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On Bäcklund Transformation Between Timelike Curves in Minkowski Space-Time

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Abstract: The aim of this paper is to define Bäcklund transformation between two timelike curves in four dimensional Minkowski space. For this purpose, we examine the transformation depending on the choice of rotation matrix which determines the relations between Frenet frames of timelike Bäcklund curves. There are three different cases for rotation matrix; two of them are spherical rotations on the spacelike hyperplane and one of them is hyperbolical rotation on the timelike hyperplane. For each case, we get the relations between curvature functions of timelike Bäcklund curves. By the way, we prove that timelike Bäcklund curves must have equal constant second torsion functions up to sign. This also means that Bäcklund transformation is a transformation which maps a timelike curve with constant second torsion to another timelike curve with constant second torsion.

Minkowski Uzay-Zamanda Timelike Eğriler Arasındaki Bäcklund Dönüşümü Üzerine

Anahtar Kelimeler
Minkowski uzay-zaman,
Timelike eğriler,
Bäcklund dönüşümü

Özet: Bu çalışmanın amacı, Minkowski uzay-zamanda timelike eğriler arasında Bäcklund dönüşümünü tanımlamaktır. Bu amaç doğrultusunda, timelike Bäcklund eğrilerin Frenet çatıları arasında ilişkiyi ortaya koyan dönme matrisinin seçimine bağlı olarak dönüşümü inceledik. İkisi spacelike hiperdüzlemde küresel dönme ve biri ise timelike hiperdüzlemde hiperbolik dönme olmak üzere üç farklı dönme matrisi durumu söz konusudur. Her durum için, timelike Bäcklund eğrilerinin eğrilik fonksiyonları arasındaki ilişki ortaya konmuştur. Bu arada, işaret farkı gözeterek timelike Bäcklund eğrilerin eşit ikinci burulma fonksiyonuna sahip olması gerektiği ispatlanmıştır. Bu aynı zamanda; Bäcklund dönüşümün bir sabit ikinci burulmaya sahip timelike eğriyi bir başka sabit ikinci burulmaya sahip timelike eğriye taşıyan dönüşüm olduğu anlamıma gelir.

1. Introduction

In mathematics, Bäcklund transformation is known as a kind of relation between partial differential equations and their solutions, which is named after the Swedish mathematician and physicists Albert Victor Bäcklund. Simply, a Bäcklund transformation can be considered as a first order partial differential equation system with two functions which depend on one more parameter. This means that the these functions satisfy partial differential equations. Then these two functions are called Bäcklund transformation of each other [1-7].

Bäcklund transformations have also origins in differential geometry by means of the transformations between pseudospherical surfaces. By the way, it can be considered as a geometrical way for generation of a new pseudospherical surface from a given surface with use of the solution of a linear differential equation. Since pseudospherical surfaces can be considered as solutions of the sine-Gordon equation, then Bäcklund transformation of surfaces is a kind of transformation between solutions of the sine-Gordon equation. Therefore, the Bäcklund transformation has important role in soliton theory. For example; Bäcklund transformation and soliton equations for KP equation were investigated in [8] and modern applications of Backlund and Darboux transformations in soliton theory were deeply discussed in [9].

In Minkowski 3-space, Bäcklund transformation is a new research area. The construction of timelike surfaces with positive Gaussian curvature and imaginary principal curvatures was established in [10] by using Bäcklund transformation. Then, the

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Bäck-lund transformation on surfaces with Gaussian curvature K = 1 in was given by Tian [11] Bäcklund transformation for pseudospherical surfaces can be restricted to give a transformation on space curves. Since Bäcklund transformation maps asymptotic curves to asymptotic curves and we know that the torsion of an asymptotic curve is constant, then Bäcklund transformation can be considered as a transformation on space curves that preserves constant torsion [12, 13]. Nemeth proved that if there is a correspondence between points of two unit speed curves α and $\tilde{\alpha}$ having the property that line joining the corresponding points $\alpha(s)$ and $\tilde{\alpha}(s)$ is the intersection of the osculating planes of these curves, then the angle between and tangent vectors of the curves $\alpha(s)$ and $\tilde{\alpha}(s)$ are the same in [12]. Moreover, Nemeth also proved that two curves $\alpha(s)$ and $\tilde{\alpha}(s)$ must have the same constant torsion $k_{n-1} = \tilde{k}_{n-1} = \frac{sin\theta}{\rho}$ in n dimensional Euclidean space. In four dimensional Euclidean space, Bäcklund transformation of two dimensional surfaces was given in [14].

On the other hand, Bäcklund transformations for nonnull and null curves in Minkowski 3-space have been also investigated. And the details of Bäcklund transformations for nonnull curves in Minkowski 3-space are explained in [15]. Furthermore, the Bäcklund transformation of a null Cartan curve in Minkowski 3-space is also investigated as a transformation which maps a null Cartan helix to another null Cartan helix, congruent to the given one in [16]. And the sufficient conditions are stated for a transformation between two null Cartan curves in the Minkowski 3-space such that these curves have equal constant torsions [16]. Furthermore, Bäcklund transformation of a pseudo null curve in Minkowski 3-space is investigated as a transformation mapping a pseudo null helix to another pseudo null helix congruent to the given one in [16, 17].

In this paper, Minkowski space-time is introduced. Then, the fundamentals of semi orthogonal matrices are discussed. Moreover, Frenet frame fields for unit speed timelike curves are defined and Serret-Frenet formulas are stated. Before starting to the main discussion, the definition of osculating hyperplane is given. Then we construct a Bäcklund transformation between two timelike curves in Minkowski space-time. Since this construction depends on the rotation between Frenet frames of the curves which occurs on the plane containing N_3, we examine the transformation with respect to the type of rotations. As a result, we investigate the relations between Frenet frames of the curves and their curvature function in three different cases. Finally, we prove that timelike Bäcklund curves must have equal constant second torsion functions up to sign. This also means that Bäcklund transformation is a transformation which maps a timelike curve with constant second torsion to another timelike curve with constant second torsion.

2. Preliminaries

Euclidean four space with Lorentzian product is called Minkowski space-time and denoted by $\mathbb{E}_1^4$. Here Lorentzian product of $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4) \in \mathbb{E}_1^4$ is defined as

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

This product classifies the vectors in Minkowski space-time as follows: if $\langle x, x\rangle_L > 0$, then x is spacelike vector; if $\langle x, x\rangle_L = 0$, then x is lightlike or null vector; if $\langle x, x\rangle_L < 0$, then x is timelike vector. And we may define the norm of vectors with Lorentzian product by $\|x\| = \sqrt{|\langle x, x\rangle_L|}$. On the other hand, we may also write the Lorentzian product of x and y in terms of matrix product as $x^T I_* y$ where $I_* = diag(-1,1,1,1)$. It is really important to note that the position of the term "-1" depends on the basis of Minkowski space-time. The set of semi orthogonal matrices in $\mathbb{E}_1^4$ can be represented as

$$O(1,3) = \{R \in M_4(\mathbb{R}):\ R^T I_* R = I_*\}.$$

For any $x, y \in \mathbb{E}_1^4$ and $R \in O(1,3)$, we may write

$$\langle Rx, Ry\rangle_L = x^T(R^T I_* R)y = x^T I_* y = \langle x, y\rangle_L$$

which means the semi orthogonal matrices preserve the Lorentzian product. The Lorentzian rotation matrices forms a subgroup of semi orthogonal matrices and defined as

$$SO(1,3) = \{R \in O(1,3):\ detR = 1\}.$$

The regular curve $\alpha: I \to \mathbb{E}_1^4$ is named after the character of its velocity vector. If $\langle \alpha'(s), \alpha'(s)\rangle_L = -1$ for all $s \in I$, then α is called unit speed timelike curve. T is the unit timelike vector field α', N_1 is the spacelike normalized vector field α'' and N_2 is spacelike unit normal component of ${N_1}'$ with respect to the plane $\{T, N_1\}$. Finally, N_3 is the unique unit spacelike vector field which is perpendicular to T, N_1 and N_2. Then $\{T, N_1, N_2, N_3\}$ corresponds to the Frenet frame field with the same orientation of$\mathbb{E}_1^4$. The Serre-Frenet formulas for unit speed timelike curve α can be given as follows:

$$\begin{bmatrix} T'(s) \\ {N_1}'(s) \\ {N_2}'(s) \\ {N_3}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 & 0 \\ \kappa(s) & 0 & \tau(s) & 0 \\ 0 & -\tau(s) & 0 & \sigma(s) \\ 0 & 0 & -\sigma(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix} \quad (1)$$

where κ is called curvature, τ is called first torsion and σ is called second torsion of α.

Definition 1. Let $\alpha: I \to \mathbb{E}_1^4$ be a regular curve. The hyperplane generating by $\{\alpha'(s), \alpha''(s), \alpha'''(s)\}$ at the point $\alpha(s)$ is called the osculating hyperplane of the curve α at $\alpha(s)$ [20].

For a given unit speed timelike α, we know that

$$\alpha'(s) = T(s),$$

$$\alpha''(s) = \kappa(s)N_1(s),$$

$$\alpha'''(s) = \kappa'(s)N_1(s) + \kappa^2(s)T(s) + \kappa(s)\tau(s)N_2(s).$$

This means that the osculating hyperplane of α at the point $\alpha(s)$ is the hyperplane generating by Frenet vectors $\{T(s), N_1(s), N_2(s)\}$ and orthogonal to $N_3(s)$.

3. Bäcklund Transformation for Timelike Curves

Assume that φ is a transformation of timelike curves α and $\tilde{\alpha}$ that is $\varphi(\alpha(s)) = \tilde{\alpha}(s)$. Now, suppose that the following properties for corresponding points of these curves are satisfied:

1. The straight line, which is combining the corresponding points of these curves, lies on the intersection of the osculating hyperplanes of the curves. The line segment $\alpha(s)$ to $\tilde{\alpha}(s)$ has a constant measurement r, that is $\left\|\overrightarrow{\alpha(s)\tilde{\alpha}(s)}\right\| = r$. We will denote the normalized unit vector $\overrightarrow{\alpha(s)\tilde{\alpha}(s)}$ by $F_1(s)$.

2. The angle between the vector field F_1 and the tangent vectors of the curves are the same and F_1 is not perpendicular to tangent vectors.

3. The Frenet frame $\{\tilde{T}, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3\}$ of $\tilde{\alpha}$ can be obtained by rotating Frenet frame $\{T, N_1, N_2, N_3\}$ of α with constant angle θ on the plane containing N_3. This means that the second binormals N_3 and $\tilde{N}_3$ forms the constant angle θ.

4. The equality $\langle T, F_i\rangle_L = -\langle N_2, F_1\rangle_L$ is for the vector field F_i which is perpendicular to intersection space of the osculating hyperplanes of the curves and is not a Frenet vector of α.

If the above properties are satisfied, then the curves α and $\tilde{\alpha}$ are called Bäcklund curves. By first property, we can define the transformation φ as

$$\varphi(\alpha(s)) = \tilde{\alpha}(s) = \alpha(s) + rF_1(s). \tag{2}$$

By third property, we can write $\tilde{X} = A^T RAX$ where $\tilde{X}^T = \{\tilde{T}, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3\}$ and $X^T = \{T, N_1, N_2, N_3\}$. Here R and $A = (a_{ij})$ are elements of $SO(1,3)$ with the property $a_{i4} = a_{4i} = \delta_{i4}$ for $i = 1,2,3,4$. Moreover, first binormal vector components of F_j should be nonzero that is that $a_{j3} \neq 0$ for $j = 1,2,3$. Otherwise, Bäcklund curves will lie on 3 dimensional subspace of $\mathbb{E}_1^4$ which means that second torsions σ and $\tilde{\sigma}$ are zero. Timelike Bäcklund curves in 3-dimensional Minkowski space are investigated and similar results on torsions τ and $\tilde{\tau}$ are obtained in [13].

Now, consider the the frame $\{F_1, F_2, F_3, N_3\}$ that is obtained by $F = AX$ where $F^T = \{F_1, F_2, F_3, N_3\}$. Therefore, we may write the following equations:

$$F_1 = a_{11}T + a_{12}N_1 + a_{13}N_2, \tag{3}$$

$$F_2 = a_{21}T + a_{22}N_1 + a_{23}N_2, \tag{4}$$

$$F_3 = a_{31}T + a_{32}N_1 + a_{33}N_2. \tag{5}$$

Moreover, the frame $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{N}_3\}$ which is obtained by $\tilde{F} = RF = RAX$ where $F^T = \{F_1, F_2, F_3, N_3\}$. These frames will help us to analyze the intersections of osculating hyperplanes of the curves.

It is time to examine the rotation matrix R with respect to kind of rotation plane. For rotation matrices in Minkowski space-time, the readers are referred to [9, 18, 19].

Therefore there are three different choice for the rotation matrix R.

Case 1. If the rotation occurs on spacelike plane spanned by the spacelike vector fields N_2 and N_3 with the constant spherical angle θ, then rotation matrix will be of the form:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\theta & -sin\theta \\ 0 & 0 & sin\theta & cos\theta \end{bmatrix}.$$

Thus, according to the transformation $\tilde{X} = A^T RAX$, the relations between Frenet vectors of Bäcklund curves can be given as:

$$\begin{aligned}\tilde{T} = {} & (a_{11}^2 + a_{21}^2 + a_{31}^2 cos\theta)T \\ & +(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}cos\theta)N_1 \\ & +(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta)N_2 \\ & -a_{31}sin\theta N_3,\end{aligned}$$

$$\begin{aligned}\tilde{N}_1 = {} & (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}cos\theta)T \\ & +(a_{12}^2 + a_{22}^2 + a_{32}^2 cos\theta)N_1 \\ & +(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}cos\theta)N_2 \\ & -a_{32}sin\theta N_3\end{aligned}$$

$$\begin{aligned}\tilde{N}_2 = {} & (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta)T \\ & +(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}cos\theta)N_1 \\ & +(a_{13}^2 + a_{23}^2 + a_{33}^2 cos\theta)N_2 - a_{33}sin\theta N_3\end{aligned} \tag{6}$$

$$\tilde{N}_3 = sin\theta(a_{31}\,T + a_{32}\,N_1 + a_{33}N_2) + cos\theta\,N_3. \tag{7}$$

On the other hand, we know that $\tilde{F} = RF$ which corresponds to the following relations:

$$\tilde{F}_1 = F_1,$$

$$\tilde{F}_2 = F_2,$$

$$\tilde{F}_3 = cos\theta F_3 - sin\theta N_3,$$

$$\tilde{N}_3 = sin\theta F_3 + cos\theta N_3.$$

The osculating hyperplane of the curves α and $\tilde{\alpha}$ are the hyperplanes spanning $\{F_1, F_2, F_3\}$ and $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}$ at the corresponding points, respectively. By above relations, the vectors $F_1$and F_2 form a frame for the intersection of osculating hyperplanes of the curves. It is easily seen that $i = 3$ and by last property of Bäcklund curves $\langle T, F_3\rangle_L = -\langle N_2, F_1\rangle_L$ should be satisfied. By Equations 3 and 5, we get $a_{31} = a_{13}$.

Note that $sin\theta \neq 0$. In the case of $sin\theta = 0$, the second binormal vector fields of the curves will be parallel and the osculating hyperplanes will be coincidence. This is a contradiction to the last property of Backlund curves, since the only vector field, which is perpendicular to intersection space of osculating hyperplanes, is the Frenet frame field N_3.

With the use of Equation 2 and 3, we see that

$$\tilde{\alpha} = \alpha + r(a_{11}T + a_{12}N_1 + a_{13}N_2).$$

Actually, the vector F_1 and the tangent vectors of the curves are the same, i.e. $\langle F_1, \tilde{T}\rangle_L = \langle F_1, T\rangle_L = -a_{11}$. By third propery, F_1 is not perpendicular to tangent vectors i.e., $a_{11} \neq 0$.

On the other hand, differentiating the equation

$$\langle \alpha - \tilde{\alpha}, \alpha - \tilde{\alpha}\rangle_L = \mp r^2$$

with respect to arc parameter length of α, we get

$$\langle rF_1, \tilde{\alpha}' - T\rangle_L = 0 \mapsto \langle F_1, \tilde{\alpha}'\rangle_L = -a_{11} = \langle F_1, \tilde{T}\rangle_L.$$

Considering $\tilde{\alpha}' = \|\tilde{\alpha}'\|\tilde{T}$, we obtain $\tilde{\alpha}' = \tilde{T}$. It means that, $\tilde{\alpha}$ is also a unit speed timelike curve.

Theorem 1. Let the curves α and $\tilde{\alpha} = \alpha + rF_1$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant spherical rotation angle in the spacelike plane between the Frenet frames of the curves. Then, the curves α and $\tilde{\alpha}$ have the same second torsion.

Proof. If we differentiate the Frenet vector $\tilde{N}_3$ of curve $\tilde{\alpha}$ in (7) and use the Frenet formulas of α in (1), then we get

$$\begin{aligned}\tilde{N}_3{}' = &(a_{31}{}'sin\theta + a_{32}\kappa sin\theta)T \\ &+(a_{32}{}'sin\theta + a_{31}\kappa sin\theta - a_{33}\tau sin\theta)N_1 \\ &+(-\sigma cos\theta + a_{33}{}'sin\theta + a_{32}\tau sin\theta)N_2 \\ &+(a_{33}\sigma sin\theta)N_3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\tilde{N}_3{}' &= -\tilde{\sigma}\tilde{N}_2 \\ &= -\tilde{\sigma}(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta)T \\ &\quad -\tilde{\sigma}(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}cos\theta)N_1 \\ &\quad -\tilde{\sigma}(a_{13}^2 + a_{23}^2 + a_{33}^2cos\theta)N_2 + \tilde{\sigma}a_{33}sin\theta N_3.\end{aligned}$$

Equality of above obtained vector equations, we obtain

$$a_{33}\sigma sin\theta = \tilde{\sigma}a_{33}sin\theta$$

which means $\tilde{\sigma} = \sigma$.

Theorem 2. Let the curves α and $\tilde{\alpha}$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant spherical rotation angle in the spacelike plane between the Frenet frames of the curves. Then the following relations are satisfied:

$$1 + ra_{11}{}' + r\kappa a_{12} = a_{11}^2 + a_{21}^2 + a_{31}^2cos\theta,$$

$$ra_{12}{}' + r\kappa a_{11} - r\tau a_{13} = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}cos\theta,$$

$$ra_{13}{}' + r\tau a_{12} = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta,$$

$$\sigma = -\frac{sin\theta}{r}.$$

Proof. If we differentiate $\tilde{\alpha} = \alpha + rF_1$ and use the Frenet formulas of α in (1), then we obtain

$$\begin{aligned}\tilde{\alpha}' = &(1 + ra_{11}{}' + r\kappa a_{12})T \\ &+(ra_{12}{}' + r\kappa a_{11} - r\tau a_{13})N_1 \\ &+(ra_{13}{}' + r\tau a_{12})N_2 + (r\sigma a_{13})N_3.\end{aligned}$$

Comparing this equality to the tangent vector field

$$\begin{aligned}\tilde{T} = &(a_{11}^2 + a_{21}^2 + a_{31}^2cos\theta)T \\ &+(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}cos\theta)N_1 \\ &+(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta)N_2 - a_{31}sin\theta N_3,\end{aligned}$$

we obtain

$$1 + ra_{11}{}' + r\kappa a_{12} = a_{11}^2 + a_{21}^2 + a_{31}^2cos\theta,$$

$$ra_{12}{}' + r\kappa a_{11} - r\tau a_{13} = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}cos\theta,$$

$$ra_{13}{}' + r\tau a_{12} = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}cos\theta,$$

$$r\sigma a_{13} = -a_{31}sin\theta.$$

Since the property $a_{13} = a_{31}$ is satisfied, then we get

$$\sigma = -\frac{sin\theta}{r}.$$

Example 1: Consider semi orthogonal matrix $A = (a_{ij})$ with the following entries

$$a_{11} = sinh(2\mu),$$

$$a_{12} = cosh(2\mu)sin\beta,$$

$$a_{13} = a_{13} = -sinh(2\mu)cos\beta,$$

$$a_{21} = sinh(2\mu)sin\beta,$$

$$a_{22} = cosh^2\mu - sinh^2\mu cos(2\beta),$$

$$a_{23} = a_{32} = -sinh^2\mu sin(2\beta),$$

$$a_{33} = cosh^2\mu + sinh^2\mu cos(2\beta),$$

$$a_{14} = a_{24} = a_{34} = a_{41} = a_{42} = a_{43} = 0, a_{44} = 1$$

such that

$$\mu: I \to \mathbb{R} - \{0\} \text{ and } \beta: I \to (0, \pi / 4)$$

are diffentiable function of parameter s. With the use of this rotation matrix, we may define the Bäcklund transformation as follows:

$$\varphi(\alpha(s)) = \tilde{\alpha}(s) = \alpha(s) + ra_{11}T(s) + ra_{12}N_1(s) + ra_{13}N_2(s)$$

$$\tilde{\alpha}(s) = \alpha(s) + rsinh(2\mu)T(s) + rcosh(2\mu)sin\beta N_1(s) - rsinh(2\mu)cos\beta N_2(s)$$

where r is the constant distance between corresponding points of α and $\tilde{\alpha}$. This transformation maps unit speed timelike curves with constant torsion to another curve with constant torsion. Now, let us consider the curve $\alpha: I \to \mathbb{E}_1^4$ parametrized by

$$\alpha(s) = (sinh(\sqrt{2}s), cosh(\sqrt{2}s), cos(s), sin(s)).$$

Then we have

$$\alpha'(s) = (\sqrt{2}cosh(\sqrt{2}s), \sqrt{2}sinh(\sqrt{2}s), -sin(s), cos(s)),$$

and

$$\langle \alpha'(s), \alpha'(s) \rangle_L = -2cosh^2(\sqrt{2}s) + 2sinh^2(\sqrt{2}s) + sin^2(s) + cos^2(s) = -1.$$

Therefore, α is a unit speed timelike curve. Frenet frame fields of α are found as follows:

$$T(s) = (\sqrt{2}cosh(\sqrt{2}s), \sqrt{2}sinh(\sqrt{2}s), -sin(s), cos(s))$$

$$N_1(s) = \left(\tfrac{2}{\sqrt{5}}sinh(\sqrt{2}s), \tfrac{2}{\sqrt{5}}cosh(\sqrt{2}s), -\tfrac{1}{\sqrt{5}}cos(s), -\tfrac{1}{\sqrt{5}}sin(s)\right)$$

$$N_2(s) = (-cosh(\sqrt{2}s), -sinh(\sqrt{2}s), \sqrt{2}sin(s), -\sqrt{2}cos(s)),$$

$$N_3(s) = \left(\tfrac{1}{\sqrt{5}}sinh(\sqrt{2}s), \tfrac{1}{\sqrt{5}}cosh(\sqrt{2}s), \tfrac{2}{\sqrt{5}}cos(s), \tfrac{2}{\sqrt{5}}sin(s)\right).$$

Moreover, the curvature and torsion functions of α are obtained as

$$\kappa(s) = \sqrt{5}, \tau(s) = \tfrac{3\sqrt{10}}{5} \text{ and } \sigma(s) = \tfrac{\sqrt{2}}{\sqrt{5}}.$$

As seen, α has constant second torsion. Thus, we get

$$\begin{aligned}\tilde{\alpha}(s) &= (sinh(\sqrt{2}s), cosh(\sqrt{2}s), cos(s), sin(s))\\ &+rsinh(2\mu)(\sqrt{2}cosh(\sqrt{2}s), \sqrt{2}sinh(\sqrt{2}s), -sin(s), cos(s))\\ &+rcosh(2\mu)sin\beta\left(\frac{2}{\sqrt{5}}sinh(\sqrt{2}s), \frac{2}{\sqrt{5}}cosh(\sqrt{2}s), -\frac{1}{\sqrt{5}}cos(s), -\frac{1}{\sqrt{5}}sin(s)\right)\\ &-rsinh(2\mu)cos\beta(-cosh(\sqrt{2}s), -sinh(\sqrt{2}s), \sqrt{2}sin(s), -\sqrt{2}cos(s))\end{aligned}$$

By choosing the constant distance

$$r = \frac{\sqrt{5}}{2},$$

then we obtain

$$sin\theta = -\frac{1}{\sqrt{2}}$$

by Theorem 2 where $\theta = \frac{5\pi}{4}$ is the constant angle between the second binormals N_3 and $\tilde{N}_3$. Morever, second torsion of the curve $\tilde{\alpha}$

$$\tilde{\sigma}(s) = \frac{\sqrt{2}}{\sqrt{5}}$$

by Theorem 1.

Case 2. If the rotation occurs on the spacelike plane spanned by the spacelike vector fields N_1 and N_3 with the constant spherical angle θ, then rotation matrix will be of the form:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & 0 & -sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & sin\theta & 0 & cos\theta \end{bmatrix}.$$

Similar to above choice of rotation matrix, we obtain the relations between Frenet vectors of Bäcklund curves are obtained as:

$$\begin{aligned}\tilde{T} = &(a_{11}^2 + a_{31}^2 + a_{21}^2cos\theta)T\\ &+(a_{11}a_{12} + a_{31}a_{32} + a_{21}a_{22}cos\theta)N_1\\ &+(a_{11}a_{13} + a_{31}a_{33} + a_{21}a_{23}cos\theta)N_2\\ &-a_{21}sin\theta N_3,\end{aligned}$$

$$\begin{aligned}\tilde{N}_1 = &(a_{11}a_{12} + a_{31}a_{32} + a_{21}a_{22}cos\theta)T\\ &+(a_{12}^2 + a_{32}^2 + a_{22}^2cos\theta)N_1\\ &+(a_{12}a_{13} + a_{32}a_{33} + a_{22}a_{23}cos\theta)N_2\\ &-a_{22}sin\theta N_3,\end{aligned}$$

$$\begin{aligned}\tilde{N}_2 = &(a_{11}a_{13} + a_{31}a_{33} + a_{21}a_{23}cos\theta)T\\ &+(a_{12}a_{13} + a_{32}a_{33} + a_{22}a_{23}cos\theta)N_1\\ &+(a_{13}^2 + a_{33}^2 + a_{23}^2cos\theta)N_2 - a_{23}sin\theta N_3,\end{aligned}$$

$$\tilde{N}_3 = sin\theta(a_{21}\,T + a_{22}N_1 + a_{23}N_2) + cos\theta\, N_3. \qquad (8)$$

Similar to the first case, we have

$$\tilde{F}_1 = F_1,$$

$$\tilde{F}_2 = cos\theta F_2 - sin\theta N_3,$$

$$\tilde{F}_3 = F_3,$$

$$\tilde{N}_3 = sin\theta F_2 + cos\theta N_3.$$

The osculating hyperplane of the curves α and $\tilde{\alpha}$ are the hyperplanes spanning $\{F_1, F_2, F_3\}$ and $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}$ at the corresponding points, respectively. By above relations, the vectors $F_1$and F_3 form a frame for the intersection of osculating hyperplanes of the curves. Therefore, $i = 2$ and by last property of Bäcklund curves, we obtain

$$\langle T, F_2\rangle_L = -\langle N_2, F_1\rangle_L.$$

This means that

$$a_{21} = a_{13}.$$

Theorem 3. Let the curves α and $\tilde{\alpha}$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant spherical rotation angle in the spacelike plane between the Frenet frames of the curves. Then, the curves α and $\tilde{\alpha}$ have the same second torsion.

Proof. Differentiating $\tilde{N}_3$ in (8) and using the relations in (1), we have

$$\begin{aligned}\tilde{N}_3{}' = &(a_{21}{}'sin\theta + a_{22}\kappa sin\theta)T\\ &+(a_{22}{}'sin\theta + a_{21}\kappa sin\theta - a_{23}\tau sin\theta)N_1\\ &+(-\sigma cos\theta + a_{23}{}'sin\theta + a_{22}\tau sin\theta)N_2\\ &+(a_{23}\sigma sin\theta)N_3.\end{aligned}$$

Moreover, we know that

$$\begin{aligned}\tilde{N}_3{}' &= -\tilde{\sigma}\tilde{N}_2\\ &= -\tilde{\sigma}(a_{11}a_{13} + a_{31}a_{33} + a_{21}a_{23}cos\theta)T\\ &\quad -\tilde{\sigma}(a_{12}a_{13} + a_{32}a_{33} + a_{22}a_{23}cos\theta)N_1\\ &\quad -\tilde{\sigma}(a_{13}^2 + a_{33}^2 + a_{23}^2cos\theta)N_2 + \tilde{\sigma}a_{23}sin\theta N_3.\end{aligned}$$

Thus, we get

$$a_{23}\sigma sin\theta = \tilde{\sigma}a_{23}sin\theta.$$

This implies $\tilde{\sigma} = \sigma$.

Theorem 4. Let the curves α and $\tilde{\alpha}$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant spherical rotation angle in the spacelike plane between the Frenet frames of the curves. Then the following relations are satisfied:

$$1 + ra_{11}{}' + r\kappa a_{12} = a_{11}^2 + a_{31}^2 + a_{21}^2cos\theta,$$

$$ra_{12}{}' + r\kappa a_{11} - r\tau a_{13} = a_{11}a_{12} + a_{31}a_{32} + a_{21}a_{22}cos\theta,$$

$$ra_{13}{}' + r\tau a_{12} = a_{11}a_{13} + a_{31}a_{33} + a_{21}a_{23}cos\theta,$$

$$\sigma = -\frac{sin\theta}{r}.$$

Proof. If we differentiate $\tilde{\alpha} = \alpha + rF_1$ and use the Frenet formulas of α in (1), then we obtain

$$\begin{aligned}\tilde{\alpha}' = &(1 + ra_{11}{}' + r\kappa a_{12})T\\ &+(ra_{12}{}' + r\kappa a_{11} - r\tau a_{13})N_1\\ &+(ra_{13}{}' + r\tau a_{12})N_2 + (r\sigma a_{13})N_3.\end{aligned}$$

On the other hand, we have $\tilde{\alpha}' = \tilde{T}$ which means

$$1 + ra_{11}{}' + r\kappa a_{12} = a_{11}^2 + a_{31}^2 + a_{21}^2cos\theta,$$

$$ra_{12}{}' + r\kappa a_{11} - r\tau a_{13} = a_{11}a_{12} + a_{31}a_{32} + a_{21}a_{22}cos\theta,$$

$$ra_{13}{}' + r\tau a_{12} = a_{11}a_{13} + a_{31}a_{33} + a_{21}a_{23}cos\theta,$$

$$r\sigma a_{13} = -a_{21}sin\theta.$$

And we know that $a_{21} = a_{13}$ and we get the proof.

Case 3. Last choice of rotation matrix is much more different from previous cases. In this case, the rotation occurs on the timelike plane spanned by the timelike vector field T and the spacelike vector field N_3 with the constant hyperbolical angle θ. The rotation matrix will be of the form:

$$R = \begin{bmatrix} cosh\theta & 0 & 0 & sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sinh\theta & 0 & 0 & cosh\theta \end{bmatrix}.$$

The relations between Frenet vectors of Bäcklund curves are stated as:

$$\begin{aligned}\tilde{T} = &(a_{21}^2 + a_{31}^2 + a_{11}^2cosh\theta)T\\ &+(a_{21}a_{22} + a_{31}a_{32} + a_{11}a_{12}cosh\theta)N_1\\ &+(a_{21}a_{23} + a_{31}a_{33} + a_{11}a_{13}cosh\theta)N_2\\ &+a_{11}sinh\theta N_3,\end{aligned}$$

$$\begin{aligned}\tilde{N}_1 = &(a_{21}a_{22} + a_{31}a_{32} + a_{11}a_{12}cosh\theta)T\\ &+(a_{22}^2 + a_{32}^2 + a_{12}^2cosh\theta)N_1\\ &+(a_{22}a_{23} + a_{32}a_{33} + a_{12}a_{13}cosh\theta)N_2\\ &+a_{12}sinh\theta N_3,\end{aligned}$$

$$\begin{aligned}\tilde{N}_2 = &(a_{21}a_{23} + a_{31}a_{33} + a_{11}a_{13}cosh\theta)T\\ &+(a_{22}a_{23} + a_{32}a_{33} + a_{12}a_{13}cosh\theta)N_1\\ &+(a_{23}^2 + a_{33}^2 + a_{13}^2cosh\theta)N_2\\ &+a_{13}sinh\theta N_3,\end{aligned}$$

$$\tilde{N}_3 = sinh\theta(a_{11}\, T + a_{12}\, N_1 + a_{13}N_2) + cosh\theta\, N_3. \quad (9)$$

The vectors F_2 and F_3 form a frame for the intersection of osculating hyperplanes of the curves. Thus, $i = 1$ and by last property of Bäcklund curves, we obtain $\langle T, F_1\rangle_L = -\langle N_2, F_1\rangle_L$ and $a_{11} = a_{13}$.

Theorem 5. Let the curves α and $\tilde{\alpha}$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant hyperbolical rotation angle in the timelike plane between the Frenet frames of the curves. Then, the curves α and $\tilde{\alpha}$ have the same second torsion except for sign, that is $\tilde{\sigma} = -\sigma$.

Proof. Differentiating $\widetilde{N}_3$ in (9) and using Serre-Frenet formulas, we obtain

$$\begin{aligned}\widetilde{N}_3{}' = &(a_{11}{}'sinh\theta + a_{12}\kappa sinh\theta)T\\ &+(a_{12}{}'sinh\theta + a_{11}\kappa sinh\theta - a_{13}\tau sinh\theta)N_1\\ &+(-\sigma cosh\theta + a_{13}{}'sinh\theta + a_{12}\tau sinh\theta)N_2\\ &+(a_{13}\sigma sinh\theta)N_3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\widetilde{N}_3{}' &= -\tilde{\sigma}\widetilde{N}_2\\ &= -\tilde{\sigma}(a_{21}a_{23} + a_{31}a_{33} + a_{11}a_{13}cosh\theta)T\\ &\quad -\tilde{\sigma}(a_{22}a_{23} + a_{32}a_{33} + a_{12}a_{13}cosh\theta)N_1\\ &\quad -\tilde{\sigma}(a_{23}^2 + a_{33}^2 + a_{13}^2 cosh\theta)N_2 - \tilde{\sigma}a_{13}sinh\theta N_3.\end{aligned}$$

Finally, we get

$$a_{13}\sigma sinh\theta = -\tilde{\sigma}a_{13}sinh\theta$$

which implies $\tilde{\sigma} = -\sigma$.

As in above cases, this is not the case that we want. That is, $a_{13} \neq 0$.

Theorem 6. Let the curves α and $\tilde{\alpha}$ be two timelike unit speed Bäcklund curves in $\mathbb{E}_1^4$ and θ be the constant hyperbolical rotation angle in the timelike plane between the Frenet frames of the curves. Then the followings are satisfied:

$$1 + ra_{11}{}' + r\kappa a_{12} = a_{21}^2 + a_{31}^2 + a_{11}^2 cosh\theta,$$

$$ra_{12}{}' + r\kappa a_{11} - r\tau a_{13} = a_{21}a_{22} + a_{31}a_{32} + a_{11}a_{12}cosh\theta,$$

$$ra_{13}{}' + r\tau a_{12} = a_{21}a_{23} + a_{31}a_{33} + a_{11}a_{13}cosh\theta,$$

$$\sigma = \frac{sinh\theta}{r}.$$

Proof. The proof can be done similar to proofs of Theorem 2 and 4 by using the property $a_{11} = a_{13}$.

4. Discussion and Conclusion

Bäcklund transformation between two timelike curves in $\mathbb{E}_1^4$ is obtained. Moreover, we state the relations between the second torsion of these two timelike Bäcklund curves in three different cases. For each case, we prove that timelike Bäcklund curves must have constant second torsion. It is essential to note that second torsion functions of timelike Bäcklund curves are equal or equal except from sign depending on the choice of the rotation matrix. If the rotation occurs in the spacelike plane, then the second torsion of the timelike Bäcklund curves α and $\tilde{\alpha}$ are exactly equal i.e.

$$\tilde{\sigma} = \sigma = -\frac{sin\theta}{r}$$

In the case of hyperbolical rotation in the timelike plane, the second torsion of the timelike Bäcklund curves have different signs that is

$$\tilde{\sigma} = -\sigma = \frac{sinh\theta}{r}$$

And it is also seen that the second torsion functions of timelike Backlund curves depends only on the angle of rotation and the constant distance r.

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