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$\mathcal{C}^3$ QUARTIC QUASI-INTERPOLANTS OVER A 6-DIRECTION MESH

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ABSTRACT. In this work, we are interested in constructing quasi-interpolants in the space of splines $\mathcal{S}_4^3(\Delta_6)$, where Δ_6 designates a triangulation of a rectangular domain generated by a uniform mesh with six directions. Firstly, we will show that we can have a subspace of $\mathcal{S}_4^3(\Delta_6)$ containing $\mathbb{P}_4$ generated by the integer translates of a box spline ϕ for which we specify the B-coefficients. We also give some main properties of this box spline. Naturally, the B-coefficients of the box spline ϕ can be obtained by convolution. However, for reasons of simplicity, we propose a method based on the subdivision schemes to determine them quickly. Finally, given the importance of this triangulation, we develop some discrete and differential quasi-interpolants, and we give numerical examples.

Keywords: Spline Approximation, Box Spline, Subdivision Scheme, Spline Quasi-Interpolation.

AMS Subject Classification: 41A15, 65D15

1. Introduction

Quasi-interpolation is an essential technique in numerical analysis and data processing, known for its efficiency and simplicity in approximating functions from discrete data. The main motivation behind developing the proposed quasi-interpolation scheme is to improve the accuracy and continuity class of the approximations while reducing computational complexity. In particular, spline-based quasi-interpolation schemes are known for their exceptional flexibility and efficiency, making them well-suited for a wide range of applications in science and engineering. The work of Buhmann and Jäger [1] provides an essential reference for understanding the theoretical foundations and recent developments in quasi-interpolation. Several authors have studied box splines and their properties on uniform meshes with 3 and 4 directions, including C.K. Chui [2] in 1988 and C. de Boor, K. Höllig, and S. Riemenschneider [5] in 1993. The key properties of these functions are clearly

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described in [2]. We also find fairly regular B-splines generalizing these classic box-splines (see [2]). In the works we have just cited, they used the Cartesian lattice. The symmetry of box-splines was not a concern for their authors. In [14], M. Kim and J. Peters presented a collection focusing on low-degree symmetric box-splines on Cartesian and hexagonal lattices (see Figure 2). In this collection, the authors introduced a symmetric box-spline, denoted as $M_{\mathrm{h}11}$ on the hexagonal lattice. As indicated in [13], the B-coefficients of this box-spline can be calculated stably and efficiently. Given the importance of the Cartesian lattice in practical applications, especially in image processing, our objective is to work on the box-spline generated by the directions $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (-1,-1)$, $e_4(-1,1)$, $e_5 = (2,1)$ and $e_6 = (-1,-2)$. In the univariate case, it is always possible to construct a basis of B-splines for the space $S_k^{k-1}(\Delta)$ where $k \geq 0$ and Δ a subdivision of an interval $[a,b]$. By expressing the monomials of the space $\mathbb{P}_k$ in the B-spline basis, we can easily construct quasi-interpolants (QIs) of degree k and class $\mathcal{C}^{k-1}$ with optimal approximation order. However, due to connectivity issues particularly on arbitrary domains extending this property to bivariate splines is nontrivial, especially for degrees exceeding 2. While the B-spline technique can be generalized to 2D via tensor products, this extension introduces additional challenges on non-rectangular domains. Recently, several authors have investigated the space $\mathcal{S}_3^2(\Delta_6)$ from different perspectives (see [3, 12]). In particular, [6] constructed discrete and differential quasi-interpolants in this space, demonstrating notable approximation properties. In this work, we aim to construct highly regular discrete and differential QIs with improved approximation orders.

The content of this work is organized as follows: In section 2, we will present the triangulation Δ_6 and discuss the dimension of the space $\mathcal{S}_4^3$ on a rectangular domain. Section 3 is devoted to the construction of the box spline ϕ, defined by the six directions $e_i,\ i = 1,\dots,6$, using the concept of subdivision schemes. We also give in this section the main properties of this box spline. In section 4, we will establish the Marsden identity and give discrete and differential QIs of optimal order. A summary table of the numerical tests relating to the different QIs developed will be given at the end of this section.

2. Quartic splines of class $\mathcal{C}^3$ on Δ_6

Let Δ_3 be the uniform triangulation of $\mathbb{R}^2$ with vertices at the integer lattice points $\mathbb{Z}^2$ and edges parallel to the three directions e_1, e_2, and e_3 (see Fig. 1). Let Δ_6 be the refinement of Δ_3 obtained by subdividing each triangle into six subtriangles using its medians (Powell-Sabin type refinement of Δ_3). This triangulation can be obtained by drawing the following six families of straight lines with equations:

$$
\begin{aligned}
&(D_1^k): \quad y-k=0,\\
&(D_2^k): \quad x-k=0,\\
&(D_3^k): \quad y-x-k=0,\\
&(D_4^k): \quad y+x-k=0,\\
&(D_5^k): \quad y-2x-k=0,\\
&(D_6^k): \quad y-\tfrac{1}{2}x-k=0, \quad k\in\mathbb{Z}.
\end{aligned}
$$

In Figure 1, we have represented the Cartesian lattice with the six directions $e_i, i = 1,\cdots,6$ used to construct our box spline on the left. On the right, we see the figure on the left multiplied by the matrix $G = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\sqrt{3} & \sqrt{3} \end{bmatrix}$. We notice that the box spline we will construct here is not a transformation of the symmetric box spline $M_{\mathrm{h}11}$ defined on the hexagonal lattice, as presented in [14]. Let M and N be two positive integers, and consider the rectangular region $\Omega_{MN} = [0, M+1]\times[0, N+1]$ provided with a triangulation

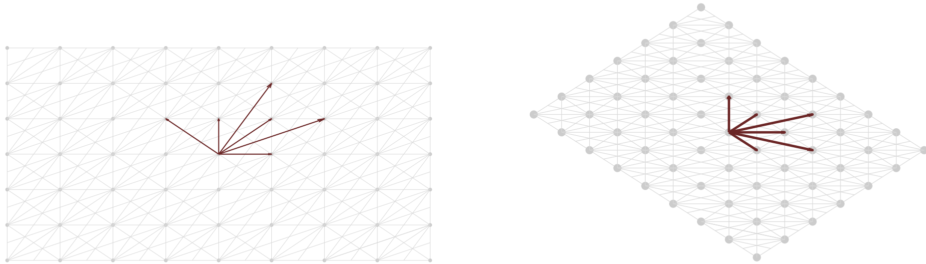


FIGURE 1. On the left: Cartesian lattice with 6 directions, on the right: hexagonal Cartesian lattice.

of type Δ_6. The obtained triangulation is denoted $\Delta_{MN}^{(6)}$. For $r < d$ in $\mathbb{Z}_+$, let us denote by $\mathrm{S}_d^r\left(\Delta_{MN}^{(6)}\right)$ vector space of class functions $\mathcal{C}^r\left(\Omega_{MN}\right)$ such that the restriction on each triangle of $\Delta_{MN}^{(6)}$ is a polynomial in the space $\mathbb{P}_d$.

Proposition 2.1. *The dimension of the space* $\mathrm{S}_4^3\left(\Delta_{MN}^{(6)}\right)$ *is equal to* $MN+5(M+N+2)$.

Proof. According to Theorem ([16] 2.1), the dimension of space $\mathrm{S}_4^3\left(\Delta_{MN}^{(6)}\right)$ is given by:

$$15+L+\sum_{i=1}^{N_v} a_4^3\left(n_i\right)$$

where N_v is the number of interior vertices $A_1, \cdots, A_{N_v}$, L is the number of interior lines, n_i is the number of lines passing through vertex A_i, and $a_d^r\left(n_i\right)$ is the number given by the following formula:

$$a_4^3(n)=\frac{1}{2}\left(1-\left[\frac{4}{n-1}\right]\right)_+\left(2(n-5)+(n-1)\left[\frac{4}{n-1}\right]\right). \tag{1}$$

$[x]$ is the integer part of x and $x_+=\max(x,0)$.

By examining triangulation $\Delta_{MN}^{(6)}$, we deduce that $L=6(M+N-1)$. We also have $(M-1)(N-1)$ interior vertices crossed by six lines, $2MN$ interior vertices crossed by three lines, and $3MN-M-N$ interior vertices crossed by two lines.
By applying the formula (1), we obtain $a_4^3(6)=1,\ a_4^3(3)=0$ and $a_4^3(2)=0$, from which, by a simple calculation, we find the result. $\square$

3. Quartic box spline of class $\mathcal{C}^3$ on triangulation Δ_6

In general, box splines are generated by directional convolutions in $\mathbb{Z}^2$ (see [8, 9]). Consider six directions e_i $(i=1,\ldots,6)$, and let ϕ be the box spline defined through convolution along these directions. The box spline ϕ serves as the basic limit function of the subdivision scheme with mask:

$$a(z):=2^{-4}\prod_{i=1}^{6}\left(1+z^{e_i}\right) \tag{2}$$

In the following, we note by $\mathcal{S}(\phi)$ the space generated by the integer translates of ϕ.

Proposition 3.1. *The box spline ϕ verified the following properties:*

(1) $\phi \in \mathrm{S}_4^3\left(\Delta_{MN}^{(6)}\right)$.

(2) *The family* $\{\phi(\cdot-\alpha), \alpha \in \mathbb{Z}^2\}$ *is linearly dependent.*
(3) $\mathbb{P}_4 \subset \mathcal{S}(\phi)$.

Proof. 1. Let $X_6 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ denote the set of directions and let d (resp. r) be the degree (resp. the class) of the box spline ϕ generated by X_6. According to de Boor and Hölling [7] we have:

$$d = \#X_6 - 2 = 6 - 2 = 4$$

and

$$r = \min\left\{\#Y : Y \subset X_6, \langle X_6 \backslash Y\rangle \neq \mathbb{R}^2\right\} - 2 = 5 - 2 = 3.$$

2. To show the linear dependence of the family $\left\{\phi(\cdot-\alpha), \alpha \in \mathbb{Z}^2\right\}$ we will proceed in the same way as de Boor and Hölling in [7]. It is enough to find a set $Y_2 \subset X_6$ such that $|\det Y_2| \neq 1$ and $\langle Y_2\rangle = \mathbb{R}^2$.
In our case, we can take $Y_2 = \{e_1, e_6\}$ which generates $\mathbb{R}^2$ and $|\det Y_2| = 2$.
3. The fact that $\phi \in \mathrm{S}_4^3\left(\Delta_{MN}^{(6)}\right)$ gives $\mathbb{P}_4 \subset \mathcal{S}(\phi)$ (see [2]). $\square$

Remark 3.1. *There exists* $(M+5)(N+5)$ *translated integers of* ϕ *whose intersection with* Ω_{MN} *is non-empty, this number is less than* $\mathrm{Dim}\left(\mathrm{S}_4^3\left(\Delta_{MN}^{(6)}\right)\right)$ *for* $M > 1$ *and* $N > 1$*. We deduce that* $\left\{\phi(\cdot-\alpha), \alpha \in \mathbb{Z}^2\right\}$ *is not a generating family of* $\mathrm{S}_4^3\left(\Delta^{(6)}\right)$*.*

The mask $a(z)$ given by equation (2) can be written explicitly as follows:

$$\begin{aligned}
a(z) =& \frac{1}{4}z^{(0,0)} + \frac{1}{8}z^{(-2,0)} + \frac{1}{4}z^{(-1,0)} + \frac{1}{4}z^{(1,0)} + \frac{1}{8}z^{(2,0)} + \\
& \frac{1}{16}z^{(-2,-3)} + \frac{1}{16}z^{(-1,-3)} + \frac{1}{8}z^{(0,-2)} + \frac{1}{16}z^{(-3,-2)} + \\
& \frac{1}{8}z^{(-2,-2)} + \frac{1}{8}z^{(-1,-2)} + \frac{1}{16}z^{(1,-2)} + \frac{1}{4}z^{(0,-1)} + \\
& \frac{1}{16}z^{(-3,-1)} + \frac{1}{8}z^{(-2,-1)} + \frac{1}{4}z^{(-1,-1)} + \frac{1}{8}z^{(1,-1)} + \\
& \frac{1}{16}z^{(2,-1)} + \frac{1}{4}z^{(0,1)} + \frac{1}{16}z^{(-2,1)} + \frac{1}{8}z^{(-1,1)} + \frac{1}{4}z^{(1,1)} + \\
& \frac{1}{8}z^{(2,1)} + \frac{1}{16}z^{(3,1)} + \frac{1}{8}z^{(0,2)} + \frac{1}{16}z^{(-1,2)} + \frac{1}{8}z^{(1,2)} + \\
& \frac{1}{8}z^{(2,2)} + \frac{1}{16}z^{(3,2)} + \frac{1}{16}z^{(1,3)} + \frac{1}{16}z^{(2,3)}.
\end{aligned}$$

It is clear that the only symmetries of this mask are with respect to the lines (D_1) : $x-y=0$ and (D_2) : $x+y=0$ (see Fig. 2). We can thus calculate ϕ on part $\left\{(x,y) \in \mathbb{R}^2, x-y \geq 0, x+y \geq 0\right\}$ and deduce the remaining parts by symmetry. This support cutting poses some difficulties when we want to schematize the B-coefficients of the box spline ϕ in a single figure.

The support of the box spline ϕ is the convex hull of the mask. To draw the graph of ϕ, we can use the cascade algorithm, which operates as follows:
1. We choose an initial function with compact support ψ_0.
2. We calculate ψ_{k+1} at step $k+1$ by $\quad \psi_{k+1} = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha \psi_k.$

Let us denote by f_α^k the value attached to the point $2^{-k}\alpha, \alpha \in \mathbb{Z}^2$. The refinement relation at step k is given by:

$$f_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^2} a_{\alpha-2\beta} f_\beta^k, \alpha \in \mathbb{Z}^2. \tag{3}$$

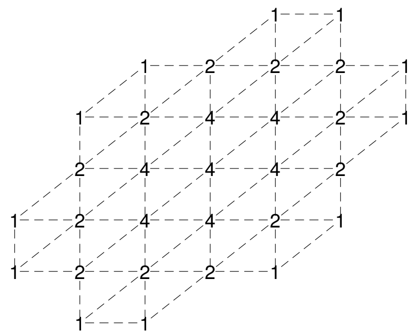
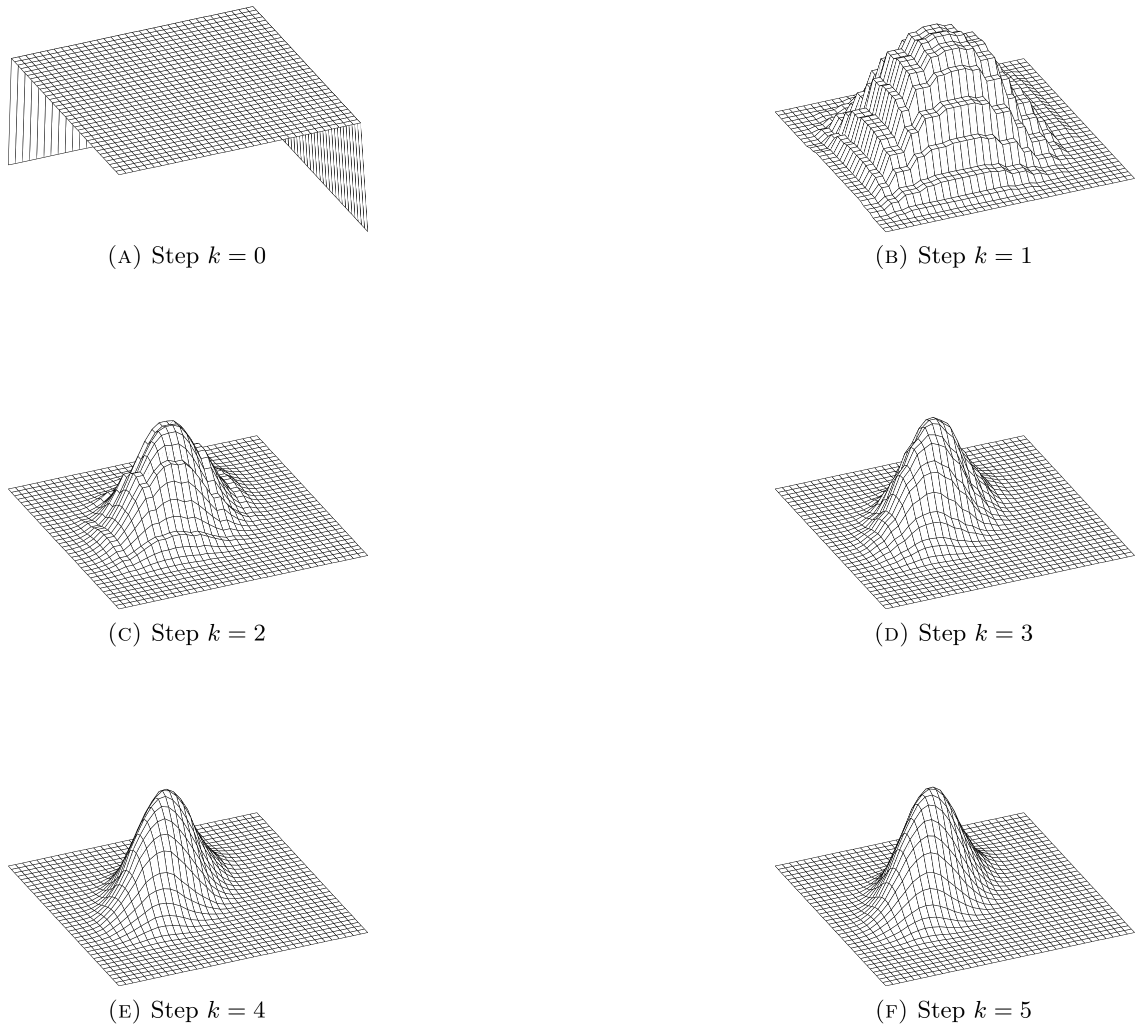


FIGURE 2. Support for the box spline, all coefficients are multiplied by 16.



(A) Step $k = 0$

(B) Step $k = 1$

(C) Step $k = 2$

(D) Step $k = 3$

(E) Step $k = 4$

(F) Step $k = 5$

FIGURE 3. Graph of ϕ using the cascade algorithm.

Specifically, for the box spline ϕ the equation (3) at step k becomes:

$$\begin{aligned}
f^{k+1}_{2\alpha} =& \frac{1}{4} f^k_{\alpha} + \frac{1}{8} f^k_{\alpha-(0,1)} + \frac{1}{8} f^k_{\alpha+(0,1)} + \frac{1}{8} f^k_{\alpha-(1,1)} + \\
& \frac{1}{8} f^k_{\alpha+(1,1)} + \frac{1}{8} f^k_{\alpha+(1,0)} + \frac{1}{8} f^k_{\alpha-(1,0)}, \\
f^{k+1}_{2\alpha+(0,1)} =& \frac{1}{4} f^k_{\alpha} + \frac{1}{4} f^k_{\alpha+(0,1)} + \frac{1}{16} f^k_{\alpha+(1,-1)} + \frac{1}{8} f^k_{\alpha+(1,1)} + \\
& \frac{1}{16} f^k_{\alpha+(1,2)} + \frac{1}{16} f^k_{\alpha-(1,1)} + \frac{1}{8} f^k_{\alpha-(1,0)} + \frac{1}{16} f^k_{\alpha+(-1,1)}, \\
f^{k+1}_{2\alpha+(1,0)} =& \frac{1}{16} f^k_{\alpha-(1,1)} + \frac{1}{4} f^k_{\alpha} + \frac{1}{8} f^k_{\alpha-(0,1)} + \frac{1}{16} f^k_{\alpha+(0,1)} + \\
& \frac{1}{16} f^k_{\alpha+(1,0)} + \frac{1}{4} f^k_{\alpha+(1,0)} + \frac{1}{8} f^k_{\alpha+(1,1)} + \frac{1}{16} f^k_{\alpha+(2,1)}, \\
f^{k+1}_{2\alpha+(1,1)} =& \frac{1}{16} f^k_{\alpha+(-1,0)} + \frac{1}{4} f^k_{\alpha} + \frac{1}{4} f^k_{\alpha+(1,1)} + \frac{1}{8} f^k_{\alpha+(0,1)} + \\
& \frac{1}{16} f^k_{\alpha+(0,-1)} + \frac{1}{8} f^k_{\alpha+(1,0)} + \frac{1}{16} f^k_{\alpha+(1,2)} + \frac{1}{16} f^k_{\alpha+(2,1)}.
\end{aligned}$$

Equation (3) can also be effectively described using subdivision masks as follows.

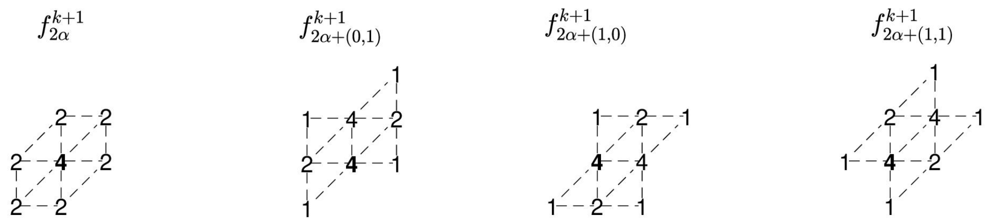


FIGURE 4. All coefficients in the subdivision masks are multiplied by 16.

While the B-coefficients of a box spline are naturally obtained through convolution, they can alternatively be computed using refinement relations.

We present an algorithm for calculating the B-coefficients of the box spline ϕ via the cascade algorithm. Since ϕ is a degree-4 polynomial on each triangular patch, we determine it through interpolation by selecting 15 points satisfying the geometric characterization (GC) condition. This geometric characterization enables a simplified expression for the Lagrange interpolant, as detailed in [4].

For the interpolation problem, we use the principal lattice (regular grid) as the underlying structure.

Given that the box spline B-coefficients are rational numbers (see Lemma 2 of [13]) and that the cascade algorithm converges, we can be confident that our algorithm will also converge to rational numbers. Therefore, on each triangle T of supp(ϕ), if we take f^N_α satisfying condition (GC), the coefficients of the resulting Lagrange polynomial - when expressed in the Bernstein basis - are approximations of rational numbers. With sufficient approximation precision, we could use numerical techniques to recover the exact fractions.

The support of the box spline ϕ is very large; even with two available symmetries, we cannot represent it in a single figure. Therefore, we divide it into nine pieces that can each be considered as separate splines (see Fig. 5).

Let us denote these splines as $\phi_{11}(x,y)$, $\phi_{12}(x,y)$, $\phi_{13}(x,y)$, $\phi_{21}(x,y)$, $\phi_{22}(x,y)$, $\phi_{23}(x,y)$, $\phi_{31}(x,y)$, $\phi_{32}(x,y)$, and $\phi_{33}(x,y)$. Examining the B-coefficients of these subsplines reveals symmetry about the center $(0,0)$. More precisely, we have the following relationships:

$$\begin{aligned}
\phi_{33}(x,y) &= \phi_{11}(-x,-y)\\
\phi_{32}(x,y) &= \phi_{12}(-x,-y)\\
\phi_{31}(x,y) &= \phi_{13}(-x,-y)\\
\phi_{23}(x,y) &= \phi_{21}(-x,-y).
\end{aligned}$$

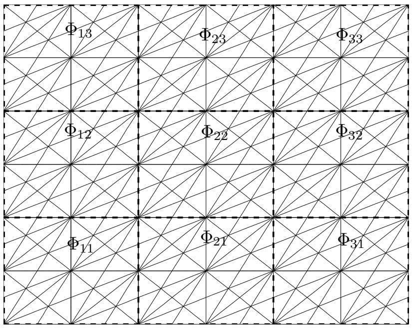


FIGURE 5. Pieces of support of spline ϕ support.

Given these symmetry properties, we provide the B-controls of the five sub-splines ϕ_{11}, ϕ_{12}, ϕ_{13}, ϕ_{21}, and ϕ_{22}. The others can be easily deduced by symmetry.

By combining the nine sub-splines, we construct the box spline ϕ (see Fig. 6f).

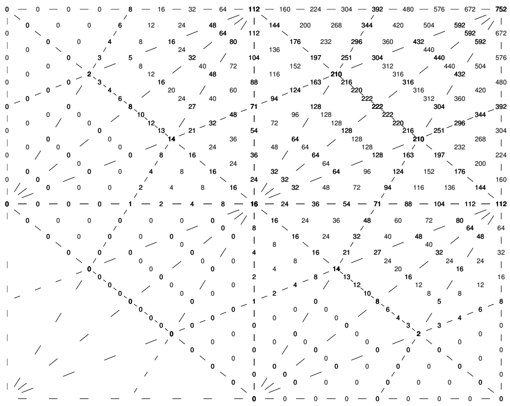
Algorithm 1 Cascade Algorithm

Require: Input:
- The support of supp(ϕ).
- An integer N for the refinement step.

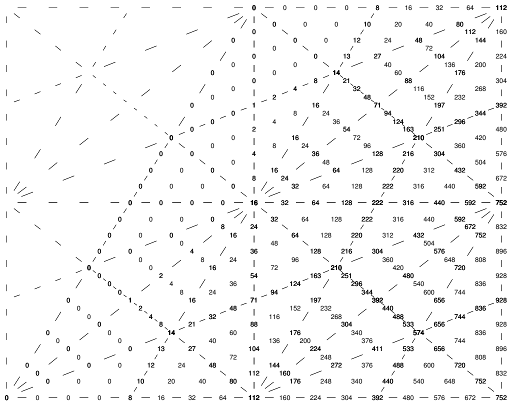
Ensure: Output:
- The B-coefficients for box spline ϕ.

Step 1: Initialization
For $k=0$, set $f^0_{(0,0)} = 1$ and $f^0_\alpha = 0$ for $\alpha \neq (0,0)$.
Step 2: Refinement
Extend the refinement up to step N.
Step 3: Interpolation
For each triangle T of supp(ϕ):
- Select 20 values f^N_α satisfying the (GC) condition.
- Compute the Lagrange interpolation polynomial p on T.

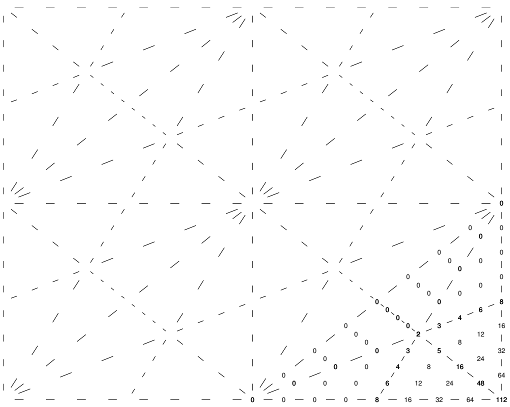
Step 4: Conversion to Bernstein Basis
Express the polynomial p in the Bernstein basis relative to the triangle T.
Deduce the B-coefficients.
Step 5: Coefficient Approximation
Approximate each B-coefficient by a fraction with the smallest possible numerator and denominator.
Return the B-coefficients.



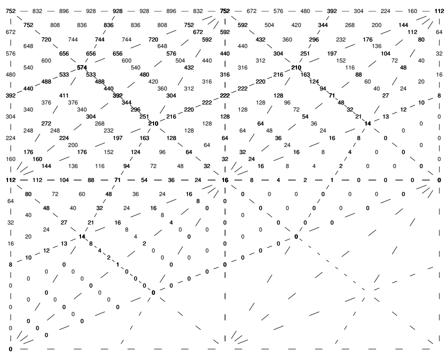
(A) B-coefficients of ϕ_{11}.



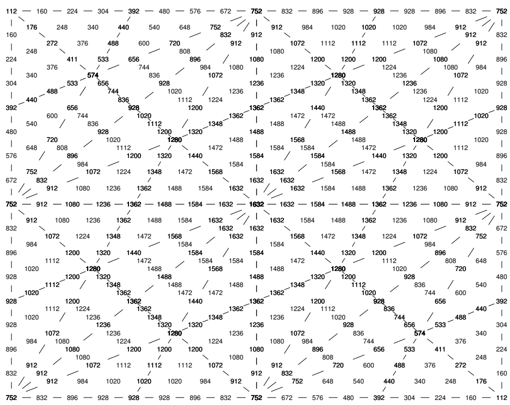
(B) B-coefficients of ϕ_{12}.



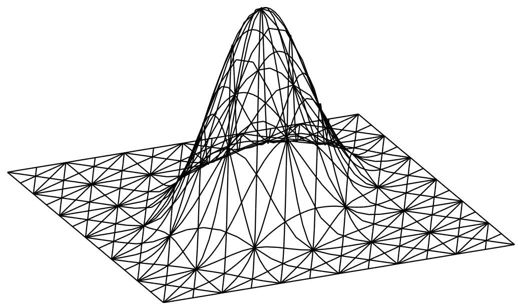
(C) B-coefficients of ϕ_{13}.



(D) B-coefficients of ϕ_{21}.



(E) B-coefficients of ϕ_{22}.



(F) Graph of ϕ plotted along the triangulation lines.

FIGURE 6. Graphs showing the B-coefficients divided by by 6912. of various ϕ functions and the triangulation lines.

4. Quasi-interpolation in $\mathcal{S}(\phi)$ space

4.1. **Reproduction of polynomials.** The concept of polynomial reproduction (also called polynomial space preservation) plays a fundamental role in approximation theory, particularly in the study of distances between functions and function spaces.

For a spline space $\mathcal{S}(\phi)$ generated by the integer translates of a compactly supported spline function ϕ, establishing the *Marsden identity* represents a crucial step. This identity provides an expansion of the monomials $m_\alpha(x) := x^\alpha$ (for $x \in \mathbb{R}^2$, $\alpha \in \mathbb{N}^2$) in terms of the integer translates of ϕ. To construct such an expansion, we utilize Appell sequences $\{P_\alpha\}_{\alpha\in\mathbb{N}^2}$, defined recursively through:

$$g_0 = m_0.$$
$$g_\alpha = m_\alpha - \sum_{j\in\mathbb{Z}^2} \phi(j) \sum_{\substack{\beta\neq\alpha \\ \beta\leq\alpha}} \frac{(-j)^{\alpha-\beta}}{(\alpha-\beta)!} g_\beta,$$

Each monomial m_α is written:

$$m_\alpha(x) = \sum_{k\in\mathbb{Z}^2} g_\alpha(k)\phi(x-k).$$

In the following, by dividing the B-coefficients by $2^8.3^2$, we will normalize the box spline ϕ, i.e., the sum of its integer translates is equal to 1.
The values of ϕ at integer points are given by:

$$\begin{aligned}
&\phi(0,0) = \frac{102}{432}, \phi(-2,-2) = \phi(-2,0) = \phi(0,-2) = \frac{1}{432},\\
&\phi(0,2) = \phi(2,2) = \phi(2,0) = \frac{1}{432},\\
&\phi(-2,-1) = \phi(-1,-2) = \phi(-1,1) = \frac{7}{432},\\
&\phi(1,-1) = \phi(1,2) = \phi(2,1) = \frac{7}{432},\\
&\phi(-1,-1) = \phi(-1,0) = \phi(0,-1) = \frac{47}{432},\\
&\phi(0,1) = \phi(1,0) = \phi(1,1) = \frac{47}{432}.
\end{aligned}$$

Therefore, we derive the expressions for the monomials of $\mathbb{P}_4$ in terms of the integer translates of ϕ for $k = (k_1, k_2) \in \mathbb{Z}^2$ as follows:

$$\begin{aligned}
1 &= \sum_{k\in\mathbb{Z}^2} \phi(x-k), \quad x_1 = \sum_{k\in\mathbb{Z}^2} k_1\phi(x-k),\\
x_2 &= \sum_{k\in\mathbb{Z}^2} k_2\phi(x-k), \quad x_1x_2 = \sum_{k\in\mathbb{Z}^2} \left(-\frac{1}{3} + k_1k_2\right)\phi(x-k),\\
x_1^2 &= \sum_{k\in\mathbb{Z}^2} \left(-\frac{2}{3} + k_1^2\right)\phi(x-k), \quad x_2^2 = \sum_{k\in\mathbb{Z}^2} \left(-\frac{2}{3} + k_2^2\right)\phi(x-k),\\
x_1^3 &= \sum_{k\in\mathbb{Z}^2} \left(-2k_1 + k_1^3\right)\phi(x-k),\\
x_1^2x_2 &= \sum_{k\in\mathbb{Z}^2} \left(-\frac{2}{3}(k_1+k_2) + k_1^2k_2\right)\phi(x-k),
\end{aligned}$$

$$x_1x_2^2 = \sum_{k\in\mathbb{Z}^2} \left(-\frac{2}{3}(k_1+k_2) + k_1k_2^2\right)\phi(x-k),$$
$$x_2^3 = \sum_{k\in\mathbb{Z}^2} \left(-2k_2 + k_2^3\right)\phi(x-k),$$
$$x_1^4 = \sum_{k\in\mathbb{Z}^2} \left(\frac{3}{2} - 4k_1^2 + k_1^4\right)\phi(x-k),$$
$$x_1^3x_2 = \sum_{k\in\mathbb{Z}^2} \left(\frac{3}{4} - k_1^2 - 2k_1k_2 + k_1^3k_2\right)\phi(x-k),$$
$$x_1^2x_2^2 = \sum_{k\in\mathbb{Z}^2} \left(\frac{3}{4} - \frac{2}{3}\left(k_1^2+k_2^2\right) - \frac{4}{3}k_1k_2 + k_1^2k_2^2\right)\phi(x-k),$$
$$x_1x_2^3 = \sum_{k\in\mathbb{Z}^2} \left(\frac{3}{4} - k_2^2 - 2k_1k_2 + k_1k_2^3\right)\phi(x-k),$$
$$x_2^4 = \sum_{k\in\mathbb{Z}^2} \left(\frac{3}{2} - 4k_2^2 + k_2^4\right)\phi(x-k).$$

4.2. **Construction of quasi-interpolants in the space $\mathcal{S}(\phi)$.** Now consider the following differential quasi-interpolant:

$$Q^*f(x) = \sum_{k\in\mathbb{Z}^2} D^*f(k)\phi(x-k),$$

where D^* is the differential operator defined by:

$$D^* = I - \frac{1}{3}\left(\partial_1^2 + \partial_1\partial_2 + \partial_2^2\right) + \frac{1}{8}\left(\partial_1^3\partial_2 + \partial_1\partial_2^3\right) + \frac{1}{16}\left(\partial_1^4 + \partial_2^4\right) + \frac{3}{16}\partial_1^2\partial_2^2,$$

where $\partial_1^i\partial_2^j = \frac{\partial^{i+j}}{\partial x^i \partial y^j}$.

From the results of the previous section, it is easy to deduce the following result:

Proposition 4.1. *The differential quasi-interpolant Q^* is exact on the space of quartic polynomials $\mathbb{P}_4$.*

Proof. It is easy to just check the exactness of Q^* on the monomials of the space $\mathbb{P}_4$. $\square$

Now, using the D^* operator, we can easily derive the Marsden identity. For any polynomial $p \in \mathbb{P}_4$, we have

$$p(x) = \sum_{k\in\mathbb{Z}^2} D^*p(k)\phi(x-k). \tag{4}$$

Following the approach of Sbibih et al. in [15], we apply the operator D^* rather than polar forms in local approximations to construct discrete and differential quasi-interpolants (QIs). In general, we establish the following proposition:

Proposition 4.2. *Let $\mathcal{I}_k, k \in \mathbb{Z}^2$ be a polynomial approximant which approximates f in a neighborhood of k and which is exact on $\mathbb{P}_4$, then the following quasi-interpolant is exact on $\mathbb{P}_4$*

$$Qf(x) = \sum_{k\in\mathbb{Z}^2} D^*\left(\mathcal{I}_kf\right)(k)\phi(x-k),$$

Proof. Let p be a polynomial of $\mathbb{P}_4$, from the exactness of the approximation operators $\mathcal{I}_k, k \in \mathbb{Z}^2$ obtaining us $\mathcal{I}_k(p) = p$, hence

$$Qp(x) = \sum_{k\in\mathbb{Z}^2} D^*\left(\mathcal{I}_k p\right)(k)\phi(x-k) \; = \sum_{k\in\mathbb{Z}^2} D^* p(k)\phi(x-k).$$

By applying the Marsden identity given by formula (4), we obtain the result. □

To construct differential QIs, we can take as operator $\mathcal{I}_k$ the Hermite interpolant in the neighborhood of k of the function f. Now let f be enough regular function, and $\mathcal{I}_k f, k \in \mathbb{Z}^2$ be the Taylor polynomial of degree 4 at the point $k = (k_1, k_2)$.
We have

$$\mathcal{I}_k f\left(x_1, x_2\right) := \sum_{0\le i_1+i_2\le 4} \frac{1}{i_1! i_2!} f^{(i_1,i_2)}(k)\left(x_1-k_1\right)^{i_1}\left(x_2-k_2\right)^{i_2},$$

where $f^{(i_1,i_2)}(k) = \frac{\partial^{i_1+i_2}}{\partial x_1^{i_1}\partial x_2^{i_2}} f(k)$.
By applying Proposition (4.2), we find the differential QI Q^*. This quasi-interpolant is given explicitly by:

$$Q^* f(x) = \sum_{k\in\mathbb{Z}^2} D^*\left(\mathcal{I}_k\right)\phi(x-k) = \sum_{k\in\mathbb{Z}^2} \mu_k(f)\phi(x-k),$$

where

$$\begin{aligned}\mu_k(f) =& f(k) - \frac{1}{3}\left(f^{(2,0)}(k) + f^{(1,1)}(k) + f^{(0,2)}(k)\right) + \frac{1}{8}\left(f^{(3,1)}(k) + f^{(1,3)}(k)\right)\\ &+ \frac{1}{16}\left(f^{(4,0)}(k) + f^{(0,4)}(k)\right) + \frac{1}{16} f^{(2,2)}(k).\end{aligned}$$

We can construct discrete quasi-interpolants of the form

$$Qf(x) = \sum_{k\in\mathbb{Z}^2} \mu_k(f)\phi(x-k), \tag{5}$$

by taking $\mathcal{I}_k$, the Lagrange operator. To simplify things, we give two discrete QIs, which are based on values satisfying the geometric characterization (CG) (see Fig. 7).
The first discrete quasi-interpolant, denoted $Q^{(1)}$, of the form (5) that we propose is based on the Lagrange polynomial evaluated at the following points:

$$\begin{aligned}&(-2,-2)+(k_1,k_2),(-1,-1)+(k_1,k_2),(k_1,k_2),(1,1)+(k_1,k_2),(2,2)+(k_1,k_2),\\ &(-2,-1)+(k_1,k_2),(-2,0)+(k_1,k_2),(-2,1)+(k_1,k_2),(-2,2)+(k_1,k_2),(1,2)+(k_1,k_2),\\ &(0,2)+(k_1,k_2),(-1,2)+(k_1,k_2),(0,1)+(k_1,k_2),(-1,0)+(k_1,k_2),(-1,1)+(k_1,k_2).\end{aligned}$$

In this case, we have the following result:

$$\begin{aligned}\mu_k^1(f) =& \frac{13}{144} f\left(k_1-2,k_2-2\right) - \frac{1}{72} f\left(k_1-2,k_2-1\right) - \frac{19}{48} f\left(k_1-2,k_2\right) + \frac{4}{3} f\left(k_1-1,k_2\right)\\ &- 2f\left(k_1-1,k_2+1\right) + \frac{47}{72} f\left(k_1-1,k_2+2\right) + \frac{61}{48} f\left(k_1,k_2\right) + \frac{4}{3} f\left(k_1,k_2+1\right)\\ &- \frac{19}{48} f\left(k_1,k_2+2\right) - \frac{49}{72} f\left(k_1+1,k_2+1\right) - \frac{1}{72} f\left(k_1+1,k_2\right) + \frac{13}{144} f\left(k_1+2,k_2+2\right)\\ &- \frac{49}{72} f\left(k_1-1,k_2-1\right) + \frac{47}{72} f\left(k_1-2,k_2+1\right) - \frac{35}{144} f\left(k_1-2,k_2+2\right).\end{aligned}$$

The second discrete quasi-interpolant, denoted $Q^{(2)}$, of the form (5) that we propose is based on the Lagrange polynomial evaluated at the following points:

$$\begin{aligned}&(-2,-2)+(k_1,k_2),(-1,-1)+(k_1,k_2),(k_1,k_2),(1,1)+(k_1,k_2),(2,2)+(k_1,k_2),\\ &(-1,-2)+(k_1,k_2),(0,-2)+(k_1,k_2),(1,-2)+(k_1,k_2),(2,-2)+(k_1,k_2),(2,1)+(k_1,k_2),\\ &(2,0)+(k_1,k_2),(2,-1)+(k_1,k_2),(1,0)+(k_1,k_2),(0,-1)+(k_1,k_2),(1,-1)+(k_1,k_2).\end{aligned}$$

For this QI, the functionals are given by:

$$\begin{aligned}\mu_k^2(f) =& \frac{13}{144} f\left(k_1-2, k_2-2\right)-\frac{1}{72} f\left(k_1-1, k_2-2\right)-\frac{49}{72} f\left(k_1-1, k_2-1\right)-\frac{19}{48} f\left(k_1, k_2-2\right) \\ & +\frac{4}{3} f\left(k_1, k_2-1\right)+\frac{61}{48} f\left(k_1, k_2\right)+\frac{47}{72} f\left(k_1+1, k_2-2\right)-2 f\left(k_1+1, k_2-1\right) \\ & +\frac{4}{3} f\left(k_1+1, k_2\right)-\frac{49}{72} f\left(k_1+1, k_2+1\right)-\frac{35}{144} f\left(k_1+2, k_2-2\right)+\frac{47}{72} f\left(k_1+2, k_2-1\right) \\ & -\frac{19}{48} f\left(k_1+2, k_2\right)-\frac{1}{72} f\left(k_1+2, k_2+1\right)+\frac{13}{144} f\left(k_1+2, k_2+2\right).\end{aligned}$$

We can also obtain discrete QIs using finite differences to discretize the differential operator Q^*. The two discrete QIs that we have just constructed use the same coefficients; we can deduce one from the other by symmetry with respect to the axis $(D_1): x-y=0$, which is the axis of symmetry of ϕ. However, they differ because the function we are approximating is not necessarily symmetric about this axis.

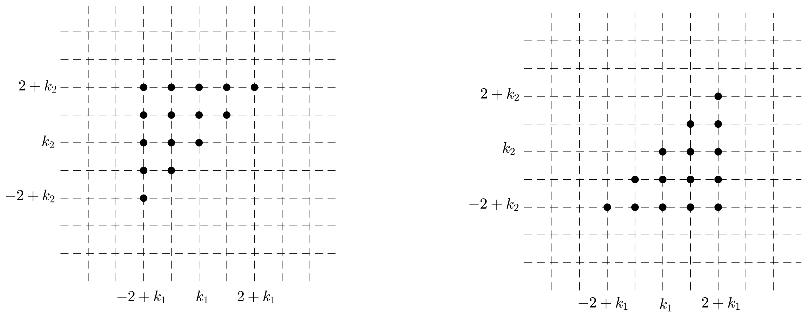


FIGURE 7. Quasi-interpolation points for discrete QIs.

A construction similar to this technique is described in [10].

4.3. **Numerical examples.** We note that our numerical calculations are performed using Matlab on data sampled from two test functions. In example 1, we will use the Franke function [11]:

$$\begin{aligned}f_1(x, y) =& \tfrac{3}{4} \exp \left(-\tfrac{(9x-2)^2+(9y-2)^2}{4}\right)+\tfrac{3}{4} \exp \left(-\tfrac{(9x+1)^2}{49}-\tfrac{(9y+1)}{10}\right) \\ & \tfrac{1}{2} \exp \left(-\tfrac{(9x-7)^2+(9y-3)^2}{4}\right)-\tfrac{1}{5} \exp \left(-(9x-4)^2-(9y-7)^2\right)\end{aligned}$$

On the domain $\Omega=[0,1]\times[0,1]$, we consider the sequence of triangulations Δ_n, associated with the vertices $(ih, jh), i, j=0, \ldots, n.$ with $h:=1/n$ and refined by a uniform Powell-Sabin refinement.

On the triangulations $\Delta_n, n=4,8,16,32$, we applied our QIs to the function f_1 and calculated the following errors: $e=\max\limits_{r,s=1,\ldots,64}\left|f_1\left(x_r, y_s\right)-Q f_1\left(x_r, y_s\right)\right|,$
where x_r, y_s are points equidistant from $[0,1]$. The numerical test results are presented in Table 1.
In example 2, we will use the function test : $f_2(x, y)=\frac{\tanh(9y-9x)+1}{9}, \quad (x, y) \in [0,1]^2$.
On the same triangulations $\Delta_n, n=4,8,16,32$, we applied our QIs to the function f_2 and calculated the errors as in example 1. The results of the numerical tests of this example are detailed in Table 2.

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$Q^{(1)}$	$2.77e-001$	$4.84e-002$	$8.44e-003$	$6.36e-004$
$Q^{(2)}$	$2.66e-001$	$5.05e-002$	$1.14e-002$	$6.37e-004$

Table 1. Errors of different quasi-interpolants of function f_1 for $\Delta_n, n = 4, 8, 16, 32$.

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$Q^{(1)}$	$2.54e-001$	$4.50e-002$	$7.44e-003$	$5.65e-004$
$Q^{(2)}$	$2.42e-001$	$4.70e-002$	$1.07e-002$	$1.06e-004$

Table 2. Errors of different quasi-interpolants of function f_2 for $\Delta_n, n = 4, 8, 16, 32$.

5. Conclusion

Our objective in this study was to construct degree 4 approximations with maximum connectivity. To achieve this, we utilized a box spline with a slightly broader support. The techniques we employed apply to various mesh types. In future work, we plan to utilize these results in image-processing applications.

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