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FUZZY AUTOMATA $\mathfrak{F}^*$-FIRST CATEGORY SUBSYSTEMS

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Abstract. In this paper, the ideas of fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces and fuzzy automata $\mathfrak{F}^*$-first category spaces are presented and reasonable examples are given. Additionally a few significant properties related with fuzzy automata ς-Baire spaces are stated. It is shown that fuzzy automata $\mathfrak{F}^*$-first category space is certainly not a fuzzy automata ς-Baire space. Further in fuzzy automata ς-Baire space, there is no fuzzy automata $\mathfrak{F}^*$-first category subsystem. At last, as an application of fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces in fuzzy T_i $(i = 0, 1)$ spaces are identified.

Keywords: Fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem, $\mathfrak{N}\mathfrak{F}^*$-dense subsystem, ς-Baire space, $\mathfrak{F}^*$-residual subsystem.

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1. Introduction

Zadeh [25] innovated the concept of a fuzzy set in 1965 and Chang [3] gave a note to the fuzzy topological space which provided a natural framework. The notion of an automaton was first fuzzified by Wee [24]. In [5], [16], [17], [19], [20], [21], [22] it is shown that certain topological and fuzzy topological concepts can be used in fuzzy automata theory to throw light on the structure of such fuzzy automata [1], particularly, to obtain certain results pertaining to their connectivity and separation properties. Z. H. Li, P. Li and Y. M. Li, [12] discussed the relationships among several types of fuzzy automata. Ignjatovic, Ciric and Simovic [8] studied the concepts of subsystems, reverse subsystems and double subsystems of a fuzzy automaton in terms of fuzzy relation inequalities and equations. Katsaras [10] introduced the idea of fuzzy norm on a linear space. In 1992, Felbin [6] introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding fuzzy metric associated to this fuzzy norm is of Kaleva and Seikkala type [9]. In 1994, Cheng and Mordeson [4] introduced another idea of a fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. In motivation

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of the paper Cheng and Mordeson, we have introduced a new definition of a fuzzy norm which is associated fuzzy automata. The novelty of this definition is the validity of this type of fuzzy norm into a family of non-empty states in fuzzy automata. The concepts of Baire spaces have been studied extensively in classical topology in [7], [15]. The concept of Baire spaces in fuzzy setting was introduced and studied by the authors in [18].

Motivated by the work done by some fuzzy topologist on general fuzzy automata, along with fuzzy automata normed linear structure space introduced by Madhuri and Amudhambigai in [13], this paper discusses several characterizations of fuzzy first category with fuzzy automata normed linear structure spaces. In this paper, the notions of fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces and fuzzy automata $\mathfrak{F}^*$-first category spaces are introduced and suitable examples are provided. Also some important properties related with fuzzy automata ς-Baire spaces are discussed. It is shown that fuzzy automata $\mathfrak{F}^*$-first category space is not a fuzzy automata ς-Baire space. Further in fuzzy automata ς-Baire space, there is no fuzzy automata $\mathfrak{F}^*$-first category subsystem.

2. Preliminaries

This section contains some basic concepts of fuzzy sets and fuzzy automaton. In addition, some related results and propositions are collected from various books and research articles. Also, this section includes almost all possible ground notions which are essential to make this paper self-contained.

Definition 2.1. [13] Let $M = (Q, X, \delta)$ be a fuzzy automaton. A fuzzy automata normed linear space is a 3-tuple (Q, N, T) where Q is non-empty set of states of M and also it is a linear space over the field $\mathbb{F}$, T is a t-norm and N is a fuzzy set on $Q \times (0, \infty)$, such that for all $p, q \in Q$ and all $s, t > 0$, the following conditions holds:

(i) $N(p, t) > 0$,
(ii) $N(p, t) = 1$, for all $t > 0$ if and only if $p = 0$,
(iii) If $\alpha \neq 0$, then $N(\alpha p, t) = N(p, \frac{t}{|\alpha|})$, $\forall t, \alpha \in \mathbb{F}$,
(iv) $T(N(p, t), N(q, s)) \leq N(p + q, t + s)$, $\forall t, s \in \mathbb{F}$,
(v) $N(p, \cdot)$ is a non-decreasing function of $\mathbb{F}$ and $lim_{t\to\infty} N(p, t) = 1$.
(vi) Assume that for all $p \neq 0, N(p, \cdot)$ is a continuous function on $\mathbb{F}$ and strictly increasing on the subset $\{\ t : 0 < N(p, t) < 1\ \}$ of $\mathbb{F}$.

Example 2.1. Let $M = (Q, X, \delta)$ be a fuzzy automaton where $Q = \mathbb{R}^2$ is a vector space over the field $\mathbb{R}$. Let $p = (p_1, p_2) \in \mathbb{R}^2$ and $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ be defined by

$$N(p, t) = \begin{cases} \frac{t^2}{(t\ +\ |p_1|)\ (t\ +\ |p_2|)}, & \text{for } t > 0 \\ \\ 0, & \text{for } t \leq 0 \end{cases}$$

and also the t-norm is defined as $T(a, b) = ab$. Then $(\mathbb{R}^2, N, T)$ is a fuzzy automata normed linear space.

Definition 2.2. [13] Let (Q, N, T) be a fuzzy automata normed linear space and let $p \in Q$, $\alpha \in (0, 1)$ and $\epsilon > 0$. The fuzzy set $\mu_\alpha(p, \epsilon)$ where $\mu_\alpha : Q \times (0, \infty) \to I$, be defined over Q by

$$\mu_\alpha(p, \epsilon)(q) = \begin{cases} 1 - \alpha, & N(p - q, \epsilon) > \alpha \\ 0, & \text{otherwise} \end{cases}$$

is said to be a fuzzy automata α-open sphere in Q if

$$c(1_Q - \mu_\alpha(p, \epsilon)) = (1_Q - \mu_\alpha(p, \epsilon)).$$

Definition 2.3. [13] Any fuzzy subsystem $\mu \in I^Q$ is called a fuzzy automata open subsystem if for $\mu(p) \geq 0$ and $c(1_Q - \mu) = (1_Q - \mu)$, there exists an $\epsilon > 0$ such that $\mu_\alpha(p, \epsilon) \leq \mu$, for some $\alpha \in [0, 1]$ and $\forall p \in Q$.

Example 2.2. Let $M = (Q, X, \delta)$ be a fuzzy automaton where Q is a trivial vector space over $\mathbb{R}$. Let $N : Q \times (0, \infty) \to [0, 1]$ be defined by

$$N(p,t) = \begin{cases} \frac{t}{\|p\|}, & \text{for } t > 0, \text{ p} \in Q, \\ \\ 0, & \text{for } t \leq 0, \text{ p} \in Q, \end{cases}$$

and also the t-norm is defined as $T(a, b) = ab$. Then (Q, N, T) is a fuzzy automata normed linear space. Let $\mu_\alpha : \mathbb{R} \times (0, \infty) \to [0, 1]$ be defined as

$$\mu_\alpha(p,\epsilon)(q) = \begin{cases} 1-\alpha, & \frac{\epsilon}{\|p-q\|} > \alpha, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mu \in I^Q$ be defined as $\mu(0) = 0.6 \geq 0$. Thus for $\alpha = 0.5$ and $\epsilon > 0$, $\mu_\alpha(p, \epsilon) \leq \mu$. Thus μ is a fuzzy automata open subsystem.

Proposition 2.1. [13] Let $M = (Q, X, \delta)$ be a fuzzy automaton and let Q be a non-empty set of states of M. Let (Q, N, T) be a fuzzy automata normed linear space. Then the family

$$\tau_{\mathscr{N}} = \{ \ \mu \in I^Q : \mu \text{ is fuzzy automata open subsystem } \}$$

is a fuzzy automata normed linear structure on Q. The members of $\tau_{\mathscr{N}}$ are called the fuzzy automata $\mathscr{N}$-open subsystems and the complement of a fuzzy automata $\mathscr{N}$-open subsystem is called a fuzzy automata $\mathscr{N}$-closed subsystem.

Example 2.3. In Example 2.2, let (Q, N, T) be a fuzzy automata normed linear space. Let $\mu_1,\ \mu_2,\ \mu_3 \in I^Q$ be formulated as follows : $\mu_1(0) = 0.6$, $\mu_2(0) = 0.7$ and $\mu_3(0) = 0.65$. For $\alpha \in [0, 1]$ and for $\epsilon > 0$, $\mu_\alpha(p, \epsilon) \leq \mu_1$, $\mu_\alpha(p, \epsilon) \leq \mu_2$ and $\mu_\alpha(p, \epsilon) \leq \mu_3$. Thus $0_Q, 1_Q, \mu_1, \mu_2, \mu_3$ are fuzzy automata open subsystems. Therefore $\tau_{\mathscr{N}} = \{ \ 0_Q, 1_Q, \mu_1, \mu_2, \mu_3 \ \}$ is a fuzzy automata normed linear structure over Q. Then the ordered pair $(Q, \tau_{\mathscr{N}})$ is a fuzzy automata normed linear structure space.

Definition 2.4. [14] Let (X, τ) be a fuzzy topological space. A fuzzy set $\mu \in I^X$ is called fuzzy irreducible if $\mu \neq 0_X$ and for all fuzzy closed sets $\gamma, \delta \in I^X$ with $\mu \leq (\gamma \vee \delta)$, it follows that either $\mu \leq \gamma$ or $\mu \leq \delta$.

Definition 2.5. [2] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be fuzzy second category.

Definition 2.6. [23] A fuzzy set μ_A is quasi-coincident with the fuzzy set μ_B iff $\exists\ x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$ (i.e.,) $\mu_A\ q\ \mu_B$.

3. Fuzzy Automata $\mathfrak{F}^*$-First Category Subsystems

In this section, the concepts of fuzzy automata irreducible, fuzzy automata $\mathscr{N}$-closed subsystems, fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystem, fuzzy automata $\mathfrak{F}^*$-first category, fuzzy automata $\mathfrak{F}^*$-second category and fuzzy automata $\mathfrak{F}^*$-residual subsystems are introduced. Some properties related with above concepts are discussed in fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces.

Definition 3.1. Let $(Q, \tau_{\mathscr{N}})$ be any fuzzy automata normed linear structure space. Any fuzzy subsystem $\mu \in I^Q$ is called fuzzy automata irreducible if $\mu \neq 0_Q$ and for all fuzzy automata $\mathscr{N}$-closed subsystems $\gamma, \delta \in I^Q$ with $\mu \leq (\gamma \vee \delta)$, it follows that either $\mu \leq \gamma$ or $\mu \leq \delta$.

Definition 3.2. Let $(Q, \tau_{\mathscr{N}})$ be any fuzzy automata normed linear structure space. Any $\lambda \in I^Q$ is said to be fuzzy automata irreducible $\mathscr{N}$-closed iff it is both fuzzy automata irreducible and fuzzy automata $\mathscr{N}$-closed.

Definition 3.3. Let $(Q, \tau_{\mathscr{N}})$ be any fuzzy automata normed linear structure space and let $\lambda \in \tau_{\mathscr{N}}$ be any fuzzy automata $\mathscr{N}$-open subsystem in $(Q, \tau_{\mathscr{N}})$. Then the collection $\mathfrak{F} = \{ \ (1_Q - \sigma) \in I^Q : \lambda \ \mathfrak{q} \ \sigma$ and σ is a fuzzy automata irreducible $\mathscr{N}$-closed subsystem in $(Q, \tau_{\mathscr{N}}) \ \}$ which is finer than the fuzzy automata normed linear topology $\tau_{\mathscr{N}}$ on Q is said to be a fuzzy automata normed linear $\mathfrak{F}$-structure on Q. A fuzzy automata normed linear $\mathfrak{F}$-structure on Q together with 1_Q is said to be a fuzzy automata normed linear $\mathfrak{F}^*$-structure on Q and it is denoted by $\mathfrak{I}_{\mathscr{N}}$. A nonempty set Q with a fuzzy automata normed linear $\mathfrak{F}^*$-structure $\mathfrak{I}_{\mathscr{N}}$ denoted by $(Q, \mathfrak{I}_{\mathscr{N}})$, is said to be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Each member of $\mathfrak{I}_{\mathscr{N}}$ is said to be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystem and the complement of each fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystem is said to be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-closed subsystem.

Example 3.1. In Example 2.3, let $(Q, \tau_{\mathscr{N}})$ be any fuzzy automata normed linear structure space. Clearly, $1_Q, (1_Q - \mu_1), (1_Q - \mu_2)$ and $(1_Q - \mu_3)$ are fuzzy automata irreducible $\mathscr{N}$-closed subsystems in $(Q, \tau_{\mathscr{N}})$. For $\mu_2 \in \tau_{\mathscr{N}}$, $\mu_2 \ \mathfrak{q} \ 1_Q$, $\ \mu_2 \ \mathfrak{q} \ (1_Q - \mu_1)$, $\ \mu_2 \ \mathfrak{q} \ (1_Q - \mu_3)$. Thus $\mathfrak{I}_{\mathscr{N}} = \{ \ 0_Q, 1_Q, \mu_1, \mu_3 \ \}$ is a fuzzy automata normed linear $\mathfrak{F}^*$-structure over Q. Then the ordered pair $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata normed linear $\mathfrak{F}^*$-structure space.

Definition 3.4. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Let $\lambda \in I^Q$ be any fuzzy subsystem. Then the fuzzy automata $\mathscr{N}\mathfrak{F}^*$-interior and fuzzy automata $\mathscr{N}\mathfrak{F}^*$-closure of λ are denoted by $\mathcal{FA}Int_{\mathscr{N}\mathfrak{F}^*}(\lambda)$ and $\mathcal{FA}Cl_{\mathscr{N}\mathfrak{F}^*}(\lambda)$ and defined as

$$\mathcal{FA}Int_{\mathscr{N}\mathfrak{F}^*}(\lambda) = \bigvee \{ \ \beta \in I^Q : \beta \leq \lambda \text{ and } \beta \text{ is fuzzy automata } \mathscr{N}\mathfrak{F}^*\text{-open } \},$$
$$\mathcal{FA}Cl_{\mathscr{N}\mathfrak{F}^*}(\lambda) = \bigwedge \{ \ \beta \in I^Q : \lambda \leq \beta \text{ and } \beta \text{ is fuzzy automata } \mathscr{N}\mathfrak{F}^*\text{-closed } \}.$$

Definition 3.5. A fuzzy subsystem $\lambda \in I^Q$ in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is called a fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystem if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $(1_Q - \lambda_i) \in \mathfrak{I}_{\mathscr{N}}$ for $i \in J$, where J is an indexed set such that $\mathcal{FA}Int_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q$.

Definition 3.6. A fuzzy subsystem $\lambda \in I^Q$ in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a called fuzzy automata $\mathfrak{F}^*$-first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, $(i \in J)$ where J is an indexed set and all $\lambda_i \in I^Q$ are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Any other fuzzy subsystem which is not fuzzy automata $\mathfrak{F}^*$-first category is said to be a fuzzy automata $\mathfrak{F}^*$-second category.

Definition 3.7. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space and let $\lambda \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-first category subsystem. Then $1_Q - \lambda$ is called a fuzzy automata $\mathfrak{F}^*$-residual subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Definition 3.8. A fuzzy subsystem $\lambda \in I^Q$ in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is called fuzzy automata $\mathfrak{F}^*$-dense if there exists no fuzzy automata $\mathscr{N}\mathfrak{F}^*$-closed subsystem $\mu \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\lambda < \mu < 1_Q$. That is, $\mathcal{FA}Cl_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 1_Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Definition 3.9. A fuzzy subsystem $\lambda \in I^Q$ in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is called fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$ for $i \in J$, where J is an indexed set.

Definition 3.10. A fuzzy subsystem $\lambda \in I^Q$ in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is called fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ if

$$\lambda = \vee_{i=1}^{\infty}(\lambda_i)$$

where $(1_Q - \lambda_i) \in \mathfrak{I}_{\mathscr{N}}$ for $i \in J$.

Proposition 3.1. If $\lambda \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-dense subsystem and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$, then $(1_Q - \lambda) \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. Let $\lambda \in I^Q$ be a fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$. Since λ is a fuzzy automata $\mathfrak{F}^*$-dense subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$,

$$\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 1_Q.$$
$$\text{Then } \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) = 1_Q.$$

But

$$1_Q = \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) \leq \wedge_{i=1}^{\infty}\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i).$$

Hence $1_Q \leq \wedge_{i=1}^{\infty}\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)$. Since $\wedge_{i=1}^{\infty}\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) \not< 1_Q$, the only possibility is $\wedge_{i=1}^{\infty}\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) = 1_Q$. Then $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) = 1_Q$ for each $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$. Since each $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$, $\lambda_i = \mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)$ and hence $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)) = 1_Q$ which implies that

$$1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)) = 0_Q, \tag{1}$$
$$\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda_i)) = 0_Q. \tag{2}$$

Since $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$, $(1_Q - \lambda_i)$ is a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-closed subsystem. Thus $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda_i) = 1_Q - \lambda_i$. Therefore from Equation (3.2), $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda_i) = 0_Q$.

Now $1_Q - \lambda = 1_Q - \wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1_Q - \lambda_i)$. Therefore $1_Q - \lambda = \vee_{i=1}^{\infty}(1_Q - \lambda_i)$ where $(1_Q - \lambda_i)$'s are fuzzy $\mathfrak{N}\mathfrak{F}^*$-dense subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Hence $1_Q - \lambda$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem. ☐

Proposition 3.2. In a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$, a fuzzy subsystem $\lambda \in I^Q$ is a fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystem if and only if $(1_Q - \lambda) \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. Let λ be a fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $1_Q - \lambda_i \in \mathfrak{I}_{\mathscr{N}}$, for $i \in J$ where J is an indexed set and $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q$. Then $1_Q - \mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 1_Q - 0_Q = 1_Q$ implies that $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda) = 1_Q$. Also $1_Q - \lambda = 1_Q - \vee_{i=1}^{\infty}(\lambda_i) = \wedge_{i=1}^{\infty}(1_Q - \lambda_i)$ where $1_Q - \lambda_i \in \mathfrak{I}_{\mathscr{N}}$, for $i \in J$. Hence $(1_Q - \lambda)$ is a fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Conversely, let $\lambda \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-dense and fuzzy $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in \mathfrak{I}_{\mathscr{N}}$, for $i \in J$. Now $1_Q - \lambda = 1_Q - \wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1_Q - \lambda_i)$. Hence $1_Q - \lambda$ is a fuzzy $\mathscr{F}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ and $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda) = 1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 1_Q - 1_Q = 0_Q$, since λ is a fuzzy automata $\mathfrak{F}^*$-dense. Therefore $1_Q - \lambda$ is a fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. ☐

Definition 3.11. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Then $(Q, \mathfrak{I}_{\mathscr{N}})$ is called fuzzy automata ς-Baire space if $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) = 0_Q$ where all $\lambda_i \in I^Q$ are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proposition 3.3. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Then the following statements are equivalent :

(i) $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space.
(ii) $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q$ for every fuzzy automata $\mathfrak{F}^*$-first category subsystem $\lambda \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$.
(iii) $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mu) = 1_Q$ for every fuzzy automata $\mathfrak{F}^*$-residual subsystem $\mu \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. **(i)⇒(ii)**

Let $\lambda \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then

$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)).$$

Since $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space, $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) = 0_Q$. Hence $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q$ for any fuzzy automata $\mathfrak{F}^*$-first category subsystem $\lambda \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$.

(ii)⇒(iii)

Let $\mu \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-residual subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $(1_Q - \mu)$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. By hypothesis,

$$\begin{aligned} \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(1_Q - \mu) &= 0_Q \\ 1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mu) &= 0_Q \\ \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mu) &= 1_Q. \end{aligned}$$

For any fuzzy automata $\mathfrak{F}^*$-residual subsystem $\mu \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$, $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\mu) = 1_Q$.

(iii)⇒(i)

Let $\lambda \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Since λ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem, $(1_Q - \lambda)$ is a fuzzy automata $\mathfrak{F}^*$-residual subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. By hypothesis,

$$\begin{aligned} \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(1_Q - \lambda) &= 1_Q \\ 1_Q - \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) &= 1_Q \\ \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) &= 0_Q \\ \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) &= 0_Q \end{aligned}$$

where λ_i's are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Hence $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space. □

Proposition 3.4. If a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space, then $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\bigwedge_{i=1}^{\infty}(\lambda_i)) = 1_Q$, where the fuzzy subsystems (λ_i)'s with $(i = 1$ to $\infty)$ are fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. Let (λ_i)'s where $(i = 1$ to $\infty)$ be fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. By Proposition 3.2, $(1_Q - \lambda_i)$'s are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Thus for $\lambda \in I^Q$, $\vee_{i=1}^{\infty}(1_Q - \lambda_i) = \lambda$. Hence λ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Now

$$\begin{aligned} \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) &= \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(1_Q - \lambda_i)) \\ &= \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(1_Q - (\wedge_{i=1}^{\infty}(\lambda_i))) \\ &= 1_Q - \mathcal{FAC}l_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) \end{aligned}$$

Since $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space, by Proposition 3.3,

$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q.$$

$$\text{Then } 1_Q - \mathcal{FAC}l_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) = 0_Q$$

$$\text{which implies } \mathcal{FAC}l_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) = 1_Q.$$

□

Proposition 3.5. If a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space, then $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(1_Q - \lambda_i)) = 0_Q$, where the fuzzy subsystems $(1_Q - \lambda_i)$'s with $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{F}^*$-first category subsystems formed from the fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems $\lambda_i \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. Let the fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata ς-Baire space and all the fuzzy subsystems $\lambda_i \in I^Q$ $(i = 1 \text{ to } \infty)$ be fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. By Proposition 3.4,

$$\begin{aligned} \mathcal{FAC}l_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) &= 1_Q \\ 1_Q - \mathcal{FAC}l_{\mathscr{N}\mathfrak{F}^*}(\wedge_{i=1}^{\infty}(\lambda_i)) &= 0_Q \\ \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(1_Q - \lambda_i)) &= 0_Q. \end{aligned}$$

Since all $\lambda_i \in I^Q$ are fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems, by Proposition 3.1, $(1_Q - \lambda_i)$'s $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{F}^*$-first category subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Hence $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\bigvee_{i=1}^{\infty}(1_Q - \lambda_i)) = 0_Q$, where the fuzzy subsystems $(1_Q - \lambda_i)$'s $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{F}^*$-first category subsystems formed from the fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems λ_i in $(Q, \mathfrak{I}_{\mathscr{N}})$. □

Definition 3.12. A fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is called fuzzy automata $\mathfrak{F}^*$-first category if the fuzzy subsystem 1_Q is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ (i.e.) $1_Q = \vee_{i=1}^{\infty}(\lambda_i)$, where all $\lambda_i \in I^Q$ are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Otherwise, $(Q, \mathfrak{I}_{\mathscr{N}})$ will be called as fuzzy automata $\mathfrak{F}^*$-second category space.

Proposition 3.6. If the fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata $\mathfrak{F}^*$-first category space, then $(Q, \mathfrak{I}_{\mathscr{N}})$ is not a fuzzy automata ς-Baire space.

Proof. Let fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata $\mathfrak{F}^*$-first category space. Then $1_Q = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Now

$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) = \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(1_Q) = 1_Q \neq 0_Q.$$

Therefore by Definition 3.11, $(Q, \mathfrak{I}_{\mathscr{N}})$ is not a fuzzy automata ς-Baire space. □

Proposition 3.7. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. If $\wedge_{i=1}^{\infty}(\lambda_i) \neq 0_Q$, where all the fuzzy subsystems $\lambda_i \in I^Q$ $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$, then $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata $\mathfrak{F}^*$-second category space.

Proof. Let (λ_i)'s where $(i = 1 \text{ to } \infty)$ be fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. By Proposition 3.2, $(1_Q - \lambda_i)$ where $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Thus

$$\wedge_{i=1}^{\infty}(\lambda_i) \neq 0_Q$$
$$1_Q - \wedge_{i=1}^{\infty}(\lambda_i) \neq 1_Q$$
$$\vee_{i=1}^{\infty}(1_Q - \lambda_i) \neq 1_Q.$$

Hence $(Q, \mathfrak{I}_{\mathscr{N}})$ is not a fuzzy automata $\mathfrak{F}^*$-first category space and therefore $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata $\mathfrak{F}^*$-second category space. □

Proposition 3.8. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. If $\lambda \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$, then there is a fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem $\delta \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\lambda \leq \delta$.

Proof. Let $\lambda \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-first category subsystem. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where all $\lambda_i \in I^Q$ with $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Now $(1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i))$ with $(i = 1 \text{ to } \infty)$ are fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\mu = \wedge_{i=1}^{\infty}(1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i))$ is a fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem. Thus

$$1_Q - \mu = 1_Q - (\wedge_{i=1}^{\infty}(1_Q - \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i))) = \vee_{i=1}^{\infty}(\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)).$$

Hence

$$\lambda = \vee_{i=1}^{\infty}(\lambda_i) \leq \vee_{i=1}^{\infty}(\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i)) = 1_Q - \mu.$$

That is, $\lambda \leq 1_Q - \mu$ and $1_Q - \mu$ is fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem. Let $\delta = 1_Q - \mu$. Hence, if λ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$, then there is a fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem $\delta \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\lambda \leq \delta$. □

Proposition 3.9. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. If $\delta \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-residual subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\eta \leq \delta$, where $\eta \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-dense and fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$, then $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space.

Proof. Let $\delta \in I^Q$ be a fuzzy automata $\mathfrak{F}^*$-residual subsystem in a fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $(1_Q - \delta) \in I^Q$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$. Now by Proposition 3.8, there is a fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem $\mu \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $(1_Q - \delta) \leq \mu$ which implies that $(1_Q - \mu) \leq \delta$. Let $\eta = 1_Q - \mu$. Then η is a fuzzy automata $\mathscr{G}\mathfrak{F}^*$-subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$ and $\eta \leq \delta$ implies that $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\eta) \leq \mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\delta)$. If $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\eta) = 1_Q$, then $\mathcal{FACl}_{\mathscr{N}\mathfrak{F}^*}(\delta) = 1_Q$. Then by Proposition 3.3, $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space. □

Proposition 3.10. If the fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space and if for $\lambda_i \in I^Q$, $\vee_{i=1}^{\infty}(\lambda_i) = 1_Q$, then there exists atleast one fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem $\lambda_i \in I^Q$ such that $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) \neq 0_Q$.

Proof. Suppose that $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) = 0_Q$, for $(i = 1 \text{ to } \infty)$, where all $\lambda_i \in I^Q$ with (i = 1 to ∞) are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \mathfrak{I}_{\mathscr{N}})$. Then $\vee_{i=1}^{\infty}(\lambda_i) = 1_Q$ which implies

$$\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) = \mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(1_Q) = 1_Q \neq 0_Q,$$

a contradiction to $(Q, \mathfrak{I}_{\mathscr{N}})$ being a fuzzy automata ς-Baire space. Hence $\mathcal{FAInt}_{\mathscr{N}\mathfrak{F}^*}(\lambda_i) \neq 0_Q$, for atleast one fuzzy automata $\mathscr{F}\mathfrak{F}^*$-subsystem $\lambda_i \in I^Q$ in $(Q, \mathfrak{I}_{\mathscr{N}})$. □

Proposition 3.11. If the fuzzy automata normed linear $\mathfrak{F}^*$-structure space $(Q, \mathfrak{I}_{\mathscr{N}})$ is a fuzzy automata ς-Baire space, then no fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystem $\lambda \in I^Q$ with $\lambda \neq 0_Q$, is a fuzzy automata $\mathfrak{F}^*$-first category subsystem in $(Q, \mathfrak{I}_{\mathscr{N}})$.

Proof. Let $\lambda \in I^Q$ with $\lambda \neq 0_Q$ be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystem in a fuzzy automata ς-Baire space $(Q, \Im_{\mathscr{N}})$. Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy automata $\mathfrak{N}\mathfrak{F}^*$-dense subsystems in $(Q, \Im_{\mathscr{N}})$. Then

$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = \mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)).$$

Since $(Q, \Im_{\mathscr{N}})$ is a fuzzy automata ς-Baire space,

$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\vee_{i=1}^{\infty}(\lambda_i)) = 0_Q$$
$$\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = 0_Q$$
$$\lambda = 0_Q$$

which is a contradiction, since $\lambda \in \Im_{\mathscr{N}}$ implies that $\mathcal{FAI}nt_{\mathscr{N}\mathfrak{F}^*}(\lambda) = \lambda \neq 0_Q$. Hence no fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open subsystem $\lambda \neq 0_Q$ is a fuzzy automata $\mathfrak{F}^*$-first category subsystem $(Q, \Im_{\mathscr{N}})$. □

4. Application of Fuzzy Automata Normed Linear $\mathfrak{F}^*$-Structure Spaces in Fuzzy T_i $(i = 0, 1)$ Spaces

In this section, the concepts of fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover, fuzzy automata $\mathscr{N}\mathfrak{F}^*$-star, fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_0$ spaces and fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_1$ spaces are introduced and some of their properties are studied.

Definition 4.1. Let $(Q, \Im_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space and $\lambda \in I^Q$. Let $\mathscr{U} = \{ \mu \in I^Q : \lambda \ \mathfrak{q} \ \mu\}$. Then fuzzy automata $\mathscr{N}\mathfrak{F}^*$-star of λ with respect to $\mathscr{U}$ is denoted by $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda, \mathscr{U})$ and it is defined as

$$FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda, \mathscr{U}) = \bigwedge\{ \mu \in \mathscr{U} : \lambda \ \mathfrak{q} \ \mu \}.$$

Example 4.1. In Example 3.1, $(Q, \Im_{\mathscr{N}})$ is a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Let $\lambda \in I^Q$ be $\lambda(0) = 0.5$. Then $\mathscr{U} = \{ \lambda < \mu \leq 1 \}$. Thus $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda, \mathscr{U}) = 0.51$.

Definition 4.2. Let $(Q, \Im_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. A family $\{ \lambda_i \in I^Q : i \in J$, where J is an indexed set $\}$ of fuzzy automata $\mathscr{N}\mathfrak{F}^*$-structure open subsystems in $(Q, \Im_{\mathscr{N}})$ is called a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover of $(Q, \Im_{\mathscr{N}})$ if $\bigvee_{i \in J} \lambda_i = 1_Q$.

Note 4.1. Let $(Q, \Im_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space and let $\lambda \in I^Q$. Suppose $\mathscr{U} = \{ \mu \in I^Q : \lambda \ \mathfrak{q} \ \mu$ and $\mu \in \Im_{\mathscr{N}} \}$ is a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover of $(Q, \Im_{\mathscr{N}})$, then $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda, \mathscr{U}) \in \Im_{\mathscr{N}}$.

Example 4.2. In Example 3.1, $(Q, \Im_{\mathscr{N}})$ is a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Let $\lambda \in I^Q$ be $\lambda(0) = 0.5$. Then $\mathscr{U} = \{ 1_Q, \mu_1, \mu_3 \}$. Thus $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda, \mathscr{U}) = \mu_1 \in \Im_{\mathscr{N}}$.

Definition 4.3. Let $(Q, \Im_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Then $(Q, \Im_{\mathscr{N}})$ is said to be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_0$ space (denoted by $FA\mathscr{N}\mathfrak{F}^*$-$T_0$ space) if for any two fuzzy subsystems λ_1, $\lambda_2 \in I^Q$ with $\lambda_1 \not\mathfrak{q} \lambda_2$, there exists a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover $\mathscr{U}$ of $(Q, \Im_{\mathscr{N}})$ such that $\lambda_1 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U})$, $\lambda_2 \not\mathfrak{q} FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U})$ or $\lambda_2 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U})$, $\lambda_1 \not\mathfrak{q} FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U})$.

Definition 4.4. Let $(Q, \Im_{\mathscr{N}})$ be a fuzzy automata normed linear $\mathfrak{F}^*$-structure space. Then $(Q, \Im_{\mathscr{N}})$ is said to be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_1$-space (denoted by $FA\mathscr{N}\mathfrak{F}^*$-$T_1$ space) if for any two fuzzy subsystems λ_1, $\lambda_2 \in I^Q$ with $\lambda_1 \not\mathfrak{q} \lambda_2$, there exist fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open covers $\mathscr{U}_1$ and $\mathscr{U}_2$ of $(Q, \Im_{\mathscr{N}})$ such that $\lambda_1 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2)$, $\lambda_2 \not\mathfrak{q} FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2)$ and $\lambda_2 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1)$, $\lambda_1 \not\mathfrak{q} FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1)$.

Proposition 4.1. Every fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_0$-space is a fuzzy T_0-space.

Proof. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_0$-space. Then for any two fuzzy subsystems $\lambda_1,\ \lambda_2 \in I^Q$ with $\lambda_1 \not\leq \lambda_2$, there exists a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover $\mathscr{U}$ of $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\lambda_1 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U})$, $\lambda_2 \not\leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U})$ or $\lambda_2 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U})$, $\lambda_1 \not\leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U})$. Since $\mathscr{U}$ is fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open cover of $(Q, \mathfrak{I}_{\mathscr{N}})$ and by Note 4.1,

$$FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}),\ FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}) \in \mathfrak{I}_{\mathscr{N}}.$$

Thus for $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}) = \mu_1$ (say), $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}) = \mu_2$ (say),

$$\lambda_1 \leq \mu_1, \lambda_2 \not\leq \mu_1 \text{ or } \lambda_2 \leq \mu_2,\ \lambda_1 \not\leq \mu_2.$$

Therefore (X, τ) is a fuzzy T_0-space. □

Proposition 4.2. Every fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_1$-space is a fuzzy T_1-space.

Proof. Let $(Q, \mathfrak{I}_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}\mathfrak{F}^*$-$T_1$-space. Then for any two fuzzy subsystems $\lambda_1,\ \lambda_2 \in I^Q$ with $\lambda_1 \not\leq \lambda_2$, there exist fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open covers $\mathscr{U}_1, \mathscr{U}_2$ of $(Q, \mathfrak{I}_{\mathscr{N}})$ such that $\lambda_1 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2)$, $\lambda_2 \not\leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2)$ and

$$\lambda_2 \leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1),$$

$\lambda_1 \not\leq FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1)$. Since $\mathscr{U}_1, \mathscr{U}_2$ are fuzzy automata $\mathscr{N}\mathfrak{F}^*$-open covers of $(Q, \mathfrak{I}_{\mathscr{N}})$ and by Note 4.1,

$$FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2) \in \mathfrak{I}_{\mathscr{N}} \text{ and } FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1) \in \mathfrak{I}_{\mathscr{N}}.$$

Thus for $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_1, \mathscr{U}_2) = \mu_1$ (say) and $FA\mathscr{N}\mathfrak{F}^*\mathfrak{St}(\lambda_2, \mathscr{U}_1) = \mu_2$ (say),

$$\lambda_1 \leq \mu_1, \lambda_2 \not\leq \mu_1 \text{ and } \lambda_2 \leq \mu_2,\ \lambda_1 \not\leq \mu_2.$$

Therefore (X, τ) is a fuzzy T_1-space. □

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6. Conclusion

The concept of fuzzy automata normed linear structure spaces can be extended by introducing fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces within the framework of fuzzy automata $\mathfrak{F}^*$-first category spaces. This research enhances, equivalent statement related to fuzzy automata Σ-Baire spaces. It has been demonstrated that fuzzy automata $\mathfrak{F}^*$-first category spaces do not constitute fuzzy automata Σ-Baire spaces. Furthermore, the application of fuzzy automata normed linear $\mathfrak{F}^*$-structure spaces in fuzzy T_i spaces (for i = 0, 1) has been presented. This work can be further extended to the study of fuzzy vector topologies using fuzzy norms, offering new directions for future research. Also, the research on fuzzy T_i spaces (for i = 0, 1) could be expanded to explore higher-index fuzzy spaces or their potential generalizations. It would also be beneficial to study how these spaces interact with various forms of fuzzy logic and fuzzy systems.

References

[1] Kumar, B., Singh, A. K., Ram, A. K. (2023), Properties of Bipolar fuzzy automata, New Math. and Nat. Comput., 20(03), 583-600.

[2] Cao, J., Greenwood, S., (2006), The ideal generated by σ-nowhere dense sets, Appl. Gen. Topol., 7(2), 253 - 264.

[3] Chang, C. L., Fuzzy topological spaces, (1968), Journal of Mathematical Analysis and Applications, 24, 182 - 190.

[4] Cheng, S. C., Mordeson, J. N., (1994), Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc., 86, 429 - 436.

[5] Das, P., (1992) A fuzzy topology associated with a fuzzy finite state machine, Fuzzy Sets and Systems, 105, 469 - 479.

[6] Felbin, C., (1992), Finite dimensional fuzzy normed linear space, Fuzzy Sets and Systems, 48, 239 – 248.

[7] Gruenhage, G., Lutzer, D., (2000),Baire and Volterra Spaces, Proc. Amer. Soc., 128, 3115 - 3124.

[8] Ignjatovic, J., Ciric, M., Bogdanovic, S., (2008) Determinization of fuzzy automata with member-ship values in complete residuated lattices, Information Sciences, 178, 64 - 180.

[9] Kaleva, O., Seikkala, S., (1984), On fuzzy metric spaces, Fuzzy Sets and Systems, 12, 215 - 229.

[10] Katsaras, A. K., (1984), Fuzzy topological vector spaces II, Fuzzy Sets and Systems, 12, 143 - 154.

[11] Kramosil, A. K., Michalek, J., (1975), Fuzzy metric and statistical metric spaces, Kybernetica, 11 , 326 - 334.

[12] Li, Z. H., Li, P., Li, Y. M., (2006), The relationships among several types of fuzzy automata, Information Sciences, 176, 2208 - 2226.

[13] Madhuri, V., Amudhambigai, B., (2019), A new view on fuzzy automata normed linear structure spaces, Iranian Journal of Fuzzy Systems, 16(6), 65 - 74.

[14] Madhuri, V., Amudhambigai, B., (2021), A new view on fuzzy F*-structure homotopy and its F*-fundamental group, Jordan Journal of Mathematics and Statistics, 14(1), 73 - 95.

[15] Neubrunn, T., (1977), A Note on Mappings of Baire spaces, Math. Slovaca, 27(2), 173 - 176.

[16] Srivastava, A. K., Tiwari, S. P., (2002), A topology for fuzzy automata, Proc. AFSS International Conference on Fuzzy Systems, Lecture Notes in Artificial Intelligence, Springer-Verlag, 2275, 485 - 491.

[17] Srivastava, A. K., Tiwari, S. P., (2003), On relationships among fuzzy approximation operators, fuzzy topology and fuzzy automata, Fuzzy Sets and Systems, 138, 197 - 204.

[18] Thangaraj, G., Anjalmose, S., (2013), On fuzzy Baire spaces, J. Fuzzy Math., 21(3), 667 – 676.

[19] Tiwari, S. P., Singh, A. K., Sharan, S., Yadav, V. K., , (2016), Bifuzzy core of fuzzy automata, Iranian Journal of Fuzzy Systems, 12(2), 63-73.

[20] Tiwari, S. P., Yadav, V. K., Singh, A. K., (2015), On Algebraic Study of fuzzy automata, International Journal of Machine Learning and Cybernetics, 6(3), 479-485.

[21] Tiwari, S. P., Singh, A. K., (2015), IF-preorder, IF-topology and IF-automata, Int. J. Mach. Learn. and Cyber, 6(2), 205 - 211.

[22] Tiwari, S. P., Singh, A. K., (2013), On bijective correspondence between IF-preorder and saturated IF-topologies, Int. J. Mach. Learn. and Cyber, 4(6), 733 - 737.

[23] Kotze, W., (1986), Quasi-coincidence and Quasi-fuzzy Hausdorff, Journal of Mathematical Analysis and Applications, 116, 465 - 472.

[24] Wee, W. G., (1967), On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph.D. Thesis, Purdue University.

[25] Zadeh, L. A., (1965), Fuzzy Sets, Information and Control, 8, 338 - 353.

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