Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 22, Sayı 3, 1151-1156, 2018

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 22, Issue 3, 1151-1156, 2018

DOI: 10.19113/sdufenbed.469483

On Concircular Curvature Tensor in Space-Times

Bahar KIRIK[*1]

[1]Marmara University, Faculty of Arts and Sciences, Department of Mathematics, Göztepe Campus 34722, İstanbul, Turkey

(Alınış / Received: 23.02.2018, Kabul / Accepted: 29.09.2018, Online Yayınlanma / Published Online: 11.10.2018)

Keywords
Concircular curvature tensor,
Bivector,
Space-time,
Holonomy theory

Abstract: The aim of this work is to examine some properties of the concircular curvature tensor on 4−dimensional manifolds admitting a Lorentz metric (so called space-times). In the first two sections, the study is introduced and the interrelated concepts together with some notations are presented. In the third section of the study, some results are obtained connected to eigenbivector structure of the concircular curvature tensor on these manifolds by taking into account the classification scheme of 2–forms (also known as bivectors) in this metric signature. Then, the known holonomy algebras on space-times are considered and some theorems are given regarding the concircular and Riemann curvature tensors. This analysis is also associated with the types of the Riemann curvature tensor on these manifolds. In the last section, the results of the study is summarized and the discussion part is presented.

Uzay-Zamanlardaki Konsörkılır Eğrilik Tensörü Üzerine

Anahtar Kelimeler
Konsörkılır eğrilik tensörü,
Bivektör,
Uzay-zaman,
Dolanım teorisi

Özet: Bu çalışmanın amacı, uzay-zaman olarak adlandırılan 4−boyutlu Lorentz metrik işaretli manifoldlar üzerinde konsörkılır eğrilik tensörünün bazı özelliklerinin incelenmesidir. İlk iki bölümde çalışma tanıtılmış ve birbiriyle ilişkili kavramlar ile bazı notasyonlar sunulmuştur. Çalışmanın üçüncü bölümünde, bu metrik işarette (bivektörler olarak da bilinen) 2–formların sınıflandırma şeması göz önüne alınarak, bu manifoldlar üzerindeki konsörkılır eğrilik tensörünün özbivektör yapısı ile ilgili bazı sonuçlar elde edilmiştir. Daha sonra, uzay-zamanlar üzerinde bilinen dolanım cebirleri dikkate alınmış, konsörkılır ve Riemann eğrilik tensörlerine ilişkin bazı teoremler verilmiştir. Söz konusu analiz, bu manifoldlar üzerindeki Riemann eğrilik tensörünün tipleri ile de ilişkilidir. Son bölümde ise, çalışmada elde edilen sonuçlar özetlenmiş ve tartışma bölümü sunulmuştur.

1. Introduction

Special transformations preserving some geometric structures have a significant place in geometry and physics (see, e.g., [1]). Moreover, a symmetry of a 4−dimensional connected manifold admitting a Lorentz metric $(-,+,+,+)$, which is called a *space-time*, preserves some geometric features of the manifold and it is an interesting subject in general relativity theory (see, e.g., [2]). Conformal transformations are one of the most important examples of such transformations and they preserve angles locally. A *concircular transformation* is a conformal transformation which transforms a geodesic circle (which is defined as a curve whose first curvature is constant and whose second curvature is identically zero) into a geodesic circle (for details, see, [3, 4]). In general, a conformal transformation does not have to preserve a geodesic circle and the geometry admitting a concircular transformation is called a *concircular geometry*. The following tensor field Z is called the *concircular curvature tensor* and it is invariant under a concircular transformation:

$$Z = Riem - \frac{r}{n(n-1)}G \tag{1}$$

where $Riem$, r and n denote the Riemann curvature tensor, the scalar curvature and the dimension of the manifold M, respectively. The tensor G is given by

$$G(U,X,Y,W) = g(U,W)g(X,Y) - g(U,Y)g(X,W) \tag{2}$$

where $U,X,Y,W \in TM$ and g denotes the metric tensor of the manifold. It is known that an Einstein manifold is mapped into itself under the concircular transformation. On the other hand, if the concircular curvature tensor vanishes identically on the manifold, then it is called *concircularly flat*. In this case, M is of constant curvature and the converse is also true, [3]. Therefore, the concircular curvature tensor can be thought as a measure of the failure of the manifold to be of constant curvature (for example, see [5]).

[*] *Corresponding author: bahar.kirik@marmara.edu.tr*

When looking at the literature, many studies were carried out on concircular curvature tensor in different manifolds, e.g., on contact metric manifolds, Einstein manifolds, Kenmotsu manifolds, pseudo-symmetric manifolds, fluid space-times (for example, see, [5–9]) and many others. Additionally, concircularly recurrent pseudo-Riemannian manifolds were studied in [10] and it was shown that such manifolds are recurrent manifolds, that is, *Riem* is recurrent.

In this paper, a space-time is considered as defined above and some properties of the concircular curvature tensor are investigated on this manifold. The eigenbivector structure of this tensor field is studied with the aid of the classification of 2−forms in Lorentz signature $(-,+,+,+)$ which will be mentioned in Section 2. Another concept discussed in this section is the theory of holonomy and it will be useful in Section 3. Other than these, the relationship between the known curvature types in space-times and the concircular curvature tensor is examined since Z crucially depends on *Riem*. In Section 3, some results about these concepts are obtained and expressed in several theorems. Finally, these results are interpreted in the conclusion section.

2. Preliminaries

Throughout the following, (M,g) will be a space-time structure. First of all, it will be useful to give some basic notations in this section. By $u.v$, we mean the inner product $g(u,v)$ arising from $g(m)$ where u and v are the members of the tangent space of the manifold at m written as T_mM. A non-zero member v of T_mM is called *timelike*, *spacelike* or *null* if $v.v<0$, $v.v>0$ or $v.v=0$, respectively. A pseudo-orthonormal basis for T_mM will be denoted by x,y,z,t where these members of T_mM are mutually orthogonal vectors and satisfy the following relations:

$$x.x = y.y = z.z = -t.t = 1. \tag{3}$$

Moreover, one can define a *null* basis l,n,x,y for T_mM where x, y are given in (3), l, n are null vectors of T_mM defined by $\sqrt{2}l = z+t$, $\sqrt{2}n = z-t$ and satisfy $l.n = 1$. For the details of this section, we refer to [2].

2.1. 2–forms in space–times

A $2-form$ (also known as a *bivector*) is a second order skew-symmetric tensor field and the space of all 2−forms at $m \in M$ will be denoted by Λ_mM. In this case, Λ_mM is a 6−dimensional vector space and it is a Lie algebra under matrix commutation. The classification of 2−forms constitutes an attractive place in the literature and it is known for all metric signatures in 4−dimensional manifolds (where the metric signature can only be positive definite $(+,+,+,+)$ or Lorentz $(-,+,+,+)$ or neutral signature $(+,+,-,-)$). For all signatures on these manifolds, a 2−form F can be classified according to the value of its rank which can only be 2 or 4 (for a non-zero member of Λ_mM) since F is skew-symmetric. If the rank of F equals 2, then F is called a *simple* $2-form$, whereas if this rank equals 4, then F is called a *non-simple* $2-form$.

A simple 2−form can be expressed as follows:

$$F^{ab} = u^a v^b - v^a u^b \tag{4}$$

where $F^{ab}(=-F^{ba})$ denotes the components of F and $u,v \in T_mM$. The 2−space spanned by $u,v \in T_mM$ is uniquely determined by this 2−form and called the *blade* of F. Then, the blade of F (or even F) is denoted by $u \wedge v$. For Lorentz signature, a simple 2−form can be spacelike (if its blade is spacelike, that is, each non-zero member of it is spacelike) or timelike (if its blade is timelike, that is, it contains exactly two distinct null directions) or null (if its blade is null, that is, it contains exactly one null direction). The classification of 2−forms in space-times is known from general relativity and the canonical forms together with corresponding Segre types, can be found, e.g., in [2, 11]. In a null basis l,n,x,y at $m \in M$, examples of these 2−forms and blades are: i) $x \wedge y$ (simple and spacelike), ii) $l \wedge n$ (simple and timelike), iii) $l \wedge y$ or $l \wedge x$ (simple and null), iv) $\alpha(l \wedge n) + \beta(x \wedge y)$ (non-simple and $\alpha,\beta \in \mathbb{R}$, $\alpha \neq 0 \neq \beta$). It is noted that one can define a 2−form metric $|\ |$ on Λ_mM given by $|F,\hat{F}| = F^{ab}\hat{F}_{ab}$ for $F,\hat{F} \in \Lambda_mM$ and $|F,F| = F^{ab}F_{ab}$ is called the *size* of F.

2.2. Holonomy theory

It will also be useful to mention about the holonomy group of a space-time (because it is a connected manifold) with respect to the Levi-Civita connection denoted by ∇ of the metric g. The *holonomy group* is the collection of all linear isomorphisms on T_mM arising from the parallel transport of each member of T_mM around a smooth, closed curve c at $m \in M$. It can be shown that since M is connected (and so, it is path-connected), the holonomy groups at any two points of the manifold are isomorphic to each other. Hence one can consider the holonomy group of (M,g) (for details of the holonomy theory, we refer to, [12] and for applications to space-times see [2]). Let Φ be this holonomy group. Since g has Lorentz signature, then Φ is isomorphic to a subgroup of the Lorentz group $\mathscr{L}$. It is known that Φ is a Lie group admitting a Lie algebra ϕ which is a subalgebra of $o(1,3)$ for Lorentz signature. The possibilities for ϕ were given in [13] in which 15 holonomy types occur and that are labelled as R_1, R_2, ..., R_{15}. Among these algebras, R_1 is the trivial case, so it will not be considered in the following. There is no 5−dimensional holonomy algebra and R_{15} is the full algebra. It is also noted that the 1−dimensional algebra R_5 cannot occur as a space-time holonomy group (for details, see [2], pages 239–240) and so it will not be studied here. The dimension and bases (in 2−form representation) for each holonomy type occurring in space-times are given as follows. Here, the symbol $<>$ denotes a spanning set.

♦ 1−dimensional holonomy algebras:

- R_2 $< l \wedge n >$
- R_3 $< l \wedge x >$ (or $< l \wedge y >$)
- R_4 $< x \wedge y >$

♦ 2−dimensional holonomy algebras:

- R_6 $< l\wedge n, l\wedge x >$
- R_7 $< l\wedge n, x\wedge y >$
- R_8 $< l\wedge x, l\wedge y >$

♦ 3−dimensional holonomy algebras:

- R_9 $< l\wedge n, l\wedge x, l\wedge y >$
- R_{10} $< l\wedge n, l\wedge x, n\wedge x >$
- R_{11} $< l\wedge x, l\wedge y, x\wedge y >$
- R_{12} $< l\wedge x, l\wedge y, l\wedge n+\omega(x\wedge y) >$ $(0\neq\omega\in\mathbb{R})$
- R_{13} $< x\wedge y, y\wedge z, x\wedge z >$

♦ 4−dimensional holonomy algebra:

- R_{14} $< l\wedge n, l\wedge x, l\wedge y, x\wedge y >$

♦ 6−dimensional holonomy algebra:

- R_{15} $o(1,3)$

The relation between 2−form representation of the subalgebras of ϕ and the Riemann curvature tensor will be given in the next subsection.

2.3. Riemann curvature tensor

Since the concircular curvature tensor is closely related to the Riemann curvature tensor, it will make sense to give more details about *Riem*. Let the components of *Riem* be denoted by $R^a{}_{bcd}$. So, one gets the type (0, 4) curvature tensor with components $R_{abcd} = g_{ae}R^e{}_{bcd}$. One can define a *curvature map* indicated by $\tilde{f}$ between the space of all 2−forms given by $F^{ab} \to R^{ab}{}_{cd}F^{cd}$. It can be shown that $\tilde{f}$ is a linear map and with the help of this map, one gives a classification of the curvature tensor at $m\in M$ as follows where five classes denoted by **A**, **B**, **C**, **D** and **O** occur, [2]:
♦ **Class A** : In this class, *Riem* is not zero at m and it is not in any of the other classes (**B**, **C** or **D** defined below). So, the rank of $\tilde{f}$, rank($\tilde{f}$) is either 2, 3, 4, 5 or 6 at $m\in M$. Hence, this type is the most general case.
♦ **Class B** : *Riem* takes this type when rank($\tilde{f}$) = 2 and the range space of $\tilde{f}$, rg($\tilde{f}$) is spanned by a spacelike and a timelike 2−form whose blades are orthogonal.
♦ **Class C** : This occurs when rank($\tilde{f}$) = 2 or 3 and rg($\tilde{f}$) can be spanned by two or three simple 2−forms. Moreover, the members of rg($\tilde{f}$) have a common (non-zero) eigenvector $k\in T_mM$ corresponding to zero eigenvalue, that is, each 2−form F' in rg($\tilde{f}$) satisfies $F'_{ab}k^b = 0$.
♦ **Class D** : In this class, rank($\tilde{f}$) = 1 and rg($\tilde{f}$) is spanned by a simple 2−form $\tilde{F}$ (in this case, $\tilde{F}$ is necessarily simple since $R_{a[bcd]} = 0$ where the square bracket around indices denotes the skew-symmetrization of the enclosed indices). Thus, $R_{abcd} = \gamma\tilde{F}_{ab}\tilde{F}_{cd}$ $(\gamma\in\mathbb{R})$ at $m\in M$.
♦ **Class O** : If *Riem* vanishes at $m\in M$, then this class occurs.

Some other notes will be helpful for the next section. Firstly, consider the following equation

$$R_{abcd}k^d = 0 \tag{5}$$

where $0\neq k\in T_mM$. It is proved that the equation (5) has no solutions if the curvature type is **A** or **B**. Additionally, if the curvature type is **C**, then (5) has a unique independent solution and if the curvature type is **D** at $m\in M$, then (5) has exactly two independent solutions (for details, see [2], pages 261–262).

It is also useful to note that i) if the holonomy type is R_2, R_3 or R_4, the curvature type at any $m\in M$ is **O** or **D**, ii) for R_6 or R_8 it is **O**, **D** or **C**, iii) for R_7 it is **O**, **D** or **B**, iv) for R_9 or R_{12} it is **O**, **D**, **C** or **A**, v) for R_{10}, R_{11} or R_{13} it is **O**, **D** or **C** and finally, vi) for R_{14} or R_{15} it can be **O**, **D**, **C**, **B**, **A**, [14].

On the other hand, it is known from the infinitesimal holonomy theory that rg($\tilde{f}$) is a subspace of the Lie algebra ϕ and so, the Riemann curvature tensor can be written as a symmetrized sum of products of members of ϕ (see, for example, [2]).

3. Results

This section is devoted to the results of the work. By using (1) and (2), for a space-time M, one has the concircular curvature tensor Z, in the local expression,

$$Z_{abcd} = R_{abcd} - \frac{r}{12}(g_{ad}g_{bc} - g_{bd}g_{ac}). \tag{6}$$

Now, consider the eigenbivector structure of Z. First of all, the following lemma will be helpful for the next step.

Lemma 3.1. *Let $0\neq F\in\Lambda_mM$. Then F is an eigenbivector of Z corresponding to eigenvalue $\alpha+\frac{r}{6}$ if and only if it is an eigenbivector of Riem corresponding to eigenvalue α.*

Proof. Suppose that $0\neq F\in\Lambda_mM$ is an eigenbivector of *Riem*. Then, $R_{abcd}F^{cd} = \alpha F_{ab}$ for some $\alpha\in\mathbb{R}$. Contracting (6) over F^{cd}, one gets

$$\begin{aligned} Z_{abcd}F^{cd} &= \alpha F_{ab} - \frac{r}{12}(F^c{}_a g_{bc} - F^c{}_b g_{ac}) \\ &= \alpha F_{ab} - \frac{r}{12}(F_{ba} - F_{ab}) \\ &= \left(\alpha + \frac{r}{6}\right)F_{ab}. \end{aligned} \tag{7}$$

Therefore, from (7), F is an eigenbivector of Z corresponding to eigenvalue $\alpha+\frac{r}{6}$.

Conversely, if F is an eigenbivector of Z with eigenvalue $\beta\in\mathbb{R}$, it can be obtained from (6) that it is an eigenbivector of *Riem* corresponding to eigenvalue $\beta-\frac{r}{6}$. This completes the proof. □

Let us now consider the 1−dimensional holonomy algebras and examine the eigenbivector structure of Z. Then, one obtains the following theorem:

Theorem 3.2. *Let (M,g) be a space-time structure and suppose that Riem does not vanish at $m\in M$. Then the following conditions hold:*

i. *For holonomy type R_2, $l \wedge n$ is an eigenbivector of Z corresponding to a non-zero eigenvalue.*

ii. *For holonomy type R_3, $l \wedge x$ (or $l \wedge y$) is an eigenbivector of Z corresponding to a zero eigenvalue.*

iii. *For holonomy type R_4, $x \wedge y$ is an eigenbivector of Z corresponding to a non-zero eigenvalue.*

For all these cases, the curvature type of Riem is **D**.

Proof. Consider $1-$dimensional holonomy algebras and assume that there exists $m \in M$ such that *Riem* does not vanish at m. Then, a local expression for the Riemann curvature tensor at $m \in M$ is written as follows:

$$R_{abcd} = \gamma F_{ab} F_{cd} \tag{8}$$

where F is a simple $2-$form and γ is not zero at $m \in M$. So, the curvature type is **D** (see, Section 2.3).

i. For holonomy type R_2, ϕ is spanned by $2-$form $F = l \wedge n$ and by using (4) and (8), one gets the expression of *Riem* as follows:

$$R_{abcd} = \gamma(l_a n_b - n_a l_b)(l_c n_d - n_c l_d). \tag{9}$$

Multiplying (9) by $F^{cd}(= l^c n^d - n^c l^d)$ and using the fact that $l.n = 1$, it can be seen that F is an eigenbivector of *Riem* with the eigenvalue -2γ. Thus, it is an eigenbivector of Z by Lemma 3.1. Contracting (9), one gets the scalar curvature $r = 2\gamma$ and putting these into (7), it follows that $F = l \wedge n$ is an eigenbivector of Z corresponding the eigenvalue $-\frac{5}{3}\gamma$ which is not zero on at $m \in M$.

ii. For holonomy type R_3, ϕ is spanned by $2-$form $F = l \wedge x$ and using (8), the scalar curvature is found to be zero. Therefore, $Z = Riem$ from (6). Since $l \wedge x$ is an eigenbivector of *Riem* with eigenvalue 0, then $R_{abcd}F^{cd} = Z_{abcd}F^{cd} = 0$. So, F is an eigenbivector of Z with eigenvalue 0.

iii. By using similar steps given above, one can observe for holonomy type R_4 that, $F = x \wedge y$ is an eigenbivector of the concircular curvature tensor with eigenvalue $\frac{5}{3}\gamma$ which is not zero at $m \in M$. Hence, the proof is completed. □

Corollary 3.3. *Let (M, g) be a space-time structure and consider $1-$dimensional holonomy algebras. Suppose that Riem does not vanish at $m \in M$. Then, Z admits timelike, null and spacelike eigenbivectors where the corresponding eigenvalues are non-zero if the eigenbivector is spacelike or timelike and zero if it is null.*

Proof. Since the blades of the bases members of the holonomy algebras R_2, R_3 and R_4 are timelike, null and spacelike, respectively, the result is clear from Theorem 3.2. □

Now consider the $2-$dimensional holonomy algebras and investigate some features of Z.

Theorem 3.4. *Let (M, g) be a space-time structure and suppose that Riem does not vanish at $m \in M$. Let the range space of the curvature map be spanned by $2-$forms $\tilde{F}, \tilde{G} \in \Lambda_m M$. A necessary and sufficient condition for the $2-$form $\tilde{F}$ be an eigenbivector of Z is that the relation*

$$\delta|\tilde{F}, \tilde{G}| + \xi|\tilde{F}, \tilde{F}| = 0$$

is satisfied for some $\delta, \xi \in \mathbb{R}$.

Proof. Suppose that $\mathrm{rg}(\tilde{f})$ is spanned by $2-$forms $\tilde{F}, \tilde{G} \in \Lambda_m M$ and that *Riem* does not vanish at $m \in M$. Then, one has the following expression of *Riem*

$$R_{abcd} = \lambda \tilde{F}_{ab}\tilde{F}_{cd} + \delta \tilde{G}_{ab}\tilde{G}_{cd} + \xi(\tilde{F}_{ab}\tilde{G}_{cd} + \tilde{G}_{ab}\tilde{F}_{cd}) \tag{10}$$

for some smooth functions λ, δ, ξ. Contracting (6) by $\tilde{F}^{cd}$ and using (10), it follows that

$$\begin{aligned} Z_{abcd}\tilde{F}^{cd} &= \lambda|\tilde{F}, \tilde{F}|\tilde{F}_{ab} + \delta|\tilde{F}, \tilde{G}|\tilde{G}_{ab} \\ &\quad + \xi(|\tilde{F}, \tilde{G}|\tilde{F}_{ab} + |\tilde{F}, \tilde{F}|\tilde{G}_{ab}) + \frac{r}{6}\tilde{F}_{ab} \\ &= (\lambda|\tilde{F}, \tilde{F}| + \xi|\tilde{F}, \tilde{G}| + \frac{r}{6})\tilde{F}_{ab} \\ &\quad + (\delta|\tilde{F}, \tilde{G}| + \xi|\tilde{F}, \tilde{F}|)\tilde{G}_{ab}. \end{aligned} \tag{11}$$

In this case, if $\tilde{F}$ is an eigenbivector of Z, it can be seen from (11) that the condition

$$\delta|\tilde{F}, \tilde{G}| + \xi|\tilde{F}, \tilde{F}| = 0 \tag{12}$$

holds since $\tilde{G} \neq 0$. Conversely, if the equation (12) is satisfied, one gets from (11) that $Z_{abcd}\tilde{F}^{cd} = \psi \tilde{F}_{ab}$ for some $\psi \in \mathbb{R}$. Therefore, $\tilde{F}$ is an eigenbivector of Z. This completes the proof. □

Example: Consider the holonomy type R_7 with algebra $< \tilde{F}, \tilde{G} >$ where $\tilde{F} = l \wedge n, \tilde{G} = x \wedge y$. Then *Riem* satisfies (10). Moreover, using the identity $R_{a[bcd]} = 0$ and (10), it can be obtained that $\xi = 0$. Thus, for this holonomy type, *Riem* takes the form

$$R_{abcd} = \lambda \tilde{F}_{ab}\tilde{F}_{cd} + \delta \tilde{G}_{ab}\tilde{G}_{cd} \tag{13}$$

for some smooth functions λ, δ. Also, one gets

$$|\tilde{F}, \tilde{G}| = 0. \tag{14}$$

Using similar steps given above and taking into account (6), (13), (14), it is seen that the equation (12) is satisfied. Hence, $\tilde{F} = l \wedge n$ is an eigenbivector of Z and so Theorem 3.4 holds for holonomy type R_7. Analogously, it can be obtained that $\tilde{G} = x \wedge y$ is also an eigenbivector of Z.

Next, assume that the equation (5) is satisfied. In this case, if one wants to investigate the solutions of

$$Z_{abcd}k^d = 0 \tag{15}$$

for $0 \neq k \in T_m M$, a contraction of (6) with k^d yields

$$r(k_a g_{bc} - k_b g_{ac}) = 0. \tag{16}$$

From (16), either $r = 0$ or

$$k_a g_{bc} - k_b g_{ac} = 0 \tag{17}$$

holds. If $r = 0$, then it follows from (6) that $Z = Riem$. However, if $r \neq 0$, multiplying (17) by g^{bc}, it follows that

$$4k_a - k_b\delta_a^b = 3k_a = 0. \quad (18)$$

Thus, $k = 0$ from (18) and hence there are no non-zero solutions of (15). So, we proved the following theorem.

Theorem 3.5. *Let (M, g) be a space-time structure and suppose that there exists $0 \neq k \in T_mM$ such that the equation (5) is satisfied and that $r \neq 0$. Then, there are no non-zero solutions of (15).*

It can be deduced from above that if the equation (15) has non-zero solutions and if $r \neq 0$, then there are no non-zero solutions of (5). Therefore, one has the following result.

Corollary 3.6. *Let (M, g) be a space-time structure and suppose that there exists $0 \neq k \in T_mM$ such that the equation (15) is satisfied and that $r \neq 0$. Then, the curvature type cannot be* **C** *or* **D** *at $m \in M$.*

Finally, it will be useful to look at the concircularly flat space-times. A contraction of (6) by g^{ad} yields the following tensor field

$$Z_{bc} = R_{bc} - \frac{r}{4}g_{bc} \quad (19)$$

which is also invariant under the concircular tranformation, [3]. Now, if Z vanishes identically on M, then it is obtained from (19) that (M, g) is an Einstein manifold. Let $r \neq 0$, then (M, g) is called a *proper* Einstein manifold and it was shown in [15] that possible holonomy algebras for a proper Einstein manifold are R_7, R_{14} or R_{15}. As for the Ricci flat case ($r = 0$), so called the "vacuum case" in general relativity, possible holonomy types are R_8, R_{14} or R_{15}, [2]. So, one obtains the following result:

Theorem 3.7. *Let (M, g) be a space-time structure. Suppose that M is concircularly flat.*

i. *If M is Ricci flat, then the holonomy type is R_8, R_{14} or R_{15}.*

ii. *If M is not Ricci flat, then the holonomy type is R_7, R_{14} or R_{15}.*

4. Discussion and Conclusion

In this paper, the concircular curvature tensor was investigated on space-times. Some properties of this tensor field were found related to the holonomy algebras of Lorentz signature $(-,+,+,+)$. Also, some remarks about the Riemann curvature tensor were given with the help of its classification on these manifolds. A link between the eigenbivector structure of the concircular tensor and Riemann curvature tensor was obtained. The classification scheme of $2-$forms was also useful in this examination. Then, these results were applied to the holonomy algebras and several properties were presented. Finally, concircularly flat space-times were investigated and by considering the known results in the literature, possible holonomy algebras for these space-times were specified.

It is noted that the result given in Lemma 3.1 is quite general and that it can be applied to all signatures and any dimension. So, these structures can be examined for the other metric signatures and $4-$dimensional manifolds. The author thinks that as an extension of this work, it would be interesting to study this tensor field on $4-$dimensional manifolds admitting a neutral metric $(+,+,-,-)$ (see, e.g., [16]) in which the classification of $2-$forms and the curvature tensor are much more complicated.

References

[1] Mikeš, J., Stepanova, E., Vanžurová, A., et al. 2015. Differential Geometry of Special Mappings. Palacký University, Olomouc.

[2] Hall, G. S. 2004. Symmetries and Curvature Structure in General Relativity. World Scientific.

[3] Yano, K. 1940. Concircular Geometry I. Concircular Transformations. Proceedings of the Imperial Academy, 16, 6, 195-200.

[4] Yano, K. 1940. Concircular Geometry II. Integrability Conditions of $\rho_{\mu\nu} = \phi g_{\mu\nu}$. Proceedings of the Imperial Academy, 16, 8, 354-360.

[5] Blair, D. E., Kim, J-S., Tripathi, M. M. 2005. On the Concircular Curvature Tensor of a Contact Metric Manifold. Journal of the Korean Mathematical Society, 42, 5, 883-892.

[6] Kühnel, W. 1988. Conformal Transformations Between Einstein Spaces. Conformal Geometry. Aspects of Mathematics / Aspekte der Mathematik, vol 12. Vieweg+Teubner Verlag, Wiesbaden, 105-146.

[7] Hong, S., Özgür, C., Tripathi, M. M. 2006. On Some Special Classes of Kenmotsu Manifolds. Kuwait Journal of Science and Engineering, 33, 2, 19-32.

[8] Hirică, I. E. 2016. Properties of Concircular Curvature Tensors on Riemann Spaces. Filomat, 30, 11, 2901-2907.

[9] Ahsan, Z., Siddiqui, S. A. 2009. Concircular Curvature Tensor and Fluid Spacetimes. International Journal of Theoretical Physics, 48, 11, 3202-3212.

[10] Olszak, K., Olszak, Z. 2012. On Pseudo-Riemannian Manifolds with Recurrent Concircular Curvature Tensor. Acta Mathematica Hungarica, 137, 1-2, 64-71.

[11] Sachs, R. K. 1961. Gravitational Waves in General Relativity. VI. The Outgoing Radiation Condition. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 264, 309-338.

[12] Kobayashi S., Nomizu K. 1963. Foundations of Differential Geometry, Interscience, vol 1., New York.

[13] Schell, J. F. 1961. Classification of Four-Dimensional Riemannian Spaces. Journal of Mathematical Physics, 2, 202-206.

[14] Hall, G. S., Lonie, D. P. 2004. Holonomy and Projective Symmetry in Spacetimes. Classical and Quantum Gravity, 21, 19, 4549-4556.

[15] Hall, G. S., Lonie, D. P. 2000. Holonomy Groups and Spacetimes. Classical and Quantum Gravity, 17, 6, 1369-1382.

[16] Wang, Z., Hall, G.S. 2013. Projective Structure in $4-$dimensional Manifolds with Metric of Signature $(+,+,-,-)$. Journal of Geometry and Physics, 66, 37-49.