Celal Bayar University Journal of Science
Volume 14, Issue 3, 2018, p 333-336

A Characterization of Approximation of Hardy Operators in VLS

Lütfi Akın*

Department of Bussiness Administration, Mardin Artuklu University, Mardin, Turkey.
*lutfiakin@artuklu.edu.tr

Received: 01 August 2018
Accepted: 21 September 2018
DOI: 10.18466/cbayarfbe.449954

Abstract

Variable exponent spaces and Hardy operator space have played an important role in recent harmonic analysis because they have an interesting norm including both local and global properties. The variable exponent Lebesgue spaces are of interest for their applications to modeling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth conditions. This studies also has been stimulated by problems of elasticity, fluid dynamics, calculus of variations, and differential equations with non-standard growth conditions. In this study, we will discuss a characterization of approximation of Hardy operators in variable Lebesgue spaces.
Keywords: Variable exponent, Hardy operator, Sobolev space.

1. Introduction

Theory of approximation with linear integral operators started with Bernstein operators [1], Bernstein operators in the space $C[0,1]$ defined by $B_n f(x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x)$ for $x \in [0,1]$ with the Bernstein basis,

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Previously, we have been working on the approach in $C[0,1]$ or $L^p[0,1]$ space functions with Bernstein type linear positive operators (see [2, 3, 4, 5])

In this paper we study a characterization of approximation of functions by Hardy operators on variable $L^{p(.)}$ spaces. Hardy type operator is defined by

$$Hf(x) = \int_0^x f(t)dt \qquad (1.1)$$

Functions are defined on an explicit subset Λ of R^d. $L^{p(.)}$ space, $L^{p(.)}(\Lambda)$ is associated with a measurable function $p: \Lambda \to [1, \infty)$. The variable exponent Lebesgue space $L^{p(.)}$ be composed of all measurable functions f on Λ such that

$$\int_\Lambda \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1$$

for any $\lambda > 0$. The norm in $L^{p(.)}$ space is the generalization of the norm in L^p space (p is constant). The norm in $L^{p(.)}$ space is defined in the following manner

$$\|f\|_{L^{p(.)}} = \inf\left\{\lambda > 0: \int_\Lambda \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\} \qquad (1.2)$$

At the same time $L^{p(.)}$ becomes a Banach space. The idea of variable exponent $L^{p(.)}$ spaces was popularized by Orlicz (see [6]). Inspired by relations to variational integrals with non-standard growth linked to design of electrorheological fluids (e.g., [7, 8, 9, 10, 11]),

2. Materials and Methods

Definition 2.1. (see Definition 1, [5]) The exponent function $p: \Lambda \to [1, \infty)$ is log-Hölder continuous. if there exist a positive constants $\tilde{B}_p > 0$ such that

$$|p(x) - p(y)| \le \frac{\tilde{B}_p}{-\log|x-y|}, x, y \in \Lambda, \quad |x-y| < \frac{1}{2} \qquad (2.1)$$

and p is log-Hölder continuous at infinity if there holds

$$|p(x) - p(y)| \le \frac{\tilde{B}_p}{-\log(e+|x|)}, x, y \in \Omega, \quad |y| \ge |x| \qquad (2.2)$$

Denote, $p_- = \inf_{y\in\Lambda} p(y)$, $p_+ = \sup_{y\in\Lambda} p(y)$. It is clear that $1 \le p_- \le p_+ < \infty$.

Uniformly of approximated functions on the variable exponent Lebesgue space $L^{p(.)}$ can be illustrated by the variable Sobolev space $W^{h,p(.)}(\Lambda)$ (e.g., [12, 13]) with a uniformly index $h \in N$ which is the Banach space of measurable functions f such that for $\alpha =$

$(\alpha_1, \alpha_2, \ldots, \alpha) \in Z_+^b$ with $|\alpha|_1 = \sum_{i=1}^{b} \alpha_i \leq h$, the partial derivative

$B^\alpha f = \frac{\partial^{|\alpha|_1}}{\partial x_1{}^{\alpha_1} \ldots \partial x_b{}^{\alpha_d}} f \in L^{p(.)}(\Lambda)$.

The norm in Sobolev space $W^{h,p(.)}(\Lambda)$ by

$$\|f\|_{h,p(.)} = \|f\|_{W^{h,p(.)}(\Lambda)} = \sum_{|\alpha|_1 \leq h} \|B^\alpha f\|_{p(.)}$$

It is clear that $W^{0,p(.)}(\Lambda) = L^{p(.)}(\Lambda)$.

Denote the seminorm,

$$|f|_{h,p(.)} = \sum_{\alpha_1 = r} \|B^\alpha f(x)\|_{p(.)}$$

Linear operators for approximating functions on R^b take the form

$$T(f,x) = \int_{R^b} \mathrm{K}(x,t) f(t) dt, \quad x \in R^b \qquad (2.3)$$

where $\mathrm{K}: R^b \times R^b \to R$ is a kernel function

$$\int_{R^b} \mathrm{K}(x,t) dt \equiv 1 \qquad (2.4)$$

and any conditions for degenerations of $\mathrm{K}(x,t)$ as $|x-t|$ increases. Lets assume that the integral operator degenerations polynomially speed in the feel that for any non-negative integer m and a constant C_m there holds

$$|\mathrm{K}(x,t)| \leq \frac{C_m}{(1+|x-t|)^m}, \forall x,t \in R^b \qquad (2.5)$$

Definition 2. 2. (see Definition 2, [5]) Let a positive integer m, we say that a linear integral operator T on $L^p(R^b)$ defined by (2.3) holds approximation order $m \in N$ if for every sufficiently smooth function $f \in L^p(R^b)$,

$\|T_d f - f\|_{p(.)} \leq \tilde{C} d^m$, as $d \to 0$ (2.6)

where the constant $\tilde{C}$ is independent of d.

Herewith the measuring operator with a measuring parameter $d > 0$ defined by $\sigma_d f = f(\frac{\cdot}{d})$, T_h is the linear operator $\sigma_d T \sigma_1 / d$.

Lemma 2. 3. (see Lemma 1, [5]) If $g \in L_{loc}^1(R^b)$ satisfies $|\Delta g| \in L_{loc}^1(R^b)$, then for any $x, t \in R^b$ and H is Hardy operator, there holds

$$|g(x) - g(t)| \leq \frac{6^b}{b}\big(H(|\Delta g|)(x) + H(|\Delta g|)(t)\big)|x-t| \qquad (2.7)$$

Lemma 2. 4. (see Lemma 2, [5]) Let $s > b$. If $g \in L_{loc}^1(R^b)$ satisfies $|\Delta g| \in L_{loc}^1(R^b)$ and H is Hardy operator, then

$$|g(x) - g(t)| \leq c_{s,b} [H(|\Delta g|^s)(x)]^{\frac{1}{s}} |x-t|, \forall x,t \in R^b \qquad (2.8)$$

Lemma 2. 5. (see Lemma 3, [5]) If Λ is an open subset of R^b and $p: \Lambda \to [1, \infty)$ satisfies $1 < p_- \leq p_+ < \infty$, and the log-Hölder conditions (2.1) and (2.2), there exist a constant $B_p > 0$ depending only on p such that

$$\|H(f)\|_{p(.)} \leq B_p \|f\|_{p(.)} \ , \forall f \in L^{p(.)}(\Lambda) \qquad (2.9)$$

Lemma 2. 6. (see Lemma 4, [5]) If Λ is an open subset of R^b and $p: \Lambda \to [1, \infty)$ satisfies $1 < p_- \leq p_+ < \infty$, then for any $h > 0$ with $hp_- \geq 1$ and $f \in L^{hp(.)}(\Lambda)$, there holds

$$\||f|^h\|_{p(.)} = \|f\|_{hp(.)}^h$$

3. Results and Discussion

Theorem 3.1. We assume that the exponent function $p: R^b \to (1, \infty)$ satisfies $1 < p_- \leq p_+ < \infty$, and the log-Hölder conditions (2.1) and (2.2). If the kernel function K holds conditions (2.4) and (2.5) with $m > b + \frac{p_-}{p_- - 1}$, then the operators $\{T_d\}_{d>0}$ on $L^{p(.)}(R^b)$ are regularity bounded by a positive constant $\widetilde{H}_p$

$\|T_h\| \leq \widetilde{H}_p, \quad \forall h > 0$ (3.1)

Proof.. **Step 1**. We will prove the regular boundedness of $\{T_d\}$ on $L^{p(.)}(R^b)$.

$$T_d(f)(x) = \frac{1}{d^b} \int_{R^d} \mathrm{K}(xd^{-1}, td^{-1}) f(t) dt, x \in R^b \qquad (3.2)$$

By the condition (2.5), we have

$$|T_d(f)(x)| \leq \frac{C_m}{d^b} \int_{R^b} \frac{1}{\left(1 + \left|\frac{x-t}{d}\right|\right)^m} |f(t)| dt = C_m \widetilde{\mathrm{K}}_d . |f|(x), x \in R^b \qquad (3.3)$$

where $\widetilde{\mathrm{K}}_d(x,t) = \frac{1}{d^b} \cdot \frac{1}{\left(1 + \left|\frac{x-t}{d}\right|\right)^m}$. From [14] we say that there exists a constant B depending on b and m such that

$\widetilde{\mathrm{K}}_d . |f|(x) \leq BH(f)(x), \forall x \in R^b, d > 0,$ (3.4).

and from Lemma 2.5, we have

$\|T_d(f)\|_{p(.)} \leq C_m B \|H(f)\|_{p(.)} \leq C_m B B_p \|f\|_{p(.)}$, (3.5)

As a result, the operators $\{T_d\}$ are regular limited with $\|T_d\| \leq C_m B B_p$, for any $d > 0$.

Step 2. From $\int_{R^b} \mathrm{K}(x,t) dt \equiv 1$, we have $T_d(1,x) \equiv 1$. So for any $f \in L^{p(.)}(R^b)$ and $g \in W^{1,p(.)}$, by the uniform boundedness of the operators $\{T_d\}$, we have

$$\|T_d(f-g)\|_{p(.)} \leq \|T_d\| \|f-g\|_{p(.)}$$

Thus for any $g \in W^{1,p(.)}$,

$$\|\mathrm{T_d(f) - f}\|_{\mathrm{p(.)}} = \|\mathrm{T_d(f-g) + T_d(g) - g + g - f}\|_{\mathrm{p(.)}} \leq (\|T_d\| + 1)\|f-g\|_{p(.)} + \|T_d(g) - g\|_{p(.)}$$

Therefore, $\|T_d(g)-g\|_{p(.)}$, for $g \in W^{1,p(.)}$.
By Lemma 2.3, for any $x \in R^b$, we have

$$|T_d(g,x)-g(x)|$$

$$=\left|\frac{1}{d^b}\int_{R^b} \mathrm{K}(xd^{-1},td^{-1})[f(t)-f(x)]dt\right|$$

$$\leq \frac{6^b}{b}\left(\int_{R^b} \widetilde{\mathrm{K}}_h(x,t)H(|\Delta g|)(x)|t-x|dt\right)$$

$$+\frac{6^b}{b}\left(\int_{R^b} \widetilde{\mathrm{K}}_d(x,t)H(|\Delta g|)(t)|t-x|dt\right)$$

$$=\frac{6^b}{b}\left(E_{1,d}(x)+E_{2,d}(x)\right)$$

Consequently,

$$\|T_d(g)-g\|_{L^{p(.)}} \leq \frac{6^b}{b}\left(\left\|E_{1,d}(x)\right\|_{p(.)}+\left\|E_{2,d}(x)\right\|_{p(.)}\right) \qquad (3.6)$$

We first estimate $\left\|E_{2,d}\right\|_{p(.)}$. Since $m > b+\frac{p_-}{p_--1}$. Let a positive number $k' > \frac{p_-}{p_--1}$ such that $m > b+k'$. Here k' is conjugate of k. $\frac{1}{k}+\frac{1}{k'}=1$. Then $1<k<p_-$. Then there hold $\frac{k}{p_-}<1$ and $kp_->1$. By the Hölder inequality, $E_{2,d}(x)$ is bounded by

$$\left(\int_{R^b} \widetilde{\mathrm{K}}_d(x,t)[H(|\Delta g|)(t)]^k dt\right)^{1/k} \times$$

$$\times\left(\int_{R^b} \widetilde{\mathrm{K}}_d(x,t)|t-x|^{k'}dt\right)^{1/k'} \qquad (3.7)$$

Since $m > b+k'$, we set the constant

$\hat{C}_m=\int_{R^b}\frac{1}{(1+|t|)^{m-k'}}dt$, and get

$$\left(\int_{R^b} \widetilde{\mathrm{K}}_d(x,t)|t-x|^{k'}dt\right)^{1/k'}$$

$$\leq d\left(\int_{R^b}\frac{d^{-b}}{\left(1+\left|\frac{x-t}{d}\right|\right)^{m-k'}}dt\right)^{1/k'}$$

$$=\hat{C}_m^{1/k'}, \forall x \in R^b \qquad (3.8)$$

By Lemma 2.5 and Lemma 2.6, from estimates (3.4) and (3.5), we have

$$\left\|\left(\int_{R^b} \widetilde{\mathrm{K}}_d(x,t)[H(|\Delta \mathrm{g}|)(t)]^k dt\right)^{1/k}\right\|_{p(.)}$$

$$=\left\|\left(\int_{R^b} K_d(x,t)[H(|\Delta g|)(t)]^k dt\right)^{1/k}\right\|_{\frac{p(.)}{k}}^{\frac{1}{k}}$$

$$\leq \left(BB_{\frac{p}{k}}\right)^{1/k}\|[H|\Delta g|]^k\|_{\frac{p(.)}{k}}^{\frac{1}{k}}=\left(BB_{\frac{p}{k}}\right)^{1/k}\|H|\Delta g|\|_{p(.)}$$

$$\leq \left(BB_{\frac{p}{k}}\right)^{1/k}B_p\||\Delta g|\|_{p(.)}$$

Combining this estimate with (3.7) and (3.8), we get

$$\left\|E_{2,d}(x)\right\|_{p(.)} \leq \left(BB_{\frac{p}{k}}\right)^{1/k}B_p\hat{C}_m^{1/k'}d\||\Delta g|\|_{p(.)} \qquad (3.9)$$

The first term $\left\|E_{1,d}(x)\right\|_{p(.)}$ is easier to estimate.

$$E_{1,d}(x)=\frac{1}{d^b}\int_{R^b}\frac{|t-x|}{\left(1+\left|\frac{x-t}{d}\right|\right)^m}dt\, H(|\Delta g|)(x)$$

$$\leq \frac{d}{d^b}\int_{R^b}\frac{|t-x|}{\left(1+\left|\frac{x-t}{d}\right|\right)^{m-1}}dt\, H(|\Delta g|)(x)$$

$$=d\int_{R^b}\frac{1}{(1+|t|)^{m-1}}dt\, H(|\Delta g|)(x)$$

Hence, $E_{1,d}(x) \leq \hat{C}_m dH(|\Delta g|)(x)$, $\forall x \in R^b$ .Thus, we have

$$\left\|E_{1,d}(x)\right\|_{p(.)} \leq \hat{C}_m d\|H(|\Delta g|)\|_{p(.)}$$

$$\leq \hat{C}_m B_p d\||\Delta g|\|_{p(.)} \qquad (3.10)$$

Putting (3.9) and (3.10) into (3.6), we finally conclude

$$\|T_d(g)-g\|_{L^{p(.)}} \leq \frac{6^b}{b}\left(\hat{C}_m B_p + \left(BB_{\frac{p}{k}}\right)^{1/k}B_p\hat{C}_m^{1/k'}\right)d\||\Delta g|\|_{p(.)}$$

for some $f \in L^{p(.)}(R^b)$, we have

$$\|T_d(f)-f\|_{L^{p(.)}} \leq \frac{6^b}{b}\left(\hat{C}_m B_p + \left(BB_{\frac{p}{k}}\right)^{1/k}B_p\hat{C}_m^{1/k'} + \|T_d\|+1\right) \leq \tilde{C}_p$$

with the constant,

$$\tilde{C}_p=\frac{6^b}{b}\left(\hat{C}_m B_p + \left(BB_{\frac{p}{k}}\right)^{1/k}B_p\hat{C}_m^{1/k'} + C_m BB_p + 1\right)$$

depending only on $p(.), b, m$ and C_m.

Theorem 3.1 has been completed.

4. Conclusion

We showed a characterization of approximation of Hardy operators in variable Lebesgue spaces

Acknowledgement

Author thanks referee for his/her valuable suggestion remarks regarding the manuscript.

References

1. Bernstein, S, N, Demonstration du teoreme de Weirerstrass, fondee sur le calcul des probabilites, *Communication Society Mathematics*, 1913, 13.

2. Berens, H, Lorentz, G, G, Inverse theorems for Bernstein polynomials, *Indiana University Mathematics Journal*, 1972, 21, 693-708.

3. Berens, H, DeVore, R,A, Quantitative Korovkin theorems for positive linear operators on Lp spaces, *Transactions American Mathematical Society*, 1978, 245, 349-361.

4. Ditzian, Z, Totik, V, Moduli of Smoothness, Springer, *Series in Computational Mathematics,* Springer-Verlag, 1987, (9).

5. Bing-Zheng, L, Bo-Lu, H, Ding-Xuan Z, Approximation on Variable Exponent Spaces by Linear Integral Operators, *Journal of Approximation Theory*, 2017, 223, 29-51.

6. Orlicz,W, Uber konjugierte Exponentenfolgen, *Studia Mathematica*, 1931, 3, 200-211.

7. Acerbi, E, Mingione, G, Regularity results for a class of functionals with nonstandard growth, *Archive for Rational Mechanics and Analysis,* 2001, 156, 121-140.

8. Blomgren, P, Chan, T, Mulet, P, Wong, C, K, Total variation image restoration: numerical methods and extensions, *Proceedings of the 1997 IEEE International Conference on Image Processing*, 1997, 3, 384-387.

9. Bollt, E, M, Chartrand, R, Esedoglu, S, Schultz, P, Vixie, K, R, Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion, *Advance Computational Mathematics*, 2009, 31, 61-85.

10. Chen,Y, Levine, S, Rao, M, Variable exponent linear growth functionals in image restoration, *SIAM Journal of Applied Mathematics*, 2006, 66, 1383-1406

11. Mamedov,F, Zeren,Y, Akın, L, Compactification of weighted Hardy operator in variable exponent Lebesgue spaces, *Asian Journal of Mathematics and Computer Science*, 2017, 17:1, 38-47.

12. Fan, X, L, Zhao, D, On the spaces Lp(x) and Wm;p(x), *Journal of Mathematic Analysis and Applications*, 2001, 263, 424-446.

13. Diening, L, Harjulehto, P, Hastö, P, Ruzicka, M, Lebesgue and Sobolev Spaces with Variable Exponents, *Springer-Verlag*, Berlin/Heidelberg, 2011.

14. Stein, E, M, Singular Integrals and Dierentiability Properties of Functions, *Princeton University Press*, Princeton, New Jersey, 1970.