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A Study on a Generalized Relaxed Curvature Energy Action

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Abstract: We investigate the variational problem of the generalized relaxed elastic line defined as the problem of finding critical points of the functional obtained by adding the twisting energy to the bending energy functional, on a non-degenerate surface in Minkowski 3-space. There arise two different situations for the curve α given on any non-degenerate surface S in Minkowski 3-space according to the absolute value expression in the curvature and torsion formulas. We study the problem for both cases and as a result we characterize the generalized relaxed elastic line with an Euler-Lagrange equation and 3 boundary conditions in both cases. Finally, we search special solutions for the differential equation system obtained with regard to the geodesic curvature, geodesic torsion and normal curvature of the curve.

Genelleştirilmiş Gevşetilmiş Eğrilik Enerji Hareketi Üzerine Bir Çalışma

Anahtar Kelimeler
Genelleştirilmiş gevşetilmiş elastik çizgi,
Euler-Lagrange denklemleri,
Varyasyonel hesap

Özet:Bükülme enerjisinin eğilme enerji fonksiyoneline eklenerek elde edildiği fonksiyonelin kritik noktalarını bulma problemi olarak tanımlanan genelleştirilmiş gevşetilmiş elastik çizgi varyasyonel problemini Minkowski 3-uzayda dejenere olmamış bir yüzey üzerinde inceledik. Minkowski 3-uzayda bulunan herhangi bir dejenere olmamış S yüzeyi üzerinde verilen α eğrisi için eğrilik ve burulma formüllerinde yer alan mutlak değer ifadesine göre iki farklı durum ortaya çıktı. Her iki durum için problemi inceledik ve sonuç olarak her iki durumda da genelleştirilmiş gevşetilmiş elastik çizgiyi bir Euler-Lagrange denklemi ve 3 sınır şart ile karakterize ettik. Son olarak eğrinin jeodezik eğriliğine, jeodezik burulmasına ve normal eğriliğine bağlı olarak elde edilen diferansiyel denklem sistemi için özel çözümler araştırdık.

1. Introduction

A relaxed elastic line is a solution of a variational problem introduced by Manning [1] in 1987 to examine mechanical features of DNA molecule. Mathematical idealization of the problem of relaxed elastic line is finding extremals of the bending energy functional $\int_0^\ell \kappa^2 ds$ under some boundary condition (see [1,2]). In [3], we have extended this problem to finding a generalized relaxed elastic line on an connected oriented surface in Euclidean 3-space. We define the generalized relaxed elastic line as a critical point of the functional which consists of the addition of twisting energy to bending energy

$$\mathscr{F} = \int_0^\ell (\kappa^2 + \lambda_2 \tau + \lambda_1) ds, \tag{1}$$

among the family of all arcs with fixed length ℓ, initial point and initial direction [3]. We consider that this problem may be useful for certain problem in non-Euclidean spaces such as Minkowski space. So, we study the problem of generalized relaxed elastic line on a connected oriented non-degenerate surface S in Minkowski 3-space $\mathbb{R}^3_1$. The main purpose of this paper is how to approach to the study of finding critical points on a non-degenerate surface S in $\mathbb{R}^3_1$, and seek what type of differences and similarities are there between both scenarios. In line with this purpose, we study the critical points corresponding to curvature energy functionals defined for non-null curves on S and derive motion equations for a generalized relaxed elastic line on S in $\mathbb{R}^3_1$. Finally,we examine whether the non-null geodesics on pseudo-plane, pseudo-sphere, pseudo-hyperbolic space and pseudo-cylinder in $\mathbb{R}^3_1$ are generalized elastic lines.

2. Basic Concepts, Variational Formulas and Partial Derivations

Let $\mathbb{R}^3_1$ denotes Minkowski 3-space with symmetric, bilinear and non-degenerate metric $\langle , \rangle$ such that for vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

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The Lorentzian vector product of x and y is defined by

$$x \wedge y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, -x_1y_2 + x_2y_1).$$

We say that a tangent vector x in $\mathbb{R}^3_1$ is a spacelike, timelike or lightlike (null) if $\langle x,x\rangle > 0$ (or $x = 0$), $\langle x,x\rangle < 0$ or $\langle x,x\rangle = 0$ and $x \neq 0$, respectively. So a curve α defined on an open interval I in $\mathbb{R}^3_1$ is a spacelike, timelike or lightlike (null) at t in I if its velocity vector $\alpha'(t)$ is a spacelike, timelike or lightlike, respectively. We consider that S is a connected oriented surface in Minkowski 3-space. Then, an immersion $\varphi : S \to \mathbb{R}^3_1$ is called spacelike, timelike or lightlike if all tangent planes $(T_pM, \varphi^* \langle,\rangle)$ are spacelike, timelike or lightlike, respectively. A surface is called as "non-degenerate" if the surface is a spacelike or timelike [4,5,6].

Let α be a non-null, smooth and regular curve on S. The Frenet frame of the curve α is given by $\{T,N,B\}$. Here T is the unit tangent vector field to α, N is the unit normal vector field and B is the binormal vector field. The derivative formulas of the Frenet frame can be written as follows

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\varepsilon}_1\kappa & 0 \\ -\varepsilon_0\kappa & 0 & \tilde{\varepsilon}_2\tau \\ 0 & -\tilde{\varepsilon}_1\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

where $\langle T,T\rangle = \varepsilon_0 = \pm 1$, $\langle N,N\rangle = \tilde{\varepsilon}_1 = \pm 1$ and $\langle B,B\rangle = \tilde{\varepsilon}_2 = \pm 1$. On the other hand $n(s)$ is the unit normal vector of S and $\varepsilon_1 Q(s) = n(s) \wedge T(s)$ is the binormal vector field, where s is the arc length parameter of α, $0 \leq s \leq \ell$, and $\varepsilon_1 = \langle Q,Q\rangle = \pm 1$. Then $\{T,Q,n\}$ is a Darboux frame on S along α with the following derivative formulas

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1\kappa_g & \varepsilon_2\kappa_n \\ -\varepsilon_0\kappa_g & 0 & \varepsilon_2\tau_g \\ -\varepsilon_0\kappa_n & -\varepsilon_1\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix} \quad (2)$$

where $\langle T,T\rangle = \varepsilon_0 = \pm 1$ and $\langle n,n\rangle = \varepsilon_2 = \pm 1$. Also, $\kappa_g = \langle T',Q\rangle$, $\kappa_n = \langle T',n\rangle$ and $\tau_g = \langle Q',n\rangle$ are the geodesic curvature, the normal curvature and the geodesic torsion of α, respectively [4, 5, 6].

From relationship between these two frame, the curvature κ and the torsion τ of α on S are found as follows

$$\kappa = \sqrt{\left|\varepsilon_1\kappa_g^2 + \varepsilon_2\kappa_n^2\right|} \quad (3)$$

and

$$\tau = \tilde{\varepsilon}_2 \left[\frac{\varepsilon_1\kappa_n^2 + \varepsilon_2\kappa_g^2}{\left|\varepsilon_1\kappa_n^2 + \varepsilon_2\kappa_g^2\right|}\tau_g + \varepsilon_1\varepsilon_2 \frac{\kappa_n'\kappa_g - \kappa_n\kappa_g'}{\left|\varepsilon_1\kappa_n^2 + \varepsilon_2\kappa_g^2\right|} \right]. \quad (4)$$

Thanks to these curvatures, it is possible to make the following definition.

Definition 1. *Any extremal of the functional*

$$\mathscr{F} = \int_0^\ell (\kappa^2 + \lambda_2\tau + \lambda_1)ds,$$

among a family of non−null curves with fixed length ℓ and initial point and direction on a non−degenerate surface S in Minkowski 3-space $\mathbb{R}^3_1$ is called a generalized relaxed elastic line.

Definition 1 shows that the problem of a generalized relaxed elastic line in $\mathbb{R}^3_1$ is an application of variational calculus because the problem concerns with finding a function for which the value of a functional is the smallest possible [7]. In this case, we usually determine a variation before beginning to solve the problem. For this reason, we first assume α lies the following coordinate path

$$X(u,v) = (x(u,v), y(u,v), z(u,v)).$$

We have the following expression for $\alpha(s) = X(u(s), v(s))$ with

$$T(s) = \alpha'(s) = \frac{du}{ds}X_u + \frac{dv}{ds}X_v$$

and

$$Q(s) = p(s)X_u + q(s)X_v$$

where $p(s)$ and $q(s)$ are some suitable scalar functions. Now we need define a family of variational arcs with fixed length ℓ. Thus we extend $\alpha(s)$ to $\alpha^*(s)$ defined for $0 \leq s \leq \ell^*$ with $\ell^* > \ell$, but sufficiently close to ℓ so that α^* lies in the coordinate path. Then, we can give the variation vector field as follow

$$\mu(s)Q(s) = \eta(s)X_u + \zeta(s)X_v \quad (6)$$

along the curve α, where $\mu(s)$ is a scalar function of class C^3, not vanishing identically. There are two restrictions placed on μ

$$\mu(0) = 0, \qquad \mu'(0) = 0 \quad (7)$$

and no other constraints on μ. Thus, the variation of α is denoted by

$$\beta(\sigma;t) = X(u(\sigma), v(\sigma)) + t(\eta(\sigma), \zeta(\sigma)), \quad (8)$$

for $0 \leq \sigma \leq \ell^*$, with $\beta(0,t) = \alpha(0)$ and $\left.\frac{\partial\beta(\sigma,t)}{\partial\sigma}\right|_{\sigma=0} = \alpha'(0)$. For $|t| < \delta_1$, (where $\delta_1 > 0$ depends upon the choice of α^* and μ) the point $\beta(\sigma;t)$ lies in the coordinate path and the variational arcs have fixed initial point and direction with α.

$\beta(\sigma;t)$, $0 \leq |t| \leq \delta \leq \delta_1$, can be constrict to an arc of length ℓ by restricting the parameter σ to an interval of $0 \leq \sigma \leq \lambda(t) \leq \ell$ by requiring

$$\int_0^{\lambda(t)} \sqrt{\left| < \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} > \right|} d\sigma = \ell,$$

where $\lambda(0) = \ell$. We next need the following derivative

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \varepsilon_0 \int_0^\ell \mu\kappa_g ds \quad (9)$$

[8].

By using (2), we obtain some partial derivatives of $\beta(\sigma;t)$ with respect to σ or t as follows

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T, \quad (10)$$

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = T' = \varepsilon_1\kappa_g Q + \varepsilon_2\kappa_n n, \tag{11}$$

$$\begin{aligned}\left.\tfrac{\partial^3\beta}{\partial\sigma^3}\right|_{t=0} = &-\varepsilon_0\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)T \\ &+\varepsilon_1\left(\kappa_g'-\varepsilon_2\kappa_n\tau_g\right)Q \\ &+\varepsilon_2\left(\varepsilon_1\kappa_g\tau_g+\kappa_n'\right)n\end{aligned} \tag{12}$$

and

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q \tag{13}$$

and some mixed derivatives of $\beta(\sigma;t)$ are obtained as

$$\left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0} = -\varepsilon_0\mu\kappa_g T + \varepsilon_2\mu\tau_g n + \mu' Q, \tag{14}$$

$$\begin{aligned}\left.\frac{\partial^3\beta}{\partial\sigma^2\partial t}\right|_{t=0} = &-\varepsilon_0\left(2\mu'\kappa_g+\mu\kappa_g'+\varepsilon_2\mu\kappa_n\tau_g\right)T \\ &+\varepsilon_2\left(2\mu'\tau_g+\mu\tau_g'-\varepsilon_0\mu\kappa_g\kappa_n\right)n \\ &+\left(\mu''-\varepsilon_0\varepsilon_1\mu\kappa_g^2-\varepsilon_1\varepsilon_2\mu\tau_g^2\right)Q\end{aligned} \tag{15}$$

and

$$\begin{aligned}\left.\frac{\partial^4\beta}{\partial\sigma^3\partial t}\right|_{t=0} = &-(3\varepsilon_0\mu''\kappa_g+3\varepsilon_0\mu'\kappa_g'+\varepsilon_0\mu\kappa_g'' \\ &+3\varepsilon_0\varepsilon_2\mu'\kappa_n\tau_g+2\varepsilon_0\varepsilon_2\mu\tau_g'\kappa_n+\varepsilon_0\varepsilon_2\mu\tau_g\kappa_n' \\ &-\varepsilon_2\mu\kappa_g\kappa_n^2-\varepsilon_1\mu\kappa_g^3-\varepsilon_0\varepsilon_1\varepsilon_2\mu\kappa_g\tau_g^2)T \\ &-(3\varepsilon_0\varepsilon_1\mu'\kappa_g^2+3\varepsilon_0\varepsilon_1\mu\kappa_g\kappa_g'+3\varepsilon_1\varepsilon_2\mu'\tau_g^2 \\ &+3\varepsilon_1\varepsilon_2\mu\tau_g\tau_g'-\mu''')Q-(3\varepsilon_0\varepsilon_2\mu'\kappa_g\kappa_n \\ &+2\varepsilon_0\varepsilon_2\mu\kappa_g'\kappa_n+\varepsilon_0\mu\kappa_n^2\tau_g-3\varepsilon_2\mu''\tau_g \\ &-3\varepsilon_2\mu'\tau_g'-\varepsilon_2\mu\tau_g''+\varepsilon_0\varepsilon_2\mu\kappa_g\kappa_n' \\ &+\varepsilon_0\varepsilon_1\varepsilon_2\mu\kappa_g^2\tau_g+\varepsilon_1\mu\tau_g^3)n.\end{aligned} \tag{16}$$

3. Motion Equations for Generalized Relaxed Elastic Lines

In this section, we investigate the generalized curvature energy action for non-null curves on a connected oriented non-degenerate surface S in Minkowski 3-space $\mathbb{R}_1^3$. We suppose that α is a smooth non-null curve on S and $\beta(\sigma;t)$, $0\le\sigma\le\lambda(t)$, $|t|<\delta$, is a variation with the variation vector field (6). Then the functional $\mathscr{F}(t)$ of arcs $\beta(\sigma;t)$ can be obtained as follows:

$$\begin{aligned}\mathscr{F}(t) = &\int_0^{\lambda(t)}\Bigg\{\Bigg|\left|<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}>\right|\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-1} \\ &-2<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial\beta}{\partial\sigma}>^2<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-3} \\ &+<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial\beta}{\partial\sigma}>^2<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>^3\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-5}\Bigg| \\ &\times\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-\frac{1}{2}}\Bigg\}d\sigma \\ &+\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\frac{\left|<\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}>\right|^{-2}<\frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^3\beta}{\partial\sigma^3}>}{\Psi}d\sigma \\ &+\lambda_1\int_0^{\lambda(t)}\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-\frac{1}{2}}d\sigma,\end{aligned}$$

where

$$\begin{aligned}\Psi = \Bigg|&\left|<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}>\right| \\ &-2<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial\beta}{\partial\sigma}>^2<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-2} \\ &+<\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial\beta}{\partial\sigma}>^2<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>^3\left|<\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}>\right|^{-5}\Bigg|.\end{aligned}$$

Assume that α be a critical value of the functional $\mathscr{F}(t)$, then we have $\frac{d\mathscr{F}}{dt}\big|_{t=0}=0$ for arbitrary μ satisfying (7) (see [7]). In calculating $\frac{dF}{dt}$, we omit

$$\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle,$$

since these terms vanishes at $t=0$ because $\langle T,T'\rangle=0$. Thus we obtain

$$\begin{aligned}\tfrac{d\mathscr{F}}{dt}=&\tfrac{d\lambda}{dt}\left\{\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-\frac{3}{2}}\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|\right\}_{\sigma=\lambda(t)} \\ &-3\int_0^{\lambda(t)}\left\langle\tfrac{\partial^2\beta}{\partial t\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle \\ &\times\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-\frac{7}{2}}d\sigma+2\int_0^{\lambda(t)}\left\langle\tfrac{\partial^3\beta}{\partial t\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle \\ &\times\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-\frac{3}{2}}d\sigma \\ &+\tilde{\varepsilon}_2\lambda_2\tfrac{d\lambda}{dt}\left\{\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-1}\left\langle\tfrac{\partial\beta}{\partial\sigma}\times\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^3\beta}{\partial\sigma^3}\right\rangle\right. \\ &\left.\times\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}\right\}_{\sigma=\lambda(t)}-2\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\left\langle\tfrac{\partial^2\beta}{\partial t\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle \\ &\times\left\langle\tfrac{\partial\beta}{\partial\sigma}\times\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^3\beta}{\partial\sigma^3}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle \\ &\times\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-3}d\sigma+\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-1} \\ &\times\left\langle\tfrac{\partial^2\beta}{\partial t\partial\sigma}\times\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^3\beta}{\partial\sigma^3}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}d\sigma \\ &+\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-1}\left\langle\tfrac{\partial\beta}{\partial\sigma}\times\tfrac{\partial^3\beta}{\partial t\partial\sigma^2},\tfrac{\partial^3\beta}{\partial\sigma^3}\right\rangle \\ &\times\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}d\sigma+\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-1} \\ &\times\left\langle\tfrac{\partial\beta}{\partial\sigma}\times\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^4\beta}{\partial t\partial\sigma^3}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-1}d\sigma \\ &-2\tilde{\varepsilon}_2\lambda_2\int_0^{\lambda(t)}\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-1}\left\langle\tfrac{\partial\beta}{\partial\sigma}\times\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^3\beta}{\partial\sigma^3}\right\rangle \\ &\times\left\langle\tfrac{\partial^3\beta}{\partial t\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\left|\left\langle\tfrac{\partial^2\beta}{\partial\sigma^2},\tfrac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|^{-3}d\sigma \\ &+\lambda_1\tfrac{d\lambda}{dt}\left\{\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-\frac{1}{2}}\right\}_{\sigma=\lambda(t)} \\ &-\lambda_1\int_0^{\lambda(t)}\left\langle\tfrac{\partial^2\beta}{\partial t\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\left|\left\langle\tfrac{\partial\beta}{\partial\sigma},\tfrac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-\frac{5}{2}}d\sigma.\end{aligned}$$

By using partial derivatives $(10-16)$, and (3), (4), (9), we obtain;

$$
\begin{aligned}
\frac{d\mathscr{F}}{dt}\Big|_{t=0} = & \int_0^\ell \mu(\varepsilon_0\kappa_g\kappa^2(\ell)+3\varepsilon_0\kappa_g\kappa^2\\
&+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}(-2\varepsilon_0\varepsilon_1\kappa_g^3-2\varepsilon_1\varepsilon_2\kappa_g\tau_g^2\\
&+2\varepsilon_2\kappa_n\tau_g'-2\varepsilon_0\varepsilon_2\kappa_g\kappa_n^2)+\lambda_2(\varepsilon_0\kappa_g\tau(\ell)\\
&+2\kappa_g\tau-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g^2\kappa_n'\kappa^{-2}-2\tilde{\varepsilon}_2\kappa_g\tau_g^3\kappa^{-2}\\
&-3\varepsilon_0\varepsilon_2\kappa_g^3\tau_g\kappa^{-2}-\tilde{\varepsilon}_2\varepsilon_1\kappa_n'\tau_g^2\kappa^{-2}-\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g'\kappa^{-2}\\
&+4\tilde{\varepsilon}_2\varepsilon_1\kappa_n\tau_g\tau_g'\kappa^{-2}+3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_n\kappa_g\kappa_g'\kappa^{-2}\\
&-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\kappa_g\kappa_n^2\tau_g\kappa^{-2}+\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g''\kappa^{-2}\\
&+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}(\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g\tau_g+2\tilde{\varepsilon}_2\varepsilon_1\kappa_g\tau_g^2\tau\kappa^{-2}\\
&+2\varepsilon_0\varepsilon_1\kappa_g^3\tau\kappa^{-2}-2\varepsilon_2\kappa_n\tau_g'\tau\kappa^{-2}+2\varepsilon_2\varepsilon_0\kappa_n^2\kappa_g\tau\kappa^{-2})\\
&+\lambda_1(\varepsilon_0+1)\kappa_g)ds+\int_0^\ell\mu'(4\varepsilon_2\kappa_n\tau_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}\\
&+\lambda_2\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}(-\tilde{\varepsilon}_2\varepsilon_0\varepsilon_2\kappa_n-4\tau_g\kappa_n\tau\kappa^{-2})\\
&-2\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g\kappa^{-2}+5\tilde{\varepsilon}_2\varepsilon_1\tau_g^2\kappa_n\kappa^{-2}\\
&+3(\varepsilon_1\varepsilon_2-1)\tilde{\varepsilon}_2\varepsilon_0\kappa_g^2\kappa_n\kappa^{-2}+3\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g'\kappa^{-2}))ds\\
&+\int_0^\ell\mu''(2\kappa_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}+\lambda_2(\tilde{\varepsilon}_2\varepsilon_2\kappa_n'\kappa^{-2}\\
&+4\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g\kappa^{-2}-2\kappa_g\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-4})ds\\
&-\lambda_2\tilde{\varepsilon}_2\varepsilon_2\int_0^\ell\mu'''\kappa_n\kappa^{-2}ds.
\end{aligned}
$$

By standart integration by parts and using (7), the first variation formula is

$$
\begin{aligned}
\frac{d\mathscr{F}}{dt}\Big|_{t=0} = & \int_0^\ell \mu(\varepsilon_0\kappa_g\kappa^2(\ell)+3\varepsilon_0\kappa_g\kappa^2\\
&+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-2}(-2\varepsilon_0\varepsilon_1\kappa_g^3-2\varepsilon_1\varepsilon_2\kappa_g\tau_g^2\\
&-2\varepsilon_2\kappa_n\tau_g'-2\varepsilon_0\varepsilon_2\kappa_g\kappa_n^2-4\varepsilon_2\kappa_n'\tau_g+2\kappa_g'')\\
&-2\left(4\varepsilon_2\kappa_n\tau_g-2\kappa_g'\right)\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-2}\\
&+2(4\varepsilon_2\kappa_n\tau_g-2\kappa_g')\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa'\\
&+4\kappa_g'(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-2}+4\kappa_g\Big(\varepsilon_1\left(\kappa_g'\right)^2\\
&+\varepsilon_1\kappa_g\kappa_g''+\varepsilon_2\left(\kappa_n'\right)^2+\varepsilon_2\kappa_n\kappa_n''\Big)\kappa^{-2}\\
&-16\kappa_g\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-3}\kappa'-4\kappa_g'(\varepsilon_1\kappa_g^2\\
&+\varepsilon_2\kappa_n^2)\kappa^{-3}\kappa'+12\kappa_g(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}\left(\kappa'\right)^2\\
&-4\kappa_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa''+\lambda_2(\varepsilon_0\kappa_g\tau(\ell)\\
&+2\kappa_g\tau-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g^2\kappa_n'\kappa^{-2}-3\varepsilon_0\varepsilon_2\kappa_g^3\tau_g\kappa^{-2}\\
&-6\tilde{\varepsilon}_2\varepsilon_1\kappa_n'\tau_g^2\kappa^{-2}-2\tilde{\varepsilon}_2\kappa_g\tau_g^3\kappa^{-2}-6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g'\kappa^{-2}\\
&-6\tilde{\varepsilon}_2\varepsilon_1\kappa_n\tau_g\tau_g'\kappa^{-2}+3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g\kappa_g'\kappa_n\kappa^{-2}\\
&+2\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g''\kappa^{-2}-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\kappa_g\kappa_n^2\tau_g\kappa^{-2}+(\varepsilon_1\kappa_g^2\\
&+\varepsilon_2\kappa_n^2)\kappa^{-2}(\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g\tau_g+2\tilde{\varepsilon}_2\varepsilon_1\kappa_g\tau_g^2\tau\kappa^{-2}\\
&+(4-2\varepsilon_2)\kappa_n\tau_g'\tau\kappa^{-2}+2\varepsilon_2\varepsilon_0\kappa_n^2\kappa_g\tau\kappa^{-2}+\tilde{\varepsilon}_2\varepsilon_0\varepsilon_2\kappa_n'\\
&+2\varepsilon_0\varepsilon_1\kappa_g^3\tau\kappa^{-2}+4\tau_g\kappa_n\tau'\kappa^{-2}+4\kappa_n'\tau_g\tau\kappa^{-2}\\
&-8\tau_g\kappa_n\tau\kappa^{-3}\kappa')-20\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g\kappa^{-3}\kappa'\\
&+6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g''\tau_g\kappa^{-2}+10\tilde{\varepsilon}_2\varepsilon_1\tau_g^2\kappa_n\kappa^{-3}\kappa'\\
&-6(\varepsilon_1\varepsilon_2-1)\tilde{\varepsilon}_2\varepsilon_0\kappa_g\kappa_g'\kappa_n\kappa^{-2}-3(\varepsilon_1\varepsilon_2-1)\tilde{\varepsilon}_2\varepsilon_0\kappa_n'\kappa^{-2}\\
&+6(\varepsilon_1\varepsilon_2-1)\varepsilon_0\kappa_g^2\kappa_n\kappa^{-3}\kappa'-10\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g'\kappa^{-3}\kappa'\\
&+2\tilde{\varepsilon}_2\varepsilon_2\kappa_n'''\kappa^{-2}-(8\varepsilon_2+2)\tilde{\varepsilon}_2\kappa_n''\kappa^{-3}\kappa'+12(\varepsilon_2\\
&+1)\tilde{\varepsilon}_2\kappa_n'\kappa^{-4}\kappa'^2-(4\varepsilon_2+4)\tilde{\varepsilon}_2\kappa_n'\kappa^{-3}\kappa''-2\kappa_g\tau''\\
&\times(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}+8\kappa_g\tau'\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa'\\
&-4\kappa_g'\tau'(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}-40\kappa_g\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-6}\kappa'^2\\
&+24\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g\kappa^{-4}\left(\kappa'\right)^2-8\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g\kappa^{-3}\kappa''\\
&+8\kappa_g\tau(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-5}\kappa''\\
&+16\varepsilon\kappa_g'\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa'\\
&+32\tau\kappa_g(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-5}\kappa'\\
&-8\tau'\kappa_g\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-4}\\
&-2\kappa_g''\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-4}+8\tau\kappa_g'(\varepsilon_1\kappa_g\kappa_g'\\
&+\varepsilon_2\kappa_n\kappa_n')\kappa^{-4}-4\tau\kappa_g(\varepsilon_1\left(\kappa_g'\right)^2+\varepsilon_1\kappa_g\kappa_g''\\
&+\varepsilon_2\left(\kappa_n'\right)^2+\varepsilon_2\kappa_n\kappa_n'')\kappa^{-4}-24\tilde{\varepsilon}_2\kappa_n\kappa^{-5}\kappa'^3\\
&+18\tilde{\varepsilon}_2\kappa_n\kappa^{-4}\kappa'\kappa''-2\tilde{\varepsilon}_2\kappa_n\kappa^{-3}\kappa'''\\
&+2(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-2}(\varepsilon_0\varepsilon_2\kappa_n+4\tau\kappa_n\tau_g\kappa^{-2})\\
&+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa'(-\varepsilon_0\varepsilon_2\tilde{\varepsilon}_2\kappa_n\\
&-4\tau\kappa_n\tau_g\kappa^{-2})+8\tau'\kappa_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa')\\
&+(\varepsilon_0+1)\lambda_1\kappa_g)ds+\mu(\ell)((4\varepsilon_2\kappa_n(\ell)\tau_g(\ell)\\
&-2\kappa_g'(\ell))\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-2}(\ell)\\
&-4\kappa_g(\ell)(\varepsilon_1\kappa_g(\ell)\kappa_g'(\ell)+\varepsilon_2\kappa_n(\ell)\kappa_n'(\ell))\kappa^{-2}(\ell)\\
&+4\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-3}(\ell)\kappa'(\ell)\\
&+\lambda_2(\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-2}(\ell)(-\tilde{\varepsilon}_2\varepsilon_0\varepsilon_2\kappa_n(\ell)\\
&-4\tau_g(\ell)\kappa_n(\ell)\tau(\ell)\kappa^{-2}(\ell))-6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'(\ell)\tau_g(\ell)\kappa^{-2}(\ell)\\
&+5\tilde{\varepsilon}_2\varepsilon_1\tau_g^2(\ell)\kappa_n(\ell)\kappa^{-2}(\ell)+3\tilde{\varepsilon}_2(\varepsilon_1\varepsilon_2-1)\\
&\times\varepsilon_0\kappa_g^2(\ell)\kappa_n(\ell)\kappa^{-2}(\ell)-\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g'(\ell)\kappa^{-2}(\ell)\\
&-2\tilde{\varepsilon}_2\varepsilon_2\kappa_n''(\ell)\kappa^{-2}(\ell)+(4\varepsilon_2+2)\tilde{\varepsilon}_2\kappa_n'(\ell)\kappa^{-3}(\ell)\kappa'(\ell)\\
&+8\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g(\ell)\kappa^{-3}(\ell)\kappa'(\ell)\\
&+2\kappa_g(\ell)\tau'(\ell)\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-4}(\ell)\\
&+2\kappa_g'(\ell)\tau(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)\\
&+4\kappa_g(\ell)\tau(\ell)(\varepsilon_1\kappa_g(\ell)\kappa_g'(\ell)+\varepsilon_2\kappa_n(\ell)\\
&\times\kappa_n'(\ell))\kappa^{-4}(\ell)-8\kappa_g(\ell)\tau(\ell)(\varepsilon_1\kappa_g^2(\ell)\\
&+\varepsilon_2\kappa_n^2(\ell))\kappa^{-5}(\ell)\kappa'(\ell)-6\tilde{\varepsilon}_2\kappa_n(\ell)\kappa^{-4}(\ell)\kappa'^2(\ell)\\
&+2\tilde{\varepsilon}_2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa''(\ell)))+\mu'(\ell)(2\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)\\
&+\varepsilon_2\kappa_n^2(\ell))\kappa^{-2}(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)\\
&+\lambda_2(2\tilde{\varepsilon}_2\varepsilon_2\kappa_n'(\ell)\kappa^{-2}(\ell)+4\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g(\ell)\kappa^{-2}(\ell)\\
&-2\tau(\ell)\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)\\
&-2\tilde{\varepsilon}_2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa'(\ell)))-\tilde{\varepsilon}_2\varepsilon_2\lambda_2\kappa_n(\ell)\kappa^{-2}(\ell)\mu''(\ell)\\
&+\tilde{\varepsilon}_2\varepsilon_2\lambda_2\kappa_n(0)\kappa^{-2}(0)\mu''(0).
\end{aligned}
\tag{17}
$$

The reader should note that κ and τ in Eq. (17) are cosidered as in Eqs. (3) and (4). Here we use the left side of the equations for writing simplicity. Now we suppose that α is a critical point of the functional (1) for all functions μ satisfying (7) with arbitrary values of $\mu(\ell)$ and $\mu'(\ell)$, then it must satisfy the following three boundary conditions;

$$
\lambda_2(\kappa_n(\ell)\kappa^{-2}(\ell)\mu''(\ell)-\kappa_n(0)\kappa^{-2}(0)\mu''(0))=0, \tag{18}
$$

$$
\begin{aligned}
&2\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-2}(\ell)\\
&+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)+\lambda_2(2\tilde{\varepsilon}_2\varepsilon_2\kappa_n'(\ell)\kappa^{-2}(\ell)\\
&+4\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g(\ell)\kappa^{-2}(\ell)\\
&-2\tau(\ell)\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)\\
&-2\tilde{\varepsilon}_2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa'(\ell))=0,
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&(4\varepsilon_2\kappa_n(\ell)\tau_g(\ell)-2\kappa_g'(\ell))\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-2}(\ell)\\
&-4\kappa_g(\ell)(\varepsilon_1\kappa_g(\ell)\kappa_g'(\ell)+\varepsilon_2\kappa_n(\ell)\kappa_n'(\ell))\kappa^{-2}(\ell)\\
&+4\kappa_g(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-3}(\ell)\kappa'(\ell)\\
&+\lambda_2(\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-2}(\ell)(-\tilde{\varepsilon}_2\varepsilon_0\varepsilon_2\kappa_n(\ell)\\
&-4\tau_g(\ell)\kappa_n(\ell)\tau(\ell)\kappa^{-2}(\ell))-6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'(\ell)\tau_g(\ell)\kappa^{-2}(\ell)\\
&+5\tilde{\varepsilon}_2\varepsilon_1\tau_g^2(\ell)\kappa_n(\ell)\kappa^{-2}(\ell)+3\tilde{\varepsilon}_2(\varepsilon_1\varepsilon_2-1)\\
&\times\varepsilon_0\kappa_g^2(\ell)\kappa_n(\ell)\kappa^{-2}(\ell)-\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g'(\ell)\kappa^{-2}(\ell)\\
&-2\tilde{\varepsilon}_2\varepsilon_2\kappa_n''(\ell)\kappa^{-2}(\ell)+(4\varepsilon_2+2)\tilde{\varepsilon}_2\kappa_n'(\ell)\kappa^{-3}(\ell)\kappa'(\ell)\\
&+8\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g(\ell)\tau_g(\ell)\kappa^{-3}(\ell)\kappa'(\ell)\\
&+2\kappa_g(\ell)\tau'(\ell)\left(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell)\right)\kappa^{-4}(\ell)\\
&+2\kappa_g'(\ell)\tau(\ell)(\varepsilon_1\kappa_g^2(\ell)+\varepsilon_2\kappa_n^2(\ell))\kappa^{-4}(\ell)\\
&+4\kappa_g(\ell)\tau(\ell)(\varepsilon_1\kappa_g(\ell)\kappa_g'(\ell)+\varepsilon_2\kappa_n(\ell)\\
&\times\kappa_n'(\ell))\kappa^{-4}(\ell)-8\kappa_g(\ell)\tau(\ell)(\varepsilon_1\kappa_g^2(\ell)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon_2\kappa_n^2(\ell))\kappa^{-5}(\ell)\kappa'(\ell)-6\tilde{\varepsilon}_2\kappa_n(\ell)\\
&\times\kappa^{-4}(\ell)\kappa'^2(\ell)+2\tilde{\varepsilon}_2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa''(\ell))=0,
\end{aligned}
\tag{20}
$$

and the differential equation at the free end:

$$
\begin{aligned}
&\varepsilon_0\kappa_g\kappa^2(\ell)+3\varepsilon_0\kappa_g\kappa^2+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\\
&\times\kappa^{-2}(-2\varepsilon_0\varepsilon_1\kappa_g^3-2\varepsilon_1\varepsilon_2\kappa_g\tau_g^2\\
&-2\varepsilon_2\kappa_n\tau_g'-2\varepsilon_0\varepsilon_2\kappa_g\kappa_n^2-4\varepsilon_2\kappa_n'\tau_g+2\kappa_g'')\\
&-2\left(4\varepsilon_2\kappa_n\tau_g-2\kappa_g'\right)\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-2}\\
&+2(4\varepsilon_2\kappa_n\tau_g-2\kappa_g')\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa'\\
&+4\kappa_g'(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-2}+4\kappa_g\Big(\varepsilon_1\left(\kappa_g'\right)^2\\
&+\varepsilon_1\kappa_g\kappa_g''+\varepsilon_2\left(\kappa_n'\right)^2+\varepsilon_2\kappa_n\kappa_n''\Big)\kappa^{-2}\\
&-16\kappa_g\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-3}\kappa'-4\kappa_g'(\varepsilon_1\kappa_g^2\\
&+\varepsilon_2\kappa_n^2)\kappa^{-3}\kappa'+12\kappa_g(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}(\kappa')^2\\
&-4\kappa_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa''+\lambda_2(\varepsilon_0\kappa_g\tau(\ell)\\
&+2\kappa_g\tau-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g^2\kappa_n'\kappa^{-2}-3\varepsilon_0\varepsilon_2\kappa_g^3\tau_g\kappa^{-2}\\
&-6\tilde{\varepsilon}_2\varepsilon_1\kappa_n'\tau_g^2\kappa^{-2}-2\tilde{\varepsilon}_2\kappa_g\tau_g^3\kappa^{-2}-6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g'\kappa^{-2}\\
&-6\tilde{\varepsilon}_2\varepsilon_1\kappa_n\tau_g\tau_g'\kappa^{-2}+3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g\kappa_g'\kappa_n\kappa^{-2}\\
&+2\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g''\kappa^{-2}-3\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\kappa_g\kappa_n^2\tau_g\kappa^{-2}+(\varepsilon_1\kappa_g^2\\
&+\varepsilon_2\kappa_n^2)\kappa^{-2}(\tilde{\varepsilon}_2\varepsilon_0\varepsilon_1\varepsilon_2\kappa_g\tau_g+2\tilde{\varepsilon}_2\varepsilon_1\kappa_g\tau_g^2\tau\kappa^{-2}\\
&+(4-2\varepsilon_2)\kappa_n\tau_g'\tau\kappa^{-2}+2\varepsilon_2\varepsilon_0\kappa_n^2\kappa_g\tau\kappa^{-2}+\tilde{\varepsilon}_2\varepsilon_0\varepsilon_2\kappa_n'\\
&+2\varepsilon_0\varepsilon_1\kappa_g^3\tau\kappa^{-2}+4\tau_g\kappa_n\tau'\kappa^{-2}+4\kappa_n'\tau_g\tau\kappa^{-2}\\
&-8\tau_g\kappa_n\tau\kappa^{-3}\kappa')-20\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g'\tau_g\kappa^{-3}\kappa'\\
&+6\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g''\tau_g\kappa^{-2}+10\tilde{\varepsilon}_2\varepsilon_1\tau_g^2\kappa_n\kappa^{-3}\kappa'\\
&-6(\varepsilon_1\varepsilon_2-1)\tilde{\varepsilon}_2\varepsilon_0\kappa_g\kappa_g'\kappa_n\kappa^{-2}-3(\varepsilon_1\varepsilon_2-1)\tilde{\varepsilon}_2\varepsilon_0\kappa_n'\kappa^{-2}\\
&+6(\varepsilon_1\varepsilon_2-1)\varepsilon_0\kappa_g^2\kappa_n\kappa^{-3}\kappa'-10\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g'\kappa^{-3}\kappa'\\
&+2\tilde{\varepsilon}_2\varepsilon_2\kappa_n'''\kappa^{-2}-(8\varepsilon_2+2)\tilde{\varepsilon}_2\kappa_n''\kappa^{-3}\kappa'+12(\varepsilon_2\\
&+1)\tilde{\varepsilon}_2\kappa_n'\kappa^{-4}\kappa'^2-(4\varepsilon_2+4)\tilde{\varepsilon}_2\kappa_n'\kappa^{-3}\kappa''-2\kappa_g\tau''\\
&\times(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}+8\kappa_g\tau'\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa'\\
&-4\kappa_g'\tau'(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-4}-40\kappa_g\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-6}\kappa'^2\\
&+24\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g\kappa^{-4}(\kappa')^2-8\tilde{\varepsilon}_2\varepsilon_1\varepsilon_2\kappa_g\tau_g\kappa^{-3}\kappa''\\
&+8\kappa_g\tau(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2)\kappa^{-5}\kappa''\\
&+16\varepsilon\kappa_g'\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa'\\
&+32\tau\kappa_g(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-5}\kappa'\\
&-8\tau'\kappa_g\left(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n'\right)\kappa^{-4}\\
&-2\kappa_g''\tau\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-4}+8\tau\kappa_g'(\varepsilon_1\kappa_g\kappa_g'\\
&+\varepsilon_2\kappa_n\kappa_n')\kappa^{-4}-4\tau\kappa_g(\varepsilon_1\left(\kappa_g'\right)^2+\varepsilon_1\kappa_g\kappa_g''\\
&+\varepsilon_2\left(\kappa_n'\right)^2+\varepsilon_2\kappa_n\kappa_n'')\kappa^{-4}-24\tilde{\varepsilon}_2\kappa_n\kappa^{-5}\kappa'^3\\
&+18\tilde{\varepsilon}_2\kappa_n\kappa^{-4}\kappa'\kappa''-2\tilde{\varepsilon}_2\kappa_n\kappa^{-3}\kappa'''\\
&+2(\varepsilon_1\kappa_g\kappa_g'+\varepsilon_2\kappa_n\kappa_n')\kappa^{-2}(\varepsilon_0\varepsilon_2\kappa_n+4\tau\kappa_n\tau_g\kappa^{-2})\\
&+\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-3}\kappa'(-\varepsilon_0\varepsilon_2\tilde{\varepsilon}_2\kappa_n\\
&-4\tau\kappa_n\tau_g\kappa^{-2})+8\tau'\kappa_g\left(\varepsilon_1\kappa_g^2+\varepsilon_2\kappa_n^2\right)\kappa^{-5}\kappa')\\
&+(\varepsilon_0+1)\lambda_1\kappa_g=0.
\end{aligned}
\tag{21}
$$

The above computation give rise to the following theorem.

Theorem 1 *Any generalized relaxed elastic line of length ℓ on a non-degenerate surface S in Minkowski 3-space $\mathbb{R}_1^3$ is characterized by the Euler-Lagrange equation* (21) *with the boundary conditions* (18), (19) *and* (20).

4. Applications

Let P be a vector plane in $\mathbb{R}_1^3$. Then P is a spacelike (resp. timelike, lightlike) plane if and only if n is a timelike (resp. spacelike, lightlike) vector, where n is an orthogonal vector to P [5]. P is known a pseudo-plane in $\mathbb{R}_1^3$. We also recall that the pseudo-sphere of radius $r>0$ in $\mathbb{R}_1^3$ is the hyperquadric

$$S_1^2(r)=\left\{\, p\in\mathbb{R}_1^3 \middle|\, <p,p>=r^2\right\}$$

and the pseudo-hyperbolic space of radius $r>0$ in $\mathbb{R}_1^3$ is the hyperquadric

$$H_0^2(r)=\left\{\, p\in\mathbb{R}_1^3 \middle|\, <p,p>=-r^2\right\}$$

[6]. On the other hand, we call the set of

$$C_1^2(r)=\left\{\,(x,y,z)\in\mathbb{R}_1^3 \middle|\, y^2-z^2=r^2,\ x\in\mathbb{R}\right\}$$

as a pseudo-cylinder [6,8]. In this section we examine whether the non-null geodesics of pseudo-plane, pseudo-sphere $S_1^2(r)$, hyperbolic space $H_0^2(r)$ and pseudo-cylinder $C_1^2(r)$ are generalized relaxed elastic lines.

Example 1 *The geodesic torsion τ_g and the normal curvature κ_n vanishes for all curves in the pseudo-plane P.*

This gives rise to the following corollary.

Corollary 1 *Any non-null geodesic on pseudo-plane P is a generalized relaxed elastic line.*

Example 2 *The pseudo-sphere $S_1^2(r)$ and the hyperbolic space $H_0^2(r)$ are non-degenerate surfaces in Minkowski 3-space $\mathbb{R}_1^3$. The pseudo-sphere $S_1^2(r)$ is a timelike surface and the hyperbolic space $H_0^2(r)$ is spacelike surface [8]. The geodesic torsion τ_g is zero and square of the normal curvature κ_n^2 is $\frac{1}{r^2}$ for all curves on $S_1^2(r)$ and $H_0^2(r)$.*

Thus, we can give the following corollary.

Corollary 2 *Any non-null geodesics of $S_1^2(r)$ and $H_0^2(r)$ are generalized relaxed elastic line in case of $\lambda_2=0$, but they are not generalized relaxed elastic line in case of $\lambda_2\neq 0$.*

Example 3 *The pseudo-cylinder $C_1^2(r)$ is a timelike surface in Minkowski 3-space and parametrized by*

$$X(u,v)=\left(u,r\cosh\left(\frac{v}{r}\right),r\sinh\left(\frac{v}{r}\right)\right),$$

where r is the radius of the lorentz circle [8].

Then, for a timelike curve α on $C_1^2(r)$ the geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g are

$$\kappa_g=-\frac{d\theta}{ds},\ \kappa_n=-\frac{1}{r}\cosh^2\theta \text{ and } \tau_g=\frac{1}{r}\sinh\theta\cosh\theta,$$

respectively, where θ is the hyperbolic angle between the $v-$ coordinate curve through $\alpha(s)$and the curve α.

Then, we have the following result;

Corollary 3 *If $\theta=0$, a timelike geodesic of the pseudo-cylinder is a generalized relaxed elastic line when $\lambda_2=0$, but it is not when $\lambda_2\neq 0$.*

For a spacelike curve α on $C_1^2(r)$, the geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g are

$$\kappa_g = \frac{d\theta}{ds},\ \kappa_n = -\frac{1}{r}\cos^2\theta \text{ and } \tau_g = \frac{1}{r}\sin\theta\cos\theta,$$

respectively, where θ *is the angle between the* $u-$*coordinate curve through* $\alpha(s)$ *and the curve* α.

Similarly we obtain the following result;

Corollary 4 *If* $\theta = 0$ *and* $\theta = \pm\pi$*, spacelike geodesics of the pseudo-cylinder are generalized relaxed elastic lines when* $\lambda_2 = 0$*, but they are not when* $\lambda_2 \neq 0$*. If* $\theta = \pm\frac{\pi}{2}$*, spacelike geodesics of the pseudo-cylinder are generalized relaxed elastic lines.*

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