{R}, E_L)$, is induced by the set of closed open intervals in $\mathbf{R}$, $\{[a, b) \,|\, a, b \in \mathbf{R} \text{ and } a < b\}$, as a base [23]. Then we obtain the product topology on $\mathbf{R}^n$, denoted by $(\mathbf{R}^n, E^n_L)$, induced by $(\mathbf{R}, E_L)$.

Let us consider the L-local rule associated with the L-topology.

3.5. Definition. [16] Under $(\mathbf{R}^n, E^n_L)$, for a point $p := (p_i)_{i\in[1,n]_\mathbf{Z}} \in \mathbf{Z}^n$ we define $N_L(p) := \{(x_i)_{i\in[1,n]_\mathbf{Z}} \,|\, x_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2})\}$. We call $N_L(p)$ the L-localized neighborhood of p associated with $(\mathbf{R}^n, E^n_L)$.

For instance, we see $N_L(p)$ in Fig.1(a) for a 2D space.

In relation to the digitization of $(\mathbf{R}^n, E^n_L)$, let us consider the following relation:

3.6. Definition. [16] For two points x, $y \in \mathbf{R}^n$, we say that x is related to y if for some point $p \in \mathbf{Z}^n$, x, $y \in N_L(p)$, denoted by '$x \sim_L y$'. Then we say that $(\mathbf{R}^n, \sim_L)$ is a relation set associated with $(\mathbf{R}^n, E^n_L)$.

3.7. Lemma. [16] *The relation '$\sim_L$' of Definition 5 is an equivalence relation.*

3.8. Remark. [16] The set $\{N_L(p) \,|\, p \in \mathbf{Z}^n\}$ is a partition of $\mathbf{R}^n$.

By Lemma 3.3, we observe that the set $\mathbf{Z}^n$ can be considered on the space obtained by identifying the points of $\mathbf{R}^n$ which belong to the same equivalence class of p. By Lemma 3.3 and Remark 3.4, we may assume $N_L(p) = [p]$. Finally, based on $(\mathbf{R}^n, E^n_L)$, we can digitize $\mathbf{R}^n$ according to the L-topology in such a way

$$(\mathbf{R}^n, E^n_L) \to (\mathbf{Z}^n, D^n) \text{ given by } N_L(p) \to p. \tag{3.2}$$

It is obvious that the process (3.2) is continuous.

Meanwhile, we may proceed the process of (3.2) in such a way:

$$(\mathbf{R}^n, E^n) \to (\mathbf{Z}^n, D^n) \text{ given by } N_L(p) \to p.$$

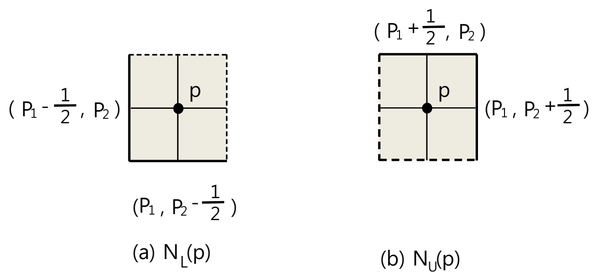

Figure 1. [16] Configuration of the local rules if the given point p in the 2D Euclidean space in terms of the L-topology (a) and the U-topology (b), where $p := (p_1, p_2)$

Then this process cannot be continuous in topological sense. This approach will be used in Section 4.

4. Establishments of a $U(k)$- and an $L(k)$-digitization

This section recalls two types of digitizations associated with the U- and the L-topology. By using the local rule proposed in Definitions 3 and 4, we establish the following:

4.1. Definition. [16] Let X be a subspace in $(\mathbf{R}^n, E^n_U)$ (*resp.* $(\mathbf{R}^n, E^n_L)$). The U- (*resp.* L-) digitization of X, denoted by $D_U(X)$ (*resp.* $D_L(X)$), is defined as follows:

$$\begin{cases} D_U(X) = \{p \in \mathbf{Z}^n \mid N_U(p) \cap X \neq \emptyset\}; \\ D_L(X) = \{p \in \mathbf{Z}^n \mid N_L(p) \cap X \neq \emptyset\} \end{cases}$$

with a k-adjacency of $\mathbf{Z}^n$ of (2.1) depending on the situation.

4.2. Remark. [16] For a set $X \subset \mathbf{R}^n$, we say that for two points $x, y \in X$, x is $\sim_U$ (*resp.* $\sim_L$) related to y according to U- (*resp.* L-) topology, as follows:

(1) $x \sim_U y$, if $x, y \in N_U(p)$ for some point $p \in \mathbf{Z}^n$ such that $X \cap N_U(p) \neq \emptyset$. The relation " $\sim_U$" is an equivalence relation (relative to X).

(2) $x \sim_L y$, if $x, y \in N_L(p)$ for some point $p \in \mathbf{Z}^n$ such that $X \cap N_L(p) \neq \emptyset$. The relation " $\sim_L$" is an equivalence relation (relative to X).

Motivated by Remark 3.2, we obviously obtain the following:

4.3. Corollary. [16] *For a non-empty nD Euclidean space (X, E^n_X), there is a partition of $\mathbf{R}^n$ associated with the space (X, E^n_X):*

$$\{N_U(p), \mathbf{R}^n \setminus \cup_{p \in D_U(X)} N_U(p) \mid p \in D_U(X)\}.$$

4.4. Definition. [16] For a space (X, E^n_X) and two points $p, q \in X$, we say that the point p is related to q if there is a point $x \in D_U(X)$ such that $p, q \in N_U(x)$. In this case we use the notation $(p, q) \in L_X$ and further, the relation set is denoted by (X, L_X).

It is clear that the relation L_X in the set (X, L_X) of Definition 7 is an equivalence relation [16].

After digitizing X in the U- and the L-topological approach (see Lemmas 3.1 and 3.3), we define the following:

4.5. Definition. (1) We say that $D_{U(k)}(X)$ is the set $D_U(X)$ with a k-adjacency. (2) We say that $D_{L(k)}(X)$ is the set $D_L(X)$ with a k-adjacency.

Using the local rule in Definition 2, we define the following:

4.6. Definition. Let $D_{U(k)} : (\mathbf{R}^n, E^n) \to (\mathbf{Z}^n, k)$ be the map defined by $D_{U(k)}(x) = p$, where $x \in N_U(p)$ and the k-adjacency depends on the situation. Then we say that $D_{U(k)}$ is a $U(k)$-digitization operator.

Indeed, the $U(k)$-digitization operator $D_{U(k)}$ is represented as follows: under $(\mathbf{R}^n, E^n_U)$, for a point $x = (x_1, ..., x_n) \in \mathbf{R}^n$ let $D_U : (\mathbf{R}^n, E^n_U) \to \mathbf{Z}^n$ be a map defined by $D_U((x_i)_{i\in[1,n]_\mathbf{Z}}) = (p_i)_{i\in[1,n]_\mathbf{Z}} := p \in \mathbf{Z}^n$ satisfying that for all $i \in [1, n]_\mathbf{Z}$, $x_i = p_i + d_i$, where $-\frac{1}{2} < d_i \le \frac{1}{2}$ (see Lemma 3.1) [16]. Furthermore, on $\mathbf{Z}^n$ consider one of the k-adjacency relations of $\mathbf{Z}^n$ of (2.1). Finally, we obtain the following process.

$$(\mathbf{R}^n, E^n) \to (\mathbf{R}^n, E^n_U) \to (\mathbf{Z}^n, D^n) \to (\mathbf{Z}^n, k) \qquad (*1)$$

Using the local rule of Definition 4, we define the following:

4.7. Definition. Let $D_{L(k)} : (\mathbf{R}^n, E^n) \to (\mathbf{Z}^n, k)$ be the map defined by $D_{L(k)}(x) = p$, where $x \in N_L(p)$, $p \in \mathbf{Z}^n$ and the k-adjacency depends on the situation. Then we say that $D_{L(k)}$ is an $L(k)$-digitization operator.

Indeed, an $L(k)$-digitization operator $D_{L(k)}$ is represented as follows: under $(\mathbf{R}^n, E^n_L)$, for a point $x = (x_1, ..., x_n) \in \mathbf{R}^n$ let $D_{L(k)} : (\mathbf{R}^n, E^n_L) \to (\mathbf{Z}^n, k)$ be the map defined by $D_L((x_i)_{i\in[1,n]_\mathbf{Z}}) = (p_i)_{i\in[1,n]_\mathbf{Z}} := p \in \mathbf{Z}^n$ satisfying that for all $i \in [1, n]_\mathbf{Z}$, $x_i = p_i + d_i$, where $-\frac{1}{2} \le d_i < \frac{1}{2}$ (see Lemma 3.3) [16]. Besides, on $\mathbf{Z}^n$ consider one of the k-adjacency relations of $\mathbf{Z}^n$ of (2.1).

Finally, we obtain the following process.

$$(\mathbf{R}^n, E^n) \to (\mathbf{R}^n, E^n_L) \to (\mathbf{Z}^n, D^n) \to (\mathbf{Z}^n, k) \qquad (*2)$$

For a non-empty set $X \subset \mathbf{R}^n$, let us now investigate some properties of a $U(k)$- and an $L(k)$-digitization.

The digitizations $D_{U(k)}(X)$ and $D_{L(k)}(X)$ of a Euclidean subspace X are proceeded according to the following algorithms.

Algorithms for the $U(k)$- and the $L(k)$-digitizing process from ETC to DTC

For $(X, E^n_X) \in ETC$ we write the following algorithms for digitizing a space (X, E^n_X) from ETC to DTC in such two ways [16]:

(Case 1): In case of the U-digitization of (X, E^n_X):

(Step 1) Read the points $p \in D_U(X)$.

(Step 2) For each point $p \in D_U(X)$ take $N_U(p) \cap X$.

(Step 3) Put $N_U(p) \cap X := \{p\}$.

(Step 4) Consider the set $D_U(X)$ with a certain k-adjacency so that we obtain $(D_U(X), k) \in DTC$.
Finally, according to $(*1)$ and (3.1), we obtain the map $D_{U(k)} : ETC \to DTC$ given by

$$D_{U(k)}((X, E^n_X)) = (D_U(X), k) \in Obj(DTC) \tag{4.1}$$

which is called a $U(k)$-digitization operator for (X, E^n_X).

(Case 2): In case of the $L(k)$-digitization of (X, E^n_X):
(Step 1) Read the points $p \in D_L(X)$.
(Step 2) For each point $p \in D_L(X)$ take $N_L(p) \cap X$.
(Step 3) Put $N_L(p) \cap X := \{p\}$.
(Step 4) Consider the set $D_L(X)$ with a certain k-adjacency so that we obtain $(D_L(X), k) \in DTC$.
Finally, according to $(*2)$ and (3.2), we obtain the map $D_{L(k)} : ETC \to DTC$ given by

$$D_{L(k)}((X, E^n_X)) = (D_L(X), k) \in Obj(DTC) \tag{4.2}$$

which is called an $L(k)$-digitization operator for (X, E^n_X).

4.8. Proposition. *For a (X, E^n_X) in $Ob(ETC)$, $D_{U(k)}(X)$ is different from $D_{L(k)}(X)$.*

Proof: As shown in Fig.2, given a space (X, E^n_X) in $Ob(ETC)$, take points $p := (p_i)_{i\in[1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ such that there is a point $x := (x_i)_{i\in[1,n]_{\mathbf{Z}}} \in X$ satisfying $x_i = p_i + d_i$, where $d_i = \frac{1}{2}$ or $d_i = \frac{-1}{2}$. Then, depending on the choice of a U- or an L-local rule of x, the point x is recognized to be a different point. Hence $D_{U(k)}(X)$ is different from $D_{L(k)}(X)$. $\Box$

4.9. Example. Consider the space (X, E^2_X) in Fig.2. After obtaining $D_{U(k)}(X)$ and $D_{L(k)}(X)$, we can see some difference between them, $k \in \{4, 8\}$

5. Developments of a $U(k)$- and an $L(k)$-map

Combining a U-localized neighborhood of Definition 2 with a k-continuous map, we define the following map which can be used to study both (X, E^n_X) and $D_{U(k)}(X)$.

5.1. Definition. Consider the map $F : (X, E^n_X) \to (Y, E^n_Y)$ in ETC.
(1) We define $D_{U(k)}(F) := f : D_{U(k)}(X) \to D_{U(k)}(Y)$ given by

$$\left\{ \begin{array}{l} \text{for } p \in D_{U(k)}(X), f \text{ maps } p \text{ to } q_i, \\ \text{where } \{q_i \in \mathbf{Z}^n \mid N_U(q_i) \cap F(N_U(p) \cap X) \neq \emptyset\} \subset D_{U(k)}(Y). \end{array} \right\}$$

Then $D_U(F)$ is called a U-digitization of F.

(2) $D_{U(k)}(F) := f$ is a k-continuous map satisfying that for any point $p \in D_{U(k)}(X), F(N_U(p) \cap X) \subset N_U(f(p)) \cap Y$.

Then we say that the map F is a lattice-based $U(k)$-continuous map (a $U(k)$-map, for short).

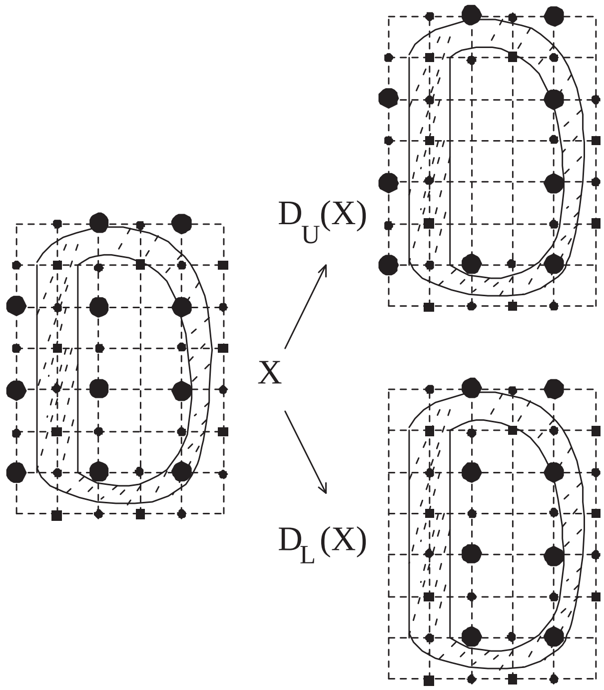

Figure 2. Comparison between $D_{U(k)}(X)$ and $D_{L(k)}(X), k \in \{4, 8\}$.

The paper denotes by UDC the category consisting of the following two sets:
$(*\ 1)$ the set of spaces $(X, E_X^n) := X$ as objects of UDC denoted by $Ob(UDC)$;
$(*\ 2)$ the set of $U(k)$-maps of every ordered pair of elements in $Ob(UDC)$ as morphisms of UDC denoted by $Mor(UDC)$.

5.2. Example. In Fig.3(a), put $X = X_1 \cup X_2 \cup X_3$, where $X_1 := (\frac{-1}{2}, \frac{-1}{4}]^2$, $X_2 := (\frac{-1}{4}, \frac{1}{4}]^2$ and $X_3 := (\frac{1}{4}, \frac{1}{2})^2$. Besides, put $Y = Y_1 \cup Y_2 \cup Y_3$, where $Y_1 = X_1$, $Y_2 := (\frac{-1}{8}, \frac{1}{8}]^2$, $X_3 = Y_3$ and $p = (0, 0)$. Then consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ given by $F(p) = p, F(X_i) \subset Y_i, i \in \{1, 2, 3\}$. Then F is a $U(k)$-map, $k \in \{4, 8\}$.

By using the method similar to the establishment of a $U(k)$-map, we can establish an $L(k)$-map: Combining an L-localized neighborhood of Definition 4 with a k-continuous map, let us now define the following map which can be used to study both (X, E_X^n) and $D_{L(k)}(X)$.

5.3. Definition. Consider the map $F : (X, E_X^n) \to (Y, E_Y^n)$ in ETC.
(1) We define $D_{L(k)}(F) := f : D_{L(k)}(X) \to D_{L(k)}(Y)$ given by

$$\left\{\begin{array}{l} \text{for } p \in D_{L(k)}(X), f \text{ maps } p \text{ to } q_i, \\ \text{where } \{q_i \in \mathbf{Z}^n \mid N_L(q_i) \cap F(N_L(p) \cap X) \neq \emptyset\} \subset D_{L(k)}(Y). \end{array}\right\}$$

Then $D_L(F)$ is called an L-digitization of F.

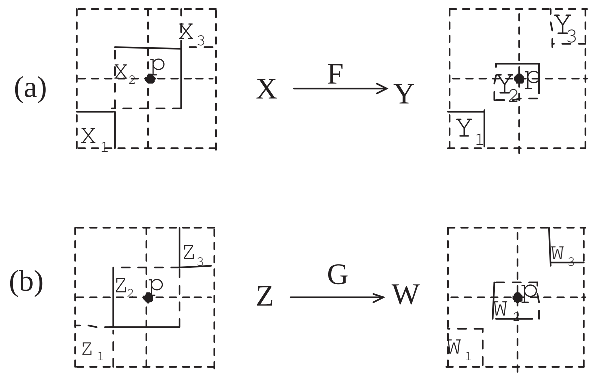

Figure 3. (a) Configuration of a $U(k)$-map in UDC; (b) configuration of an $L(k)$-map in LDC.

(2) $D_{L(k)}(F) := f$ is a k-continuous map satisfying that for any point $p \in D_{L(k)}(X), F(N_L(p) \cap X) \subset N_L(f(p)) \cap Y$.

Then we say that the map F is a lattice-based $L(k)$-continuous map (an $L(k)$-map, for short).

The paper denotes by LDC the category consisting of the following two sets:
($*$ 1) the set of spaces $(X, E^n_X) := X$ as objects of LDC denoted by $Ob(LDC)$;
($*$ 2) the set of $L(k)$-maps of every ordered pair of elements in $Ob(LDC)$ as morphisms of LDC denoted by $Mor(LDC)$.

5.4. Example. In Fig.3(b), put $Z = Z_1 \cup Z_2 \cup Z_3$, where $Z_1 := (\frac{-1}{2}, \frac{-1}{4})^2$, $Z_2 := [\frac{-1}{4}, \frac{1}{4})^2$ and $Z_3 := [\frac{1}{4}, \frac{1}{2})^2$. Besides, put $W = W_1 \cup W_2 \cup W_3$, where $W_1 = Z_1$, $W_2 := [\frac{-1}{8}, \frac{1}{8})^2$, $Z_3 = W_3$ and $p = (0, 0)$. Then consider the map $G : (Z, E^2_Z) \to (W, E^2_W)$ given by $G(p) = p, G(Z_i) \subset W_i, i \in \{1, 2, 3\}$. Then G is an $L(k)$-map, $k \in \{4, 8\}$.

Owing to Proposition 4.3, we obtain the following:

5.5. Proposition. *For a given map $f : (X, E^m_X) \to (Y, E^n_X)$ in $Mor(ETC)$, a $U(k)$-map is different from an $L(k)$-map.*

5.6. Definition. (1) Let $f : (X, E^m_X) \to (Y, E^n_Y)$ be a $U(k)$-map. Then consider a map $D_{U(k)}(f) : D_{U(k)}(X) \to D_{U(k)}(Y)$ induced by the given map f. Then we say that $D_{U(k)}(f)$ is a $U(k)$-digitization induced by the map f of Definition 11.

(2) Let $f : (X, E^m_X) \to (Y, E^n_Y)$ be an $L(k)$-map. Then consider a map $D_{L(k)}(f) : D_{L(k)}(X) \to D_{L(k)}(Y)$ induced by the given map f. Then we say that $D_{L(k)}(f)$ is an $L(k)$-digitization induced by the map f of Definition 12.

6. $U(k)$- and $L(k)$-homotopic properties in UDC and LDC

This section addresses the questions (Q1) and (Q2) posed in Section 1. First of all, let us investigate a relation among $f \in Mor(ETC)$, a $U(k)$-map and an $L(k)$-map, as follows:

6.1. Lemma. *A map $f \in Mor(ETC)$ need not induce a $U(k)$-map and an $L(k)$-map.*

Proof: By using a counterexample, we prove the assertion (see Fig.4(a)). Put $X := \{(t,1) \mid 0 \lneq t \leq 1\} \cup \{(1,t) \mid 0 \lneq t \leq 1\}$ and $Y := \{(t,2) \mid 0 \lneq t \leq 2\} \cup \{(2,t) \mid 0 \lneq t \leq 2\}$ (see Fig.4(a)).

Let us consider the map $f : (X, E^2_X) \to (Y, E^2_Y)$ given by

$$\begin{cases} f((t,1)) = (4t,2) \text{ if } 0 \lneq t \lneq \dfrac{1}{2}, \\ f((t,1)) = (2,2) \text{ if } \dfrac{1}{2} \leq t \leq 1, \\ f((1,t)) = (2,4t) \text{ if } 0 \lneq t \lneq \dfrac{1}{2}, \\ f((1,t)) = (2,2) \text{ if } \dfrac{1}{2} \lneq t \leq 1. \end{cases}$$

Then the map f is a continuous map in $Mor(ETC)$. But it is clear that the map f is neither a $U(k)$-map nor an $L(k)$-map, $k \in \{4,8\}$.

To be specific, based on the given map f, we cannot have its $U(k)$- and $L(k)$-maps which are denoted by $D_{U(k)}(f)$ and $D_{L(k)}(f)$ induced by the map f, respectively. contrary to the properties of Definitions 11 and 12, respectively. Namely,

$$\begin{cases} D_{U(k)}(f) : D_{U(k)}(X) \to D_{U(k)}(Y) \text{ is not a } k\text{-continuous map}, k \in \{4,8\}; \\ D_{L(k)}(f) : D_{L(k)}(X) \to D_{L(k)}(Y) \text{ is not a } k\text{-continuous map}, k \in \{4,8\}. \end{cases}$$

For instance, we observe that $D_{U(k)}(f)$ (*resp.* $D_{L(k)}(f)$) is not a $U(k)$-map (*resp.* an $L(k)$-map) at the point $(0,0), k \in \{4,8\}$. $\square$

6.2. Remark. (1) Unlike the given map f in Lemma 6.1, as shown in Fig.4(c), it is clear that the given map $g : Z \to W$ given by $g(t) = 2t$ is a (Euclidean topologically) continuous map, where $Z := (0, \frac{1}{4})$ and $W := (0, \frac{1}{2})$. But we see that its digitization $D_{U(2)}(g)$ (*resp.* $D_{L(2)}(g)$) is a $U(2)$-map (*resp.* an $L(2)$-map).

(2) By Lemma 6.1 and Proposition 5.3, it turns out that none of a map $f \in Mor(ETC)$, a $U(k)$-map in $Mor(UDC)$ and an $L(k)$-map in $Mor(LDC)$ implies the other.

In view of Lemma 6.1, we need to propose a certain homotopy suitable for studying a $U(k)$- and an $L(k)$-digitization. To do this work, first of all, we need to recall the notion of a k-homotopy [2]. For a space $X \in Ob(DTC)$ let B be a subset of X. Then (X, B) is called a digital image pair [7]. Furthermore, if B is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital image in $Ob(DTC)$. To study homotopic properties of $D_{U(k)}(X)$, in this section we use the notions of a k-homotopy relative to a subset $B \subset X$ [10] and a k-homotopy equivalence [6, 15]. Based on the pointed digital homotopy in [2], the following notion of a k-homotopy relative to a subset $A \subset X$ is often used in studying a k-homotopic thinning and a strong k-deformation retract of a digital image (X, k) in $\mathbf{Z}^n$ [9].

6.3. Definition. [9] (see also [10]) Let $((X, A), k_0)$ and (Y, k_1) be a digital image pair and a digital image, respectively. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such

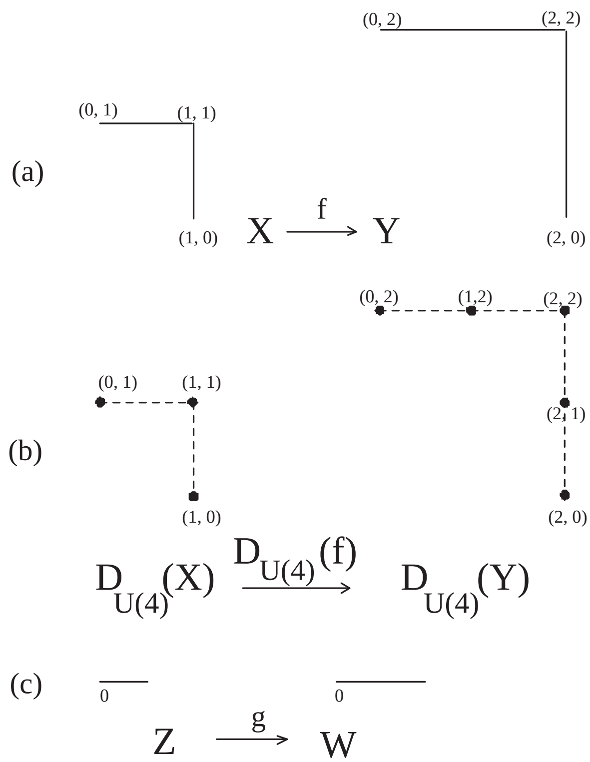

Figure 4. Comparison among a map $f \in Mor(ETC)$, a $U(k)$-map and an $L(k)$-map

that
• for all $x \in X, F(x,0) = f(x)$ and $F(x,m) = g(x)$;
• for all $x \in X$, the induced function $F_x : [0,m]_{\mathbf{Z}} \to Y$ given by $F_x(t) = F(x,t)$ for all $t \in [0,m]_{\mathbf{Z}}$ is $(2,k_1)$-continuous;
• for all $t \in [0,m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x,t)$ for all $x \in X$ is (k_0,k_1)-continuous.
Then we say that F is a (k_0,k_1)-homotopy between f and g [2].
• Furthermore, for all $t \in [0,m]_{\mathbf{Z}}$, $F_t(x) = f(x) = g(x)$ for all $x \in A$.
Then we call F a (k_0,k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0,k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1)relA} g$ in symbols.

In Definition 14, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0,k_1)-homotopy at $\{x_0\}$ [2]. When f and g are pointed (k_0,k_1)-homotopic in Y, we use the notation that $f \simeq_{(k_0,k_1)} g$. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that f and g are pointed k_0-homotopic in Y and we use the notation that $f \simeq_{k_0} g$ and $f \in [g]$ which denotes the k_0-homotopy class of g.

Based on this digital k-homotopy, to study some relations between $D_{U(k)}(X)$ and (X, E_X^n) from the viewpoint of homotopy theory, after combining an ordinary homotopy in ETC and a k-homotopy in DTC, we develop the following $U(k)$-homotopy.

6.4. Definition. Consider $(X, E_X^n) := X$ and $(Y, E_Y^n) := Y$ and $(B, E_B^n) := B$ which is a subspace of (X, E_X^n). Let $f, g : X \to Y$ be $U(k)$-maps. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that
• for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
• for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by
$F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is a $U(k)$-map;
• for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is a $U(k)$-map.
Then we say that F is a $U(k)$-homotopy between f and g.
• Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, assume that $F_t(x) = f(x) = g(x)$ for all $x \in B$. Then we call F a $U(k)$-homotopy relative to B between f and g, and we say that f and g are $U(k)$-homotopic relative to B in Y, $f \simeq_{U(k) rel. B} g$ in symbol.

To study some relations between $D_{L(k)}(X)$ and (X, E_X^n) from the viewpoint of homotopy theory, combining an ordinary homotopy in ETC and a k-homotopy in DTC, we develop the following $L(k)$-homotopy.

6.5. Definition. Consider $(X, E_X^n) := X$ and $(Y, E_Y^n) := Y$ and $(B, E_B^n) := B$ which is a subspace of (X, E_X^n). Let $f, g : X \to Y$ be $L(k)$-maps. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that
• for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
• for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by
$F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is an $L(k)$-map;
• for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is an $L(k)$-map.
Then we say that F is an $L(k)$-homotopy between f and g.
• Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, assume that $F_t(x) = f(x) = g(x)$ for all $x \in B$. Then we call F an $L(k)$-homotopy relative to B between f and g, and we say that f and g are $L(k)$-homotopic relative to B in Y, $f \simeq_{L(k) rel. B} g$ in symbol.

Owing to Lemma 6.1 and Remark 6.2, we obtain the following related to the queries (Q1)-(Q2):

6.6. Proposition. *An ordinary homotopy in ETC does not induce a $U(k)$-homotopy in UDC and an $L(k)$-homotopy in LDC*

Let us now investigate relations among a $U(k)$-homotopy, an $L(k)$-homotopy and a k-homotopy. To do this work, we recall some notions related to a $U(k)$- and an $L(k)$-digitization. The paper [16] studied the following:

6.7. Lemma. [16] *If (X, E_X^n) is connected, then both $D_{U(k)}(X)$ and $D_{L(k)}(X)$ are $(3^n - 1)$-connected.*

Let us prove that a $U(k)$- and an $L(k)$-homotopy induces a k-homotopy in DTC, as follows:

6.8. Theorem. *Consider two $U(k)$-maps $f, g : (X, E_X^n) \to (Y, E_Y^n)$ and their $U(k)$-digitizations $D_{U(k)}(f), D_{U(k)}(g) : D_{U(k)}(X) \to D_{U(k)}(Y)$, where (X, E_X^n) and (Y, E_Y^n) are connected. If there is a $U(k)$-homotopy between f and g, then we* *obtain a k-homotopy between $D_{U(k)}(f)$ and $D_{U(k)}(g)$ induced by the given $U(k)$-homotopy.*

Proof: Assume a $U(k)$-homotopy H in UDC between two $U(k)$-maps $f, g : (X, E^n_X) \to (Y, E^n_Y)$, *i.e.*

$$H : X \times [0, m]_{\mathbf{Z}} \to Y \text{ such that } H(x, 0) = f(x) \text{ and } H(x, m) = g(x)$$

satisfying the property of Definition 15. By Remark 3.2, we obtain

$$D_{U(k)}(H) : D_{U(k)}(X) \times [0, m]_{\mathbf{Z}} \to D_{U(k)}(Y) \text{ such that}$$

• for all $x' \in D_{U(k)}(X)$

$$\left\{ \begin{array}{l} D_{U(k)}(H)(x', 0) = D_{U(k)}(f)(x') \text{ and} \\ D_{U(k)}(H)(x', m) = D_{U(k)}(g)(x'); \end{array} \right\}$$

• for all $x' \in D_{U(k)}(X)$, the induced function $H_{x'} : [0, m]_{\mathbf{Z}} \to D_{U(k)}(Y)$ given by $H_{x'}(t) = H(x', t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is a $(2, k)$-continuous map;

• for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $H_t : D_{U(k)}(X) \to D_{U(k)}(Y)$ given by $H_t(x') = H(x', t)$ for all $x' \in D_{U(k)}(X)$ is a k-continuous map,
which implies that H is a k-homotopy between the above k-continuous maps $D_{U(k)}(f)$ and $D_{U(k)}(g)$. □

By Definition 16 and Remark 3.4, by using the method similar to Theorem 6.5, we obtain the following:

6.9. Corollary. *Consider two $L(k)$-maps $f, g : (X, E^n_X) \to (Y, E^n_Y)$ and their $L(k)$-digitizations $D_{L(k)}(f), D_{L(k)}(g) : D_{L(k)}(X) \to D_{L(k)}(Y)$. If there is an $L(k)$-homotopy between f and g, then we obtain a k-homotopy between $D_{L(k)}(f)$ and $D_{L(k)}(g)$ induced by the given $L(k)$-homotopy.*

6.10. Remark. In view of Propositions 5.3 and 6.3, it turns out that none of an ordinary homotopy in ETC, a $U(k)$- and an $L(k)$-homotopy implies the other.

In view of Theorem 6.5 and Corollary 6.6, we can answer the questions (Q3)-(Q4) affirmatively. Finally, it turns out that both a $U(k)$- and an $L(k)$-homotopy can play an important role in studying both (X, E^n_X) and its $U(k)$-digitized space $D_{U(k)}(X)$ and its $L(k)$-digitized space $D_{L(k)}(X)$.

7. A comparison among an ordinary homotopy equivalence, a $U(k)$- and an $L(k)$-homotopy equivalence and a k-homotopy equivalence

In this section, after proposing the notions of a $U(k)$- and an $L(k)$-homotopy equivalence, we compare among an ordinary homotopy equivalence, a $U(k)$- and an $L(k)$-homotopy equivalence, and a k-homotopy equivalence.

7.1. Definition. [6](see also [15]) In DTC, for two spaces X and Y, if there are k-continuous maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h$ is k-homotopic to 1_X and $h \circ l$ is k-homotopic to 1_Y, then the map $h : X \to Y$ is called a k-homotopy equivalence. Then we use the notation $X \simeq_{k \cdot h \cdot e} Y$.

7.2. Theorem. [6] *The composition also preserves a k-homotopy equivalence in DTC. Namely, if $X \simeq_{k \cdot h \cdot e} Y$ and $Y \simeq_{k \cdot h \cdot e} Z$, then $X \simeq_{k \cdot h \cdot e} Z$.*

Motivated by several types of digital versions of homotopy equivalences in [6, 9, 10], let us propose the notion of a $U(k)$-homotopy equivalence in UDC.

7.3. Definition. For two spaces (X, E_X^n) and (Y, E_Y^n), if there are $U(k)$-maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h$ is $U(k)$-homotopic to 1_X and $h \circ l$ is $U(k)$-homotopic to 1_Y, then the map $h : X \to Y$ is called an $U(k)$-homotopy equivalence. Then we use the notation $X \simeq_{U(k) \cdot h \cdot e} Y$.

7.4. Example. Consider the two spaces (X, E_X^2) and (Y, E_Y^2) in Fig.5(a) and (b), where $p := (\frac{-1}{2}, \frac{3}{2})$ and $q := (\frac{-1}{2}, \frac{1}{2})$ in Fig.5. In addition, the spaces X and Y are assumed to contain the point p and do not have the point q, respectively. While they are quite different from each other up to an ordinary homotopy equivalence, they are $U(k)$-equivalent, $k \in \{4, 8\}$. Indeed, in this case we see $D_{U(k)}(X) = D_{U(k)}(Y)$.

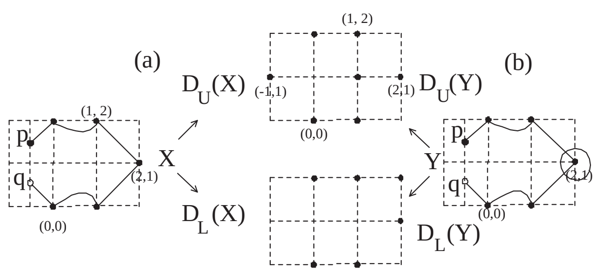

Figure 5. Comparison among a homotopy equivalence in ETC, a $U(k)$-, an $L(k)$-and a k-homotopy equivalence.

Comparing a $U(k)$-homotopy equivalence and an ordinary homotopy equivalence in [27], we can observe that a $U(k)$-homotopy equivalence has some merits in approximation theory.

7.5. Theorem. *The composition also preserves a $U(k)$-homotopy equivalence in UDC. Namely, if $X \simeq_{U(k) \cdot h \cdot e} Y$ and $Y \simeq_{U(k) \cdot h \cdot e} Z$, then $X \simeq_{U(k) \cdot h \cdot e} Z$.*

Proof: It is clear.

By using the method similar to that of Definition 18, we now establish the notion of an $L(k)$-homotopy equivalence in LDC.

7.6. Definition. For two spaces (X, E_X^n) and (Y, E_Y^n), if there are $L(k)$-maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h$ is $L(k)$-homotopic to 1_X and $h \circ l$ is $L(k)$-homotopic to 1_Y, then the map $h : X \to Y$ is called an $L(k)$-homotopy equivalence. Then we use the notation $X \simeq_{L(k) \cdot h \cdot e} Y$.

7.7. Example. Consider the two spaces (X, E_X^2) and (Y, E_Y^2) in Fig.5 (a) and (b). While they are quite different from each other up to an ordinary homotopy equivalence, they are $L(k)$-homotopy equivalent, $k \in \{4, 8\}$. Indeed, in this case we see $D_{L(k)}(X) = D_{L(k)}(Y)$, $k \in \{4, 8\}$.

Let us now compare among an ordinary homotopy equivalence in ETC, a $U(k)$- and an $L(k)$-homotopy equivalence.

7.8. Theorem. *None of a homotopy equivalence in ETC and a $U(k)$-homotopy equivalence in UDC implies the other.*

Proof: Consider two Euclidean topological spaces (X, E^n_X) and (Y, E^n_Y) in Fig.6 and their $U(k)$-spaces $D_{U(k)}(X)$ and $D_{U(k)}(Y)$. In addition, we assume both X and Y contain the point p and they do not have the point q. Besides, in X, we assume $p := (\frac{-1}{2}, \frac{7}{8})$ and $q := (\frac{-7}{8}, \frac{1}{2})$; in Y, we assume $p := (\frac{-1}{2}, \frac{1}{2})$ and $q := (\frac{-1}{2}, \frac{-1}{2})$ First of all, by Lemma 6.1 and Remark 6.2, it is clear that none of a homotopy equivalence between (X, E^n_X) and (Y, E^n_Y) in ETC and a k-homotopy equivalence in DTC implies the other. For instance, consider the spaces (X, E^2_X) in Fig.6(a) and (Y, E^2_Y) in Fig.6(b). While they are homotopy equivalent to each other, they are not $U(k)$-homotopy equivalent, $k \in \{4, 8\}$. To be specific, comparing $D_U(X)$ in Fig.6(a) and $D_U(Y)$ in Fig.6(b), we obviously see that $(D_U(X), k)$ in Fig.6(a) is not k-homotopy equivalent to $(D_U(Y), k)$ in Fig.6(b), $k \in \{4, 8\}$. Hence the given space (X, E^n_X) cannot be $U(k)$-homotopy equivalent to (Y, E^n_Y), $k \in \{4, 8\}$.

Conversely, consider the spaces (Y, E^2_Y) in Fig.6(b) and (Z, E^2_Z) in Fig.6(c). While the $U(k)$-spaces $D_{U(k)}(Y)$ in Fig.6(a) and $D_{U(k)}(Z)$ in Fig.6(c) are 8-homotopy equivalent to each other, it is clear that the space (Y, E^2_Y) is not homotopy equivalent to (Z, E^2_Z) in ETC, which means that a $U(k)$-homotopy equivalence of (Y, E^2_Y) and (Z, E^2_Z) in UDC does not imply their homotopy equivalence in ETC. □

Let us now compare between a $U(k)$-homotopy equivalence in UDC and a k-homotopy equivalence in DTC.

7.9. Theorem. *A $U(k)$-homotopy equivalence between (X, E^n_X) and (Y, E^n_Y) in UDC implies a k-homotopy equivalence between $D_{U(k)}(X)$ and $D_{U(k)}(Y)$ in DTC.*

Proof: Consider two topological spaces (X, E^n_X) and (Y, E^n_Y) in $Ob(UDC)$ and their $U(k)$-spaces $D_{U(k)}(X)$ and $D_{U(k)}(Y)$. By Theorem 6.5, we conclude that an $U(k)$-homotopy equivalence between (X, E^n_X) and (Y, E^n_Y) in UDC implies a k-homotopy equivalence between $D_{U(k)}(X)$ and $D_{U(k)}(Y)$ in DTC. □

By the method similar to Theorem 7.5, we obtain the following:

7.10. Corollary. *None of a homotopy equivalence in ETC and an $L(k)$-homotopy equivalence in LDC implies the other.*

By the method similar to Theorem 7.6, we obtain the following:

7.11. Corollary. *An $L(k)$-homotopy equivalence between (X, E^n_X) and (Y, E^n_Y) in LDC implies a k-homotopy equivalence between $D_{L(k)}(X)$ and $D_{L(k)}(Y)$ in DTC.*

In view of Proposition 5.3, we obtain the following:

7.12. Proposition. *None of $U(k)$- and $L(k)$-homotopy equivalence implies the other.*

Proof: By Proposition 4.3, the proof is trivial. □

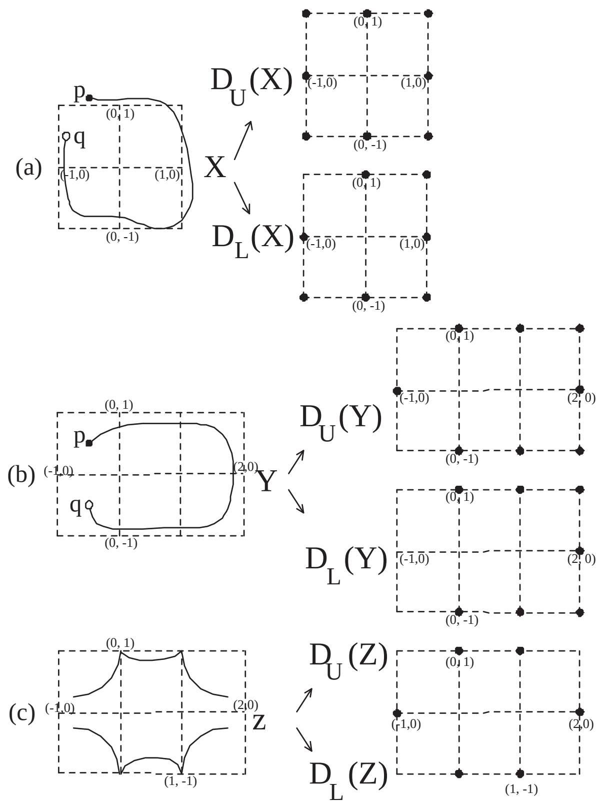

Figure 6. Comparison among the homotopy equivalence in ETC, the $U(k)$- and $L(k)$-homotopy equivalence, and the k-homotopy equivalence.

7.13. Example. Consider the spaces (Y, E_Y^2) in Fig.6(b) and (Z, E_Z^2) in Fig.6(c). Then, while (Y, E_Y^2) is $U(8)$-homotopy equivalent to (Z, E_Z^2), they are not $L(8)$-homotopy equivalent.

7.14. Remark. In view of Theorem 6.5, we obtain the following:
(1) the notion of a $U(k)$-homotopy equivalence in UDC can be used to study both (X, E_X^n) and its $U(k)$- space $D_{U(k)}(X)$ from the viewpoint of homotopy theory.

(2) In view of Corollary 6.6, the notion of an $L(k)$-homotopy equivalence in LDC can be used to study both (X, E_X^n) and its $L(k)$-space $D_{L(k)}(X)$ from the viewpoint of homotopy theory.

8. Summary and further works

Comparing with the usual topology on $\mathbf{R}^n$, we found that the U- and the L-topology has some merits of digitizations of (X, E_X^n). Thus we have studied various properties of an $L(k)$-homotopy and an $L(k)$-homotopy equivalence. Besides, comparing a Euclidean topological continuous map with an $L(k)$-map, we observed

H.E. in ETC --> U(k)-H.E	H.E. in ETC --> k-H.E	U(k)-H.E --> H.E in ETC	U(k)-H.E --> k-H.E	k-H.E. --> H.E. in ETC
NO	NO	NO	YES	NO

H.E. in ETC --> L(k)-H.E	H.E. in ETC --> k-H.E	L(k)-H.E --> H.E in ETC	L(k)-H.E --> k-H.E	k-H.E. --> H.E. in ETC
NO	NO	NO	YES	NO

Figure 7. Comparison among a homotopy equivalence in ETC, a $U(k)$- an $L(k)$-and a k-homotopy equivalence.

that an $L(k)$-map has strong merits of digitizing (X, E_X^n). Furthermore, comparing a Euclidean homotopy with both a $U(k)$-homotopy and an $L(k)$-homotopy, we concluded that a $U(k)$-homotopy and an $L(k)$-homotopy are suitable homotopies for studying both ETC, UDC and LDC. Besides, the paper investigated some relations between subspaces (X, E_X^n) and their $U(k)$-spaces $D_{U(k)}(X)$ in terms of an $U(k)$-homotopy equivalence and a k-homotopy equivalence (see Fig.7).

Recently, the paper[13] improved the LMA-map in [14] as follows: Let us now develop the notion of a generalized LMA-map as follows:

8.1. Definition. [13] Consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ such that $D_{MA}(F) := f : D_{MA}(X) \to D_{MA}(Y)$ is an MA-map, where $D_{MA}(F) := f$ is induced by F satisfying that for any point $p \in D_{MA}(X)$

$$\left\{ \begin{array}{l} F(N_M(p) \cap X) \subset N_M(f(p)) \cap Y, \text{ and} \\ f \text{ maps } p \text{ to } q_i, \text{ where} \\ \{q_i \in \mathbf{Z}^2 \mid N_M(q_i) \cap F(N_M(p) \cap X) \neq \emptyset\} \subset D_{MA}(Y). \end{array} \right\}$$

Then we say that the map F is a generalized LMA-map.

It turns out that [13] this version is both a kind of a generalization of an LMA-map in [14] and an improved and corrected version of an LMA-map in [14]. Thus the LMA-map of the paper [14] can be replaced by the current generalized LMA-map. Hereafter, we will call the map F in Definition 20 an LMA-map instead of *a generalized LMA-map*[13].
Besides, the paper[13] also improved the LA-map in [12] as follows:

8.2. Definition. [13] Consider the map $F : (X, E_X^n) \to (Y, E_Y^n)$ such that $D_{KA}(F) := f : D_{KA}(X) \to D_{KA}(Y)$ is an A-map, where $D_{KA}(F) := f$ is induced by F satisfying that for any point $p \in D_{KA}(X)$

$$\left\{\begin{array}{l} F(N_K(p)\cap X)\subset N_K(f(p))\cap Y \text{ and} \\ f \text{ maps } p \text{ to } q_i, \text{ where} \\ \{q_i\in \mathbf{Z}^n \mid N_K(q_i)\cap F(N_K(p)\cap X)\neq\emptyset\}\subset D_{KA}(Y). \end{array}\right\}$$

Then we say that the map F is a generalized LA-map.

It turns out that [13] this version is both a kind of a generalization of an LA-map in [12] and an improved and corrected version of an LA-map in [12]. Hereafter, we will call the map F in Definition 21 an LA-map instead of *a generalized LA-map*[13]. Thus the LA-map of the paper [12] can be replaced by the current generalized LA-map.
As a further work, we can compare among digitizations based on several kinds of digital topological structures in terms of the above LMA-map, LA-map, $U(k)$-map, and $L(k)$-map and further, find their own features and utilities.

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