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A Note On Convergence of Nonlinear General Type Two Dimensional Singular Integral Operators

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Abstract

The object of this study is to present both the pointwise convergence and the rate of convergence of the nonlinear integral operators given by

$$V_\zeta(x,y;f) = \iint_\Omega K_\zeta\big(t,s,x,y;f(t,s)\big)dsdt\,,(x,y)\in\Omega,\qquad \zeta\in E$$

where $\Omega = \langle a,b\rangle \times \langle c,d\rangle$ is arbitrary bounded region in $\mathbb{R}^2$ or $\Omega = \mathbb{R}^2$, morever, E is a set of nonnegative numbers, ζ_0 is an accumulation point of E, and the function f is Lebesgue- integrable function on Ω.

Keywords: Lebesgue point, nonlinear singular integral, Lipschitz condition, pointwise convergence.

1. INTRODUCTION

In [1], Musielak introduced a new type convergence problem using the nonlinear integral operatorsin the following form

$$T_\omega f(t) = \int_a^b K_\omega\big(s-t,f(s)\big)\,ds, t\in\langle a,b\rangle \qquad (1)$$

and by assuming the whose kernel K_ω satisfies the generalized Lipschitz condition.

By using this idea, Bardaro et al. ([1], [3]) and in [4], Karsli studied the special cases of equation (1) in some Lebesgue spaces. Also, in [5], Swiderski and Wachnicki gave the theorems on pointwise aproximation of the operators of equation (1) in the class of integrable and periodic functions. For further informations on nonlinear integral operators, we mention some of studies as [1]-[12]. Also, the several approximation properties of many new type integral operators have been studied and discussed by some authors, see [13]-[15].

In this note, first, we present the pointwise convergence, and in the sequel, we give the rate of convergence of general type nonlinear two dimensional singular integral operators of the following type:

$$V_\zeta(x,y;f) = \iint_\Omega K_\zeta\big(t,s,x,y;f(t,s)\big)dsdt,$$

$$(x,y)\in\Omega,\ \zeta\in E \qquad (2)$$

under various assumptions on $f(t,s)$ and $(K_\zeta)_{\zeta\in E}$. By $L_1(\Omega)$, we denote the class of all functions $f(t,s)$ Lebesgue-integrable over the rectangle

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$\Omega = \langle a,b\rangle \times \langle c,d\rangle$ or $\Omega = \mathbb{R}^2$ and E is a set of positive numbers and ζ_0 is an accumulation point of E.

First, we shall give the basic concepts which are used in this paper.

Definition 1.1. A point $(x_0, y_0) \in \Omega$ is a μ-generalized Lebesgue point of function $f \in L_1(\Omega)$ if

$$\lim_{(h,k)\to(0,0)} \frac{1}{\mu(h,k)} \int_0^h \int_0^k \left|f(x_0 \pm t, y_0 \pm s) - f\big((x_0, y_0)\big)\right| dsdt = 0$$

where $0 < h \le b-a$, $0 < k \le d-c$ and $\mu(h,k) = \int_0^h \int_0^k \rho(t,s)dsdt$

is non-negative function provided $\rho(t,s)$ is nonnegative Lebesgue integrable function defined on $[0, b-a] \times [0, d-c]$ (see [16]).

We have created the following definition by taking advantage of the article [11] and [17].

Definition 1.2. If the family of functions $(K_\zeta)_{\zeta\in \mathrm{E}}$,$K_\zeta\colon \mathbb{R}^2 \times \mathbb{R}^2 \times \mathrm{E} \to \mathbb{R}$, holds the following condition and then we say that $(K_\zeta)_{\zeta\in\mathrm{E}}$ belongs to class $\mathcal{A}$:

a) $K_\zeta(t,s,x,y,0) = 0$, for every $t,s,x,y \in \mathbb{R}$ and $\zeta \in \mathrm{E}$.

b) Let $T_\zeta\colon \mathbb{R}^2 \times \mathbb{R}^2 \times \Lambda \to \mathbb{R}_0^+$ be a function such that

$$\left|K_\zeta(t,s,x,y;u) - K_\zeta(t,s,x,y;v)\right| \le T_\zeta(t,s;x,y)|u-v|,$$

for every $t,s,x,y \in \mathbb{R}$ and $\zeta \in E$. Moreover, for any fixed $(x,y) \in \Omega$, $T_\zeta(t,s,x,y)$ is integrable function of (t,s).

c) For every $u \in \mathbb{R}$ and any fixed $(x,y) \in \Omega$

$$\lim_{\zeta\to\zeta_0} \left| \iint\limits_{\mathbb{R}^2} K_\zeta(t,s,x,y;u)dsdt - u \right| = 0$$

d) For any fixed $(x,y) \in \Omega$ and for every $\delta > 0$

$$\lim_{\zeta\to\zeta_0} \left[\sup_{(t,s)\in\mathbb{R}^2\setminus N_\delta(x,y)} T_\zeta(t,s;x,y) \right] = 0$$

where

$N_\delta(x,y) = (x-\delta, x+\delta) \times (y-\delta, y+\delta)$.

e) For any fixed $(x,y) \in \Omega$

$$\lim_{\zeta\to\zeta_0} \iint\limits_{\mathbb{R}^2\setminus N_\delta(x,y)} T_\zeta(t,s,x,y,u)dsdt = 0$$

f) For every $\zeta \in \mathrm{E}$

$$\left\|T_\zeta(.)\right\|_{L_1(\mathbb{R}^2)} \le M < \infty,$$

g) For any fixed $x \in \langle a,b\rangle$, $T_\zeta(t,s;x,y)$ is non-increasing as a function of t on $\langle x-\delta, x]$ and non-decreasing function on $[x, x+\delta\rangle$, for each fixed $\lambda \in \Lambda$. Similarly, for any fixed $y \in \langle c,d\rangle$, $T_\zeta(t,s,x,y)$ is non-increasing as a function of s on $\langle y-\delta, y]$ and non-decreasing function on $[y, y+\delta\rangle$ for each fixed $\zeta \in \mathrm{E}$.

Analogously, for any fixed $(x,y) \in \Omega$ and fixed $\zeta \in \mathrm{E}$, $T_\lambda(t,s,x,y)$ is bimonotonically increasing with respect to (t,s) on both $\langle x, x+\delta\rangle \times \langle y, y+\delta\rangle$ and $\langle y-\delta, y] \times \langle x-\delta, x]$. Similiarly, $T_\zeta(t,s;x,y)$ is bimonotonically increasing with respect to (t,s) on both $[x, x+\delta\rangle \times \langle y-\delta, y]$ and $\langle x-\delta, x] \times [y, y+\delta\rangle$.

2. POINTWISE CONVERGENCE

Now we shall prove the existence of the integral operators in equation (2) by the Theorem 2.1.

Theorem 2.1. If $f \in L_1(\Omega)$, then for every $\zeta \in \mathrm{E}$, $V_\zeta \in L_1(\Omega)$ and $\left\|V_\zeta\right\|_{L_1(\Omega)} \le M\|f\|_{L_1(\Omega)}$

Proof. We define a function

$$h(t,s) = \begin{cases} f(t,s), & (t,s) \in \Omega \\ 0, & (t,s) \in \mathbb{R}\setminus\Omega \end{cases} \tag{3}$$

Using Fubini's Theorem (see, e.g., [18]) and conditions (a), (b) and (f) of class $\mathcal{A}$ we get the following inequalities:

$$\left\|V_\zeta(x,y;f)\right\|_{L_1(\Omega)}$$

$$= \left| \iint\limits_{\Omega} K_\zeta\big(t,s,x,y;f(t,s)\big)dsdt \right| dydx$$

$$\le \iint\limits_{\Omega} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t,s)| T_\zeta(t,s,x,y)\, dsdt \right) dydx$$

$$\le \|f\|_{L_1(\Omega)} \left\|T_\zeta\right\|_{L_1(\mathbb{R}^2)}.$$

The proof is completed for this case.

Now we assume that $\Omega = \mathbb{R}^2$. Following similar steps, as in the first case, we have

$$\left\|\boldsymbol{V}_\zeta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{f})\right\|_{\boldsymbol{L}_1(\boldsymbol{\Omega})} \leq \left\|\boldsymbol{T}_\zeta\right\|_{\boldsymbol{L}_1(\mathbb{R}^2)}\|\boldsymbol{f}\|_{\boldsymbol{L}_1(\mathbb{R}^2)}$$

$$\leq M\|f\|_{L_1(\mathbb{R}^2)}.$$

The proof is completed.

We shall show to pointwise convergence of the operator (2) at the μ-generalized Lebesgue point.

For $C > 0$, let P_C denote the set

$$\left\{\begin{matrix}(x,y,\zeta) \in \mathbb{R}^2 \times E: \\ \int\limits_{x_0-\delta}^{x_0+\delta}\int\limits_{y_0-\delta}^{y_0+\delta} \{T_\zeta(t,s;x_0,y_0) \times \rho(|t-x_0|,|s-y_0|)dsdt\} < C\end{matrix}\right\}$$

Theorem 2.2. Let (x_0, y_0) be a $\mu-$generalized Lebesgue point of function $f \in L_1(\Omega)$ and functions K_ζ satisfies the assumptions listed in class $\mathcal{A}$, then for any $C > 0$ and $(x,y,\zeta) \in P_C$

$$\lim_{\zeta\to\zeta_0} V_\zeta(x_0,y_0;f) = f(x_0,y_0).$$

Proof. Suppose that $(x_0,y_0) \in \Omega$ is the $\mu-$generalized Lebesgue point of function $f \in L_1(\Omega)$ and

$0 < x_0 - x < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$ for all $\delta > 0$ satisfying

$x_0 + \delta < b, x_0 - \delta > a,$

$y_0 + \delta < d, y_0 - \delta > c.$

For the remaining cases, the proof follows a similar line. From Definition 1.1, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all h and k satisfying $0 < h, k \leq \delta$ the inequality:

$$\int\limits_{x_0}^{x_0+h}\int\limits_{y_0-k}^{y_0} |f(t,s) - f(x_0 - y_0)|dsdt < \varepsilon\mu(h,k)$$

holds.

By conditions (b) and (c) of class $\mathcal{A}$, and from equation (3) using the extention $g(t,s)$ of $f(t,s)$, we get the following inequality:

$$\left|V_\zeta(x_0,y_0;f) - f(x_0,y_0)\right|$$

$$\leq \iint\limits_{\Omega} |f(t,s) - f(x_0 - y_0)|T_\zeta(t,s;x_0,y_0)dsdt$$

$$+|f(x_0,y_0)| \iint\limits_{\mathbb{R}^2\setminus\Omega} T_\zeta(t,s;x_0,y_0)dsdt$$

$$+\left|\iint_{\mathbb{R}^2} K_\zeta\big(t,s;x_0,y_0;f(x_0,y_0)\big)dsdt - f(x_0,y_0)\right|$$

$$= I_1 + I_2 + I_3.$$

The necessity of (e) and (c) of class $\mathcal{A}$ provides the $I_2 \to 0$ and $I_3 \to 0$ as $\zeta \to \zeta_0$, respectively.

Splitting I_1 into two parts, we get the following:

$$I_1 = \iint\limits_{\Omega\setminus N_\delta} |f(t,s) - f(x_0,y_0)|T_\zeta(t,s;x_0,y_0)dsdt$$

$$+ \iint\limits_{N_\delta} |f(t,s) - f(x_0,y_0)|T_\zeta(t,s;x_0,y_0)dsdt$$

$$= I_{11} + I_{12}$$

where $N_\delta = (x_0 - \delta\,, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)$ stands for the family of all neigborhoods of (x_0, y_0) in $\mathbb{R}^2$.

For the integral I_{11} we may write the following

$$I_{11} = \iint\limits_{\Omega\setminus N_\delta} |f(t,s) - f(x_0,y_0)|T_\zeta(t,s;x_0,y_0)dsdt$$

$$\leq \left(\sup_{(t,s)\in\mathbb{R}^2\setminus N_\delta(x,y)} T_\zeta(t,s;x_0,y_0)\right)$$

$$\times \left(\|f\|_{L_1(\Omega)} + f(x_0,y_0)|b-a||d-c|\right)$$

by condition (d), $I_{11} \to 0$ as ζ tends to ζ_0.

Now, we focus on the integral I_{12}. It is easy to see that I_{12} can be written in the following form:

$$I_{12} = \left\{\int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0-\delta}^{y_0} + \int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0} + \int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0}^{y_0+\delta} + \int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0}^{y_{0+\delta}}\right\}$$

$$|f(t,s) - f(x_0,y_0)|L_\zeta(t,s;x_0,y_0)dsdt$$

$$= I_{121} + I_{122} + I_{123} + I_{124}.$$

We shall prove $I_{12} \to 0$ as $\zeta \to \zeta_0$. It is enough to show that the integrals I_{121}, I_{122}, I_{123} and I_{124} tend to zero as $\zeta \to \zeta_0$ on P_C.

Let us define a new function as such:

$$G(t,s) := \int_{x_0}^{t}\int_{s}^{y_0} |f(u,v) - f(x_0,y_0)| dv du.$$

Then, for all t and s satisfying $t - x_0 \leq \delta$ and $y_0 - s \leq \delta$ we have

$$|G(t,s)| \leq \varepsilon\mu(t - x_0, y_0 - s) \qquad (4)$$

The following equality holds for the integral I_{121}:

$$|I_{121}| = \left| \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} |f(t,s) - f(x_0,y_0)| T_\zeta(t,s;x_0,y_0) ds dt \right|$$

$$= \left| (LS) \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} T_\zeta(t,s;x_0,y_0) d_t d_s[-G(t,s)] \right|,$$

where (LS) means Lebesgue-Stieltjes integral.

Using the integration by parts (see [20]) to the Lebesgue-Steltjes integral I_{121}, we have the following inequality:

$$|I_{121}| \leq \left| \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} G(t,s) d_t d_s [T_\zeta(t,s;x_0,y_0)] \right|$$

$$+ \left| \int_{x_0}^{x_0+\delta} G(t, y_0 - \delta) d_t [T_\zeta(t, y_0 - \delta; x_0, y_0)] \right|$$

$$+ \int_{y_0-\delta}^{y_0} G(x_0 + \delta, s) d_s [T_\zeta(x_0 + \delta, s; x, y)]$$

$$+ |G(x_0 + \delta, y_0 - \delta)[T_\zeta(x_0 - \delta, y_0 - \delta; x, y)]|.$$

from equation (4), we have the following inequality:

$$I_{121} \leq \varepsilon\{ \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} \mu(t - x_0, y_0 - s) d_t d_s [T_\zeta(t,s;x_0,y_0)]\}$$

$$+\varepsilon \int_{x_0}^{x_0-\delta} \mu(t - x_0, \delta) |d_t[T_\zeta(t, y_0 - \delta; x_0, y_0)]|$$

$$+\varepsilon \int_{y_0-\delta}^{y_0} \mu(\delta, y_0 - s) |d_s[T_\zeta(x_0 + \delta, s; x_0, y_0)]|$$

$$+\varepsilon\mu(\delta,\delta) T_\zeta(x_0 + \delta, y_0 - \delta; x_0, y_0)$$

$$= i_1 + i_2 + i_3 + i_4.$$

Using the condition (g) and applying the integration by parts to each integral, the following inequality is obtained (for the rest of the operations see [19] and [20]).

$$|I_{121}| \leq \varepsilon \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} T_\zeta(t,s;x_0,y_0) \rho(|t - x_0|, |s - y_0|) ds dt.$$

Computing the integrals I_{122}, I_{123} and I_{124} with the same method, and combining the obtained inequalities we have the following inequality

$$|I_{12}| \leq \varepsilon \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} T_\zeta(t,s;x_0,y_0) \rho(|t - x_0|, |s - y_0|) ds dt$$

which in viev of the definition of the set P_C tends to zero as $\zeta \to \zeta_0$.

Thus the proof of the theorem is completed.

Theorem 2.3. Let $(x_0, y_0) \in \mathbb{R}^2$ be a μ −generalized Lebesgue point of function $f \in L_1(\mathbb{R}^2)$ and function K_ζ satisfies the assumptions listed in class $\mathcal{A}$, then for any $C > 0$ and $(x, y, \zeta) \in P_C$

$$\lim_{\zeta\to\zeta_0} V_\zeta(x_0, y_0; f) = f(x_0, y_0).$$

Proof. The proof can be shown analogous to the proof of Theorem 2.2.

3. RATE OF CONVERGENCE

In this part, we establish the rate of pointwise convergence which we got in the Section 2.

Theorem 3.1. Suppose that the hypothesis of Theorem 2.2 is satisfied. Let

$$\Delta(\zeta,\delta,x_0,y_0)=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} T_\zeta(t,s;x_0,y_0)$$

$$\rho(|t-x_0|,|s-y_0|)dsdt$$

for $0<\delta<\delta_0$ and the following assumptions be satisfied:

i. $\Delta(\zeta,\delta,x_0,y_0)\to 0$ as $\zeta\to\zeta_0$ for some $\delta>0$.

ii. For $(x_0,y_0)\in\Omega$

$$\iint_{\mathbb{R}^2\setminus N_\delta} T_\zeta(t,s;x_0,y_0)dsdt=o\big(\Delta(\zeta,\delta,x_0,y_0)\big)$$

as $\zeta\to\zeta_0$.

iii. For $(x_0,y_0)\in\Omega$

$$\left|\iint_{\mathbb{R}^2} K_\zeta(t,s;x,y,u)dsdt-u\right|$$
$$=o\big(\Delta(\zeta,\delta,x_0,y_0)\big)$$

as $\zeta\to\zeta_0$

iv. For $(x_0,y_0)\in\Omega$

$$\sup_{(t,s)\in\mathbb{R}^2\setminus N_\delta} T_\zeta(t,s;x_0,y_0)=o\big(\Delta(\zeta,\delta,x_0,y_0)\big)$$

$\delta>0$, as $\zeta\to\zeta_0$.

Then, at each μ −generalized Lebesgue point of $f\in L_1(\Omega)$ and we have as $\zeta\to\zeta_0$.

$$|V_\zeta(x_0,y_0;f)-f(x_0,y_0)|=o\big(\Delta(\zeta,\delta,x_0,y_0)\big).$$

Proof. Omitted.

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