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COMPLEX RAYS AND APPLICATIONS

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Abstract. Complex rays are a fascinating aspect of modern diffraction theory, typically sought as complex solutions to the eikonal equation. Traditionally, these solutions are obtained by analytically continuing real rays into the complex domain. However, this approach demands the analyticity of initial data, significantly limiting its applicability to many practical problems. Additionally, unlike real rays, complex rays cannot be visualized in space, presenting another drawback. In this paper, we present an alternative interpretation of complex rays, as introduced in [1], and describe a novel approach to two model diffraction problems and Gaussian beams.

Keywords: wave equation, eikonal equation, complex rays, Minkowski space, Gaussian beams.
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1. Introduction

Following Schrödinger's groundbreaking formulation of his seminal equation, numerous efforts were made to precisely solve this equation. However, exact solutions remained elusive, achievable only for the most elementary scenarios. With powerful numerical methods and computational capabilities yet to emerge, researchers sought approximate solutions that could closely mimic exact outcomes. In 1926, Wentzel, Kramers, and Brillouin independently devised an asymptotic approach known as the Wentzel-Kramers-Brillouin (WKB) method, offering a means to approximate solutions. This method proved invaluable not only for tackling the Schrödinger equation but also for analyzing many other complex wave behaviors at high frequencies. Over time, it has become an indispensable tool across various realms of physics. The primary advantage of the WKB method lies in its ability to simplify the study of wave phenomena, characterized by small parameter. For this the field (electromagnetic, aucustic, seismic ets.) under question is assumed to be represented in the asymptotic form

$$u\left(x,y,z;k\right)\sim e^{ikS}\sum_{n=0}^{\infty}\frac{A_n}{\left(ik\right)^n}, \tag{1}$$

called ray anstaz, where $k=\omega/c$ is a wavenumber, $S=S(x,y,z)$ is a phase function and $A_n=A_n\left(x,y,x\right)$ are amplitudes no more depending on k. It is well known that in nonhomogenous and isotropic medium each component of electrical and magnetic vectors in the classical Maxwell equations satifies the scalar Helmholtz equation (time factor $e^{-i\omega t}$ suppressed)

$$\Delta u+k^2n^2u=0, \tag{2}$$

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where Δ is the Laplace operator, $n = n(x, y, z)$ is the refractive index of media. To find the ray solutions or high frequency asymptotic solutions of (2) subject to a rapidly oscillating boundary condition

$$u(x, y, 0) = U_0(x, y)\, e^{ik\varphi(x,y)}, \tag{3}$$

where $\varphi(x, y) = S(x, y, 0)$ is the initial phase, we substitute (1) into (2) and after equating the corresponding degrees of k we get an equation

$$(\nabla S)^2 = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2 \tag{4}$$

for determining $S = S(x, y, z)$, called eikonal equation, and a sequence of equations

$$\begin{gathered} 2\nabla A_0 \cdot \nabla S + A_0\Delta^2 S = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 2\nabla A_n \cdot \nabla S + A_n\Delta^2 S = A_{n-1} \end{gathered} \tag{5}$$

for determining amplitudes $A_n(x, y, z)$, called transport equations. The zero approximation in (1),

$$u(x, y, z) = A_0(x, y, z)\, e^{ikS(x,y,z)}, \tag{6}$$

is essentially the primary equation of the geometrical optics approximation. Since the eikonal equation (4) does not involve the amplitudes, it can be solved separately and independently of the transport equations. In fact, (4) is no other than Hamilton-Jacobi equation for the variational principle (Fermat's Principle)

$$\min \int_L n(x, y, z)\, ds \tag{7}$$

of geometrical optics and is a nonlinear first-order partial differential equation. It is well known that the extremals (light rays) of equation (7) satisfy the Euler-Lagrange system of ordinary differential equations; see, for example, [2], [3] and other standard textbooks on classical geometrical optics. Notably, in contrast to the general theory of Hamilton–Jacobi equations, equation (4) may admit complex-valued solutions, known as complex rays. These solutions describe the field in regions where the real solutions of (4) do not exist, such as shadow regions. In what follows, we aim to describe the field in these domains. For other applications of complex rays, see [7-11] and the extended bibliography therein.

2. Mathematical Background

The fundamental property of equations of the type (4) is that finding their solutions can be reduced to solving a system of ordinary differential equations, called characteristic system. Here, we focus on solutions of (4) in the z-direction. Studying wave propagation along a specified direction is more realistic than analyzing radiation from an ideal isotropic source, as truly isotropic emitters do not exist in nature. This physical fact is typically reflected in the mathematical formulation by imposing appropriate initial conditions on the governing equations. In the considerations that follow, in our case of geometrical optics this becomes crucial, since the resulting geometry differs substantially from the usual Euclidean space; instead, it gives rise to a three-dimensional pseudo-Riemannian geometry, analogous to the well-known four-dimensional pseudo-Riemannian structure employed in the general theory of relativity. Remember that the three-dimensional affine coordinate space $\mathbb{R}^3$ becomes a metric space with indefinite metric if we intruduce a scalar product of two vektors $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$ in $\mathbb{R}^3$ as

$$\left\langle \vec{\xi}, \vec{\eta} \right\rangle = \xi_3\eta_3 - n^2 (\xi_1\eta_1 + \xi_2\eta_2), \tag{8}$$

where $n = n(x, y, z) > 0$ is a smooth function defined on $\mathbb{R}^3$. Then the length of a vector with respect to the scalar product (8) is defined to be

$$\left|\vec{\xi}\right| = \sqrt{\xi_3^2 - n^2 \left(\xi_1^2 + \xi_2^2\right)}. \tag{9}$$

A manifold whose the tangent space TM_x at each point x is furnished by the scalar product (8), we shall indicate as $M_{1,2}^3$. Now solving (4) with respect to $\frac{\partial S}{\partial z}$, we get

$$r - \sqrt{n^2 - p^2 - q^2} = 0, \qquad r = \frac{\partial S}{\partial z},\ p = \frac{\partial S}{\partial x},\ q = \frac{\partial S}{\partial y}. \tag{10}$$

Then the characteristic system for (4) reads as

$$\frac{dx}{dz} = \frac{p}{\sqrt{n^2 - p^2 - q^2}}, \qquad \frac{dy}{dz} = \frac{q}{\sqrt{n^2 - p^2 - q^2}}, \tag{11a}$$

$$\frac{dr}{dz} = \frac{nn_z}{\sqrt{n^2 - p^2 - q^2}}, \qquad \frac{dS}{dz} = \frac{n^2(x, y, z)}{\sqrt{n^2 - p^2 - q^2}}, \tag{11b}$$

$$\frac{dp}{dz} = \frac{nn_x}{\sqrt{n^2 - p^2 - q^2}}, \qquad \frac{dq}{dz} = -\frac{nn_y}{\sqrt{n^2 - p^2 - q^2}} \tag{11c}$$

and should be solved subject to the initial conditions

$$\left(x_0(\xi, \eta), y_0(\xi, \eta), p_0(\xi, \eta), q_0(\xi, \eta),\ r_0(\xi, \eta), S_0(\xi, \eta)\right)$$

at $z = 0$, where z is the parameter along the rays as it is supposed to be and, (ξ, η) are coordinates on the initial surface $S(\xi, \eta, 0) = \varphi(\xi, \eta)$. The solutions of the system (11a-c) under these conditions provide solutions (at least, locally) of (4) in parametric form :

$$x = x(\xi, \eta; z), y = y(\xi, \eta; z), z = z, p = p(\xi, \eta; z), q = q(\xi, \eta; z), r = (\xi, \eta; z), S = S(\xi, \eta; z) \tag{12}$$

called Lagrange manifold of ray configurations in space (to be exact, the first six equatios). Now, to find the solution of (4) in explicit form $S = S(x, y; z)$, one should solve the first two equations of (12) with respect to x, y and substitute into the last equation of (12) to get an explicit solution $S = S(x, y, z)$, or in the terms of classical Hamilton mechanics (or symplectic geometry), project Lagrange manifold onto the configuration space of the ray configuration. If the Jacobian of (12)

$$J(\xi, \eta; z) = \frac{\partial(x, y; z)}{\partial(\xi, \eta; z)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{13}$$

doesn't vanish at some point (x_0, y_0) (therefore, at some vicinity of (x_0, y_0)) then (13) is reversiable and ξ and η can be expressed via the variables x and y in the form $\xi = (x, y; z), \eta = (x, y; z)$. Incorporating this into the last equation of (12) we get the solution in explicit form $S = S(x, y; z)$. A special case occurs at points for where $J(\xi, \eta; z) = 0$. These points form caustics of wave fronts. On the plane $z = z_0$ they represent intersection curves of caustics with the $z = z_0$ plane.

After solving equation (4) for S, one substitutes this solution into system (5) to determine the amplitudes A_n of the irradiation in a successive manner. A fundamental property used in these considerations is that the rays remain orthogonal to the initial surface with respect to the scalar product (8), and that the wavefronts constitute equidistant surfaces from the initial surface in the metric specified by (9). This geometric structure is lost if the rays are interpreted in the standard Euclidean space. From (11a-11c) we observe that if $p^2 + q^2 > n^2$ then the solutions become complex. Usually, for finding complex rays, the analytical continuation with respect to coordinates is used which requires analiticity of all functions involved. For example, in the extendent survey [10] on complex rays (about 100 pages), the definition of the complex rays in two dimensional case is given as follows:

A complex ray is the set of complex points (x, y) *corresponding to fixing a complex value of* s *in the ray equation of 11a)-c) with boundary values* $(x_0(\xi,\eta), y_0(\xi,\eta), p_0(\xi,\eta), q_0(\xi,\eta), r_0(\xi,\eta), S_0(\xi,\eta))$ *necessarily analytic functions and then allowing the distance parameter* z *to range over all complex values with* $z = 0$ *coinciding with the initiation point on the boundary.*

Along with the hard conditions imposed on the boundary values, an analytical continuation is rather harmful and geometrically unclear operation in its own way. In fact, if instead of complexification of coordinates one admits complex distances, that is, if the equations (11a-c) are interpreted in non Euclidean spaces, this difficulty can be avoided.

Geometrical Interppretation Of Eikonal Equation

Since we are going to use tensor notations, for conviniency we set $x = x^1, y = x^2, z = x^3$. Let $\mathbb{R}^1_2$ be the Riemann manifold endowed by a pseudo-riemann metric tensor

$$ds^2 = g_{ij}dx^i dx^j = n^2\left(x^1, x^2, x^3\right)\left(dx^3\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2, \tag{14}$$

that is, $g_{11} = g_{22} = -1, g_{33} = n^2\left(x^1, x^2, x^3\right)$, and $g_{ij} = 0$ if $i \neq j$. The equation for geodesics with respect to the metric (9) reads as

$$\frac{d^2x^k}{ds^2} = -\Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}, \qquad (i, j, k = 1, 2, 3), \tag{15}$$

where Γ^k_{ij} $(i, k = 1, 2, 3)$ are connection coefficients with respect to the metric (9). They are defined via the metric tensor as

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(g_{jl,i} + g_{il,j} - g_{ij,l}\right), \qquad (i, j, k, l = 1, 2, 3), \tag{16}$$

where $g_{ij,m} = \frac{\partial g_{ij}}{\partial x^m}$ and g^{kl} are the contravariant coordinates of the metric tensor. Notice that in (14)-(16), the contraction rule is used: if the same symbol appears twice (one upper and one lower) within a term, it means that over this pair a summation occures. Since $g_{ij} = 0$ if $i \neq j$ and $g^{33} = \frac{1}{n^2}, g^{11} = g^{22} = -1, \Gamma^k_{ij} = \Gamma^k_{ji}$ $(i, j, k = 1, 2, 3)$, then, among the coefficients Γ^k_{ij} non zero are only those, which involve two 3 symboles. Straightforward calculations show that

$$\Gamma^1_{33} = nn_x, \quad \Gamma^2_{33} = nn_y, \quad \Gamma^3_{33} = \frac{n_z}{n}, \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{n_x}{n}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{n_y}{n}.$$

Returning back to the cartesian coordinates, (15) becomes as

$$\frac{d^2x}{ds^2} = -nn_x\left(\frac{dz}{ds}\right)^2, \quad \frac{d^2y}{ds^2} = -nn_y\left(\frac{dz}{ds}\right)^2, \quad \frac{d^2z}{ds^2} = -\frac{1}{n}\frac{dn}{ds}\frac{dz}{ds} \tag{17}$$

and $ds = \sqrt{n^2\left(dx^3\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2}$. In (17) we recognize the standart Euler-Lagrange equations for the varitional problem (7) which are equivalent to 11a)-11c) in Hamilton's formalizm of geometrical optics ([2-3]). Thus we arrive at the following theorem.

Theorem 1. *The rays given by the solutions of the system 11a)-c) are geodesics in* $\mathbb{M}^3_{1,2}$ *with respect to the metric (14).*

In $\mathbb{M}^3_{1,2}$ equidistance surfaces along the geodesics are perpendicular to the surfaces. Hence, solving the initial value problem (1)-(2) is equivalent to drawing normals in $\mathbb{M}^3_{1,2}$ to the initial value function $u(x, y, 0) = \varphi(x, y)$ at each point. Notice that in usual space, rays are not perpendicular to the initial surface unless the initial surface it self is not a wave front. At each point of space where the ray arrives, two rays are in fact present: one lies inside the light cone and is a real ray with real length, while the other lies outside the light cone and is a complex ray with complex length. Moreover, these rays are perpendicular to each other in the sense of the metric (9).

3. Diffraction in Linear Layer and Complex Rays

We illustrate this with a simple model problem, where the direction of propagation is the z-axis, and the oncoming wave is a plane wave with unit amplitude in a medium with a refractive index $n = 1$:

$$u\,(x, z) = x \sin\theta + z \cos\theta. \tag{18}$$

It is assumed that the wave undergoes refraction at the $z = 0$ plane and progresses toward positive z. Suppose that the refractive index of the region $z > 0$ is $n^2(z) = 1 - \frac{z}{z_0}$, where $z_0 > 0$. For simplicity, consider the two-dimensional i.e. cylindrical system:.

$$q - \sqrt{n^2 - p^2} = 0, \quad q = \frac{\partial S}{\partial z}, \quad p = \frac{\partial S}{\partial x}.$$

Then the characteristic system for 11a)-c) reads as:

$$\frac{dx}{d\tau} = p, \quad \frac{dz}{d\tau} = q, \quad \frac{dp}{d\tau} = 0, \quad \frac{dq}{d\tau} = -\frac{1}{2z_0}, \tag{19}$$

$$\frac{dS}{d\tau} = n^2 = 1 - \frac{z_0}{z}, \tag{20}$$

where τ is a parameter along the rays. Integrating under the initial conditions $(\xi, 0, \sin\theta, \xi \sin\theta)$, keeping in mind that ξ stands for x on the initial plane and serves as the initial point of the ray, we get:

$$p(\xi, \tau) = \sin\theta, \quad q(\xi, \tau) = \cos\theta - \frac{\tau}{2z_0} \tag{21}$$

and

$$\text{a) } x(\xi, \tau) = \xi + \tau \sin\theta, \quad \text{b) } z(\xi, \tau) = \tau \cos\theta - \frac{\tau^2}{4z_0}, \tag{22}$$

$$\text{c) } S(\xi, \tau) = \xi \sin\theta + \tau - \frac{\cos\theta}{2z_0}\tau^2 - \frac{1}{12z_0^2}\tau^3. \tag{23}$$

To find the ray trajectories in the (z, x) coordinates, eliminating τ from a) and b):

$$(x - \xi - z_0 \sin 2\theta)^2 = 4z_0 \sin^2\theta \left(z_0 \cos^2\theta - z\right). \tag{24}$$

Now setting $a = \xi + z_0 \sin 2\theta, b = -4z_0 \sin^2\theta, c = z_0^2 \sin^2 2\theta$, we get:

$$(x - a)^2 = -bz + c, \tag{25}$$

which represents a family of parabolas whose vertices are located at $z = z_0 \cos^2\theta$, and the axes are parallel to the z-axis. The equation $z = z_0 \cos^2\theta$ defines a plane perpendicular to the z-axis. Consequently, rays have turning points on this plane; after touching it, they return and intersect the initial plane where q is negative in (7). Thus, the plane $z = z_0 \cos^2\theta$ serves as a caustic for the rays. After finding S from (23), it can be incorporated into the system (5) to determine the amplitudes A_n of the irradiation successively. In fact, the first equation in (5) is no other than the energy conservation law in a small ray tube. For small $\varepsilon > 0$ and for $0 < z < z_0 \cos^2\theta - \varepsilon$, it reads as:

$$A(z, \theta) = A(0, \theta) \left| \frac{x'_z(z, \xi)}{x'_z(0, \xi)} \right|^{\frac{1}{2}} = A(0, \theta) \left| \frac{\sqrt{z_0} \cos\theta}{\sqrt{z_0 \cos^2\theta - z}} \right|. \tag{26}$$

If $z \to z_0 \cos^2\theta$, the ray tube vanishes, and the field's amplitude becomes infinite. Clearly, this violates the energy conservation law and is physically meaningless. Below, by slightly modifying the Maslov method, we will eliminate this singularity and calculate the amplitude on the caustic (see also [11]). Here, we are interested in the case $z_0 \cos^2\theta - z < 0$, where the rays are complex.. Let us rewrite the system (22)-(23) in the following form:

$$\begin{bmatrix} z(\xi,\tau) \\ x(\xi,\tau) \\ S(\xi,\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \xi \\ \xi\sin\theta \end{bmatrix} + \tau \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix} + \tau^2 \begin{bmatrix} -1/4z_0 \\ 0 \\ -\cos\theta/2z_0 \end{bmatrix} +$$

$$+\tau^3 \begin{bmatrix} 0 \\ -0 \\ -1/12z_0 \end{bmatrix}, \tag{27}$$

where τ is a parameter along the ray. To interpret (27), it is convient to introduce three dimensional pseudo Riemann space (z, x, S) endowed with indefinite scaler product

$$<\mathbf{u},\mathbf{v}>=S_1S_2 - n^2(x_1x_2 + z_1z_2),$$

and length of a vector endowed with $\|\mathbf{u}\|^2 = S^2 - n^2(x^2 + z^2)$. The angle between two vectors is defined to be

$$\cos^2\psi = \frac{<\mathbf{u},\mathbf{v}>}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \geq 1.$$

Therefore, in this geometry $\cos\psi < -1$ or $\cos\psi > 1$. With each point (z_0, x_0, S_0) in space we associate a cony $(S - S_0)^2 - n^2(z)\left[(x - x_0)^2 + (z - z_0)^2\right] = 0$, called light cony. At every point this cony divides the space into two regions where $(S - S_0)^2 - n^2(z)\left[(x - x_0)^2 + (z - z_0)^2\right] > 0$ and $(S - S_0)^2 - n^2(z)\left[(x - x_0)^2 + (z - z_0)^2\right] < 0$. The vectors, whose initial points are at (z_0, x_0, S_0) with positive length, that is they lay in the light cony, are called timelike and they are called spacelike if lay outside of the cone and have negative lengths. The vectors whose length are zero are called izotrop vectors. Thus we have defined a cony field in space. Remember that, in this spase the unit normal vector to a surface $\varphi(x, z)$ is defined to be

$$\mathbf{n} = \left(\frac{\varphi_x}{\sqrt{n^2 - \varphi_x^2}}, \frac{\varphi_y}{\sqrt{n^2 - \varphi_x^2}}, \frac{n^2}{\sqrt{n^2 - \varphi_x^2}}\right).$$

In our case $\varphi(x) = S(0, x) = x\sin\theta$, $\varphi_x = \sin\theta$, $\varphi_z = 0$, $n^2(0) = 1$. Since the tangent vector to $\varphi(x, z)$ at $z = 0$ is $\mathbf{t} = (0, 1, \sin\theta)$ and in the mentioned metric it is perpendicular to the vector $\mathbf{e} = (\cos\theta, \sin\theta, 1)$. Moreover, $\mathbf{e}$ is an izotropic vector: $\|\mathbf{e}\|^2 = 1 - \cos^2\theta - \sin^2\theta = 0$. (27) shows that in the mentioned space wave behavier depends on the second power of τ.Since we are interested propogation in the z direction, let us express τ via z variable. Solving (22b) for τ we obtain:

$$\tau = 2z_0\left(\cos\theta - \sqrt{z_0\cos^2\theta - z}\right),$$

from which we observe that for the values $z < z_0\cos^2\theta$ parameter τ is real which means that the ray is real

$$\tau^2 = \left(4z_0^2\cos^2\theta - z_0\cos\theta + z\right) - \left(8z_0^2\sqrt{z_0\cos^2\theta - z}\right)i$$

or for small z/z_0 :

$$\tau^2 = \left(4z_0^2\cos^2\theta - z_0\cos\theta + z\right) - \left(8z_0^{\frac{5}{2}}\cos\theta - 4z_0^{\frac{3}{2}}\cos^3\theta\right)zi.$$

Now ignoring cubic term in (27), for the phase function $S(z, x)$ we get

$$S(z, x) = (x - \sin\theta)\sin\theta + (a + bi)\frac{\cos\theta}{2z_0} = x\sin x - \sin^2\theta +$$

$$+\frac{a\cos\theta}{2z_0}+\frac{b\cos\theta}{2z_0}i=c+di,$$

where $a=4z_0^2\cos^2\theta-z_0\cos\theta+z,\ b=-8z_0^2\sqrt{z_0\cos^2\theta-z}, c=(x-\sin\theta)\sin\theta+a\cos\theta/2z_0$, $d=b\cos\theta/2z_0$. Substituting $S(x,z)$ into (6) and keeping in mind that we consider two dimensional case, for $u(x,z)$ we get

$$u(x,z)=A(0,x)\left|\frac{\sqrt{z_0}\cos\theta}{\sqrt{z_0\cos^2\theta-z}}\right|e^{iS(z,x)\}}=$$

$$=A(0,x)\left|\frac{\sqrt{z_0}\cos\theta}{\sqrt{z_0\cos^2\theta-z}}\right|e^{i(c+di)},$$

or

$$u(x,z)=A(0,x)\left|\frac{\sqrt{z_0}\cos\theta}{\sqrt{z_0\cos^2\theta-z}}\right|e^{-d}e^{ic}=A_1e^{-d}e^{ic}, \tag{28}$$

where

$$A_1(x,z)=A(0,x)\left|\frac{\sqrt{z_0}\cos\theta}{\sqrt{z_0\cos^2\theta-z}}\right|e^{ic}$$

Thus, the field exponentially decreases behind the caustics, since after touching the caustic d becomes is positive. From (28) we observe that at the caustic $z=z_0\cos^2\theta$ the amplitude steel remains unbounded. Eleminating the appearing singularity is one of the subtle problems of asymptotic approximation method and has a deep connection with the catastrophe theory, symplectic geometry and differential topology ([12-13]). The problem is that a geometric optical configuration mathematically can be described by means of six dimensional symplectic geometry stemming from Hamiltonian mechanics. A symplectic geomery is an even dimentional (in our case it is $\mathbb{R}_6$ or $\mathbb{R}_4$) smooth manifold furnished by nondegenerate and closed skew symmetric bilineer form on its cotangent bundle. In this geometry the length of every vector on every tangent space is zero (are isotrop) and an angle between vectors is a meaningless. Since position of a ray is uniquely determined by its initial point (x,y,z) and by direction (p,q,r) in space, then the point $(x,y,z;p,q,r)$ in six dimensional space $\mathbb{R}_6$ will completely describe the position of the ray (in our case the corresponding simplectic manifold is $\mathbb{R}_4$). A submanifold of a symplectic manifold is said to be an isotrop manifold if the scew symmetric form vanishes at it. The maximal dimension of a isotrop subspace don't exceed the half of the dimension of the manifold and an isotrop submanifold of maximal dimension is called Lagrange manifold. It turns out that solution of the eikonal equation (4),

$$x=x(\xi,\tau,\sigma),\ y=y(\xi,\tau,\sigma),z=z(\xi,\tau,\sigma),$$
$$p=p(\xi,\tau,\sigma),\ q=y(\xi,\tau,\sigma),p=p(\xi,\tau,\sigma) \tag{29}$$

is a three dimensional Lagrange manifold of $\mathbb{R}_6$, that is the symplectic form vanishes at this submanifold. To find the phase function $S(x,y,z)$ we should express p,q,z via x,y,z and substitute them into

$$S(x,y,z)=\varphi(x,y,z)+\int\limits_L pdx+qdy+rdz,$$

where L is a ray (a characteristic) joining an initial point with (x,y,z). The main difficulty in this process is that expressing p,q,z via x,y,z, in other words, projecting Lagrange manifold over (x,y,z) space might fail at some points or curves, that is, the jacobian

$$J=\frac{\partial(p,q,r)}{\partial(\xi,\tau,\sigma)}$$

vanishs at these points. In our case these points excactly are caustics:

$$z=\sqrt{z_0\cos^2\theta}$$

Now, let

$$z = z_0 \cos^2\theta - t^2.$$

Substituting into (28), we get

$$S(x,t) = -\frac{2}{3}\frac{1}{\sqrt{z_0}}t^3 + x\sin\theta + \frac{2}{3}z_0\cos^3\theta,\ x(\xi,t) =$$

$$-\left(2\sqrt{z_0}\sin\theta\right)t + z_0\sin 2\theta,\ z(t) = z_0\cos^2\theta - t^2,$$

or

$$A(\xi,t) = A(0)\left|\frac{x'_t(\xi,t)}{x'_t(0,t)}\right|^{\frac{1}{2}} = A(0)\left|\frac{\left(2\sqrt{z_0}\sin\theta\right)}{\left(2\sqrt{z_0}\sin\theta\right)}\right|^{\frac{1}{2}} = A(0).$$

therefore the singularity on the caustics disappears.

4. Gaussian Beams

Usually, the Gaussian beams expression is derived from the paraxial wave equation. Complex rays allow a convenient framework for describing Gaussian beams. Below, we present a simple geometric method for obtaining the Gaussian beam based on the considerations outlined above. Let us consider again two dimensional case and let $n(z,x) = 1$, $A_0 = 1/R$, $\varphi(x) = S(0,x) = -\sqrt{x^2 - R^2}$. Notise that in the (x,S) plane the curve $\varphi(x) = -\sqrt{x^2 - R^2}$ describes the left branch of the unit circle $S^2 - x^2 = -R^2$ of imaginary radius iR in two dimensional Minkowski plane. We have

$$\sqrt{1 - \varphi_x^2(x)} = \frac{iR}{\sqrt{x^2 - R^2}}$$

Then (9) becomes as

$$z = \tau\sqrt{1 - \varphi_x^2(x)} \quad \xi = x + \tau\varphi_x, \qquad S = \varphi(x) + \tau,$$

and we get

$$\begin{bmatrix} z(\xi,\tau) \\ x(\xi,\tau) \\ S(\xi,\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \xi \\ -\sqrt{\xi^2 - R^2} \end{bmatrix} + \tau \begin{bmatrix} \sqrt{1-\varphi_x^2} \\ \varphi_x \\ 1 \end{bmatrix}$$

or by projecting on the $x - S$ plane and eliminating τ we have

$$\begin{bmatrix} x \\ S \end{bmatrix} = \begin{bmatrix} \xi \\ -\sqrt{\xi^2 - R^2} \end{bmatrix} + z \begin{bmatrix} -\frac{\varphi_x}{\sqrt{1-\varphi_x^2}} \\ \frac{1}{\sqrt{1-\varphi_x^2}} \end{bmatrix} = \begin{bmatrix} \xi \\ -\sqrt{\xi^2 - R^2} \end{bmatrix} + z \begin{bmatrix} -\frac{\xi}{iR} \\ \frac{\sqrt{\xi^2-R^2}}{iR} \end{bmatrix}. \tag{30}$$

Since the vector

$$\overrightarrow{n} = \begin{bmatrix} -\frac{\xi}{iR} \\ \frac{\sqrt{\xi^2-R^2}}{iR} \end{bmatrix}$$

is the unit normal to initial phase front $\varphi(x)$, (30) can be written in the form:

$$\begin{bmatrix} x \\ S \end{bmatrix} = \begin{bmatrix} \xi \\ -\sqrt{\xi^2 - R^2} \end{bmatrix} + z\overrightarrow{n}. \tag{31}$$

(31) persists to asume that rays are perpendicular to $\varphi(x)$ in the mentioned geomety and z serves as the parameter along the rays (Fig.1). In contrast to describing Gaussian beams in usual Euclidean space rays are not perpendicular to wave fronts (Fig.2). Now excluding ξ from (30) to find the phace function $S(x,z)$, we get

$$S(x,z) = -\sqrt{x^2 + (R + iz)^2},$$

and incorporation into (6) we obtain

$$u(x,z) = \frac{1}{R - iz}e^{-ik\sqrt{x^2 + (R+iz)^2}}.$$

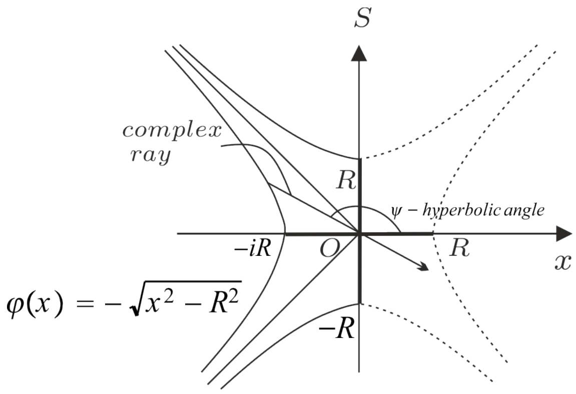

Fig.1. Gaussian beam in *x-S* plane. Complex rays are perpendicular to wave fronts.

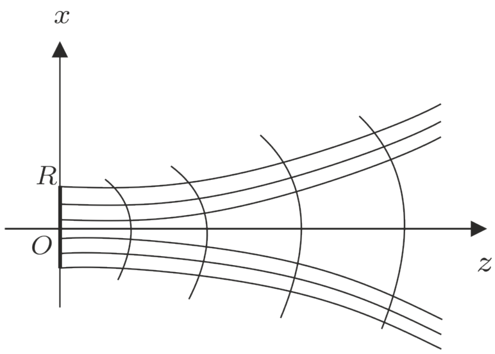

Fig.2. Gaussian beam in *z-x* plane. Rays are not perpendicular to wave fronts.

In the last equation we recognize the standart Gaussian beam with waist R.

5. Diffraction on Half Plane

Now consider the following two-dimensional model problem

$$\Delta u + k^2 u = 0, \qquad u = u(x, z), \qquad z > 0, \tag{32}$$

and

$$u(x, 0) = e^{ik\frac{x^2}{2}} \qquad \text{on} \qquad z = 0 \tag{33}$$

previously studied in [3] in under a slightly different aspect. The corresponding ray solution is

$$x = s + \sqrt{1 - x^2}\tau, \; z = \tau\sqrt{1 - x^2}, \; u = \frac{x^2}{2} + \tau.$$

The equation for centers of curvature (i.e. the caustic) in (z, s) coordinates is

$$z = -\left(1 - s^2\right)^{\frac{3}{2}} \tag{34}$$

which is real if $|s| < 1$ and pure imaginary otherwise. After eliminating τ and using (34) we obtain the parametric equation of the caustic in (x, z, u) space as

$$x = s^3, \; z = -\left(1 - s^2\right)^{\frac{3}{2}}, \; u = \frac{3}{2}s^2 - 1 \tag{35}$$

which is real only for $|s| < 1$. But the projection of this curve onto the "extended" (x, u) space (furnished with the metric of $\mathbb{R}^1_{1,n}$) regardless of z whether is real or complex, provides

$$u = \frac{3}{2}x^{\frac{2}{3}} - 1 \tag{36}$$

which is real for all s and therefore for all x. For $|x| < 1$ it represents real part of the caustics and for $|x| > 1$ the complex part of the caustic. Now studying Figure 1 we may make some conclusions about the ray picture of the problem (32)-(33).

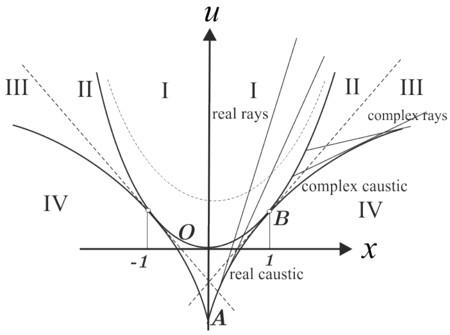

Fig. 3 Ray picture of the problem (32)-(33)

The wave fronts are propagating inside the parabola but the caustic lies outside the parabola. Both real and complex portions of the caustic are represented by one equation (36) in real coordinates. However along the complex part of the caustic the curve parameter is pure imaginary. The figure predicts the number of rays passing through each point in the (x, u) plane. In zone I there is one real ray through each point. At each point of zone II meet two rays: one real and one complex. In zone III through each point passes one complex ray and no comes into zone IV. Point B is the point of switching real rays to complex.

Conclusions

It turns out that the eikonal equation provides a convinient metric in pseudo-Riemann geometry and in this geometry complex rays become visiable as their real counterparts. Copmplex reys and real rays are perpendicular that allows to trace their trajectory in the same space.

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E. Hasanoglu for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.5, No.1.