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α-Topological Vector Spaces

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Abstract

The notion of $\alpha-$topological vector space is introduced and several properties are studied. A complete comparison between this class and the class of topological vector spaces is presented. In particular, $\alpha-$topological vector spaces are shown to be independent from topological vector spaces. Finally, a sufficient condition for α–regularity of $\alpha-$topological vector spaces is given.

Keywords: *α-open set, $\alpha-$irresolute map, $\alpha-$topological vector space.*

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1. Introduction

A topological vector space (TVS) is a vector space with a topological structure such that the algebraic operations; addition and scalar multiplication, are continuous, see for example Jarchow [15] and Köthe [16], Al-hawary and Al-Nayef [8, 7]. The theory of topological vector spaces often clarifies results in many branches of functional analysis such as the theory of normed spaces. Let $(X, \mathfrak{T})$ be a topological space. A subset $A \subseteq X$ is α-open if $A \subseteq \overline{A^o}^o$, where $\bar{A}$ denotes the closure of A in X and A^o denotes the interior of A. The collection of all α-open sets in $(X,\mathfrak{T})$ is denoted by $\alpha O(X)$ and the pair $(X, \alpha O(X))$ is called the *α-topological space* associated with $(X,\mathfrak{T})$. We remark that $(X, \alpha O(X))$ is a topological space.

Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{T}^{'})$ be topological spaces. A map $f : X \to Y$ is *$\alpha-$irresolute* if the inverse image of every α-open set in Y is α-open in X, see Maheshwari and Thakur [17] and Takashi [18]. A map $f : X \to Y$ is *pre-α-open* if the image of any α-open set in X is α-open in Y. Two topological spaces $(X,\mathfrak{T})$ and $(Y,\mathfrak{T}^{'})$ are *α-homeomorphic* if there exists a map $h : X \to Y$ which is bijective, α-irresolute and pre-α-open. Such an h is called *α-homeomorphism*. We shall call a map $f : X \to Y$ *inverse α–continuous* if the inverse of every α–open subset of Y is open in X. We refer the reader interested in more

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details about the preceding notions to Al-Hawary [9, 10, 5, 6, 11, 2, 1, 3, 4], Crossley and Hildebrand [12, 13], Maheshwari and Thakur [17] and Takashi [18].

Our main goal in this paper is to introduce and study properties of what we call $\alpha-$*topological vector spaces* (αTVSs). This concept is shown to be totally different from the concept of TVS. We give necessary and sufficient conditions for each of them to imply the other. Section 2 is devoted to discuss the relation between TVSs and αTVSs, while in Section 3, several properties of αTVSs are discussed. In Section 4, we define the notion of *strong α–topological vector space* (SαTVS) and show that it is a stronger notion than that of αTVS and every SαTVS is *α–regular*.

We next recall two necessary results:

1.1. Lemma. [18] *Let X and Y be topological spaces. Then*

a) Every $\alpha-$irresolute map $f : X \to Y$ is $\alpha-$continuous, but the converse need not be true.

b) A subset A of X is α-open if and only if there exists an open set V in X such that $V \subseteq A \subseteq \overline{V}^o$.

The proof of the following fundamental result follows from the definitions.

1.2. Lemma. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^{'})$ be topological spaces. Then a subset $U \subseteq X \times Y$ is an α-open set (with respect to the product topology) if and only if $U = A \times B$ where A and B are α-open sets in X and Y, respectively.*

2. αTVSs versus TVSs

We begin this section with the definition of the notion of an $\alpha-$topological vector space.

2.1. Definition. Let X be a vector space over the field of real numbers, and let $\mathcal{T}$ be a topology on X such that the addition map $S_X : X \times X \to X$ and the scalar multiplication map $M_X : \mathbb{R} \times X \to X$ are $\alpha-$irresolute. Then $(X, \alpha O(X))$ is called *$\alpha-$topological vector space (αTVS).*

If every subset of a topological vector space is open (closed) or every α-open set is clopen, then clearly every αTVS is a TVS. By Lemma 1.1 (a), an $\alpha-$irresolute map need not be α-continuous and hence need not be continuous. Thus an αTVS need not be a TVS.

Next, an interesting example of an αTVS will be given. For that, we prove the following result:

2.2. Lemma. *Let $f : X \to Y$ be a continuous and open map. Then f is α-irresolute.*

Proof. For every α-open subset of Y, by Lemma 1.1, there exists an open set V in Y such that $V \subseteq A \subseteq \overline{V}^o$. Thus $f^{-1}(V) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{V}^o)$. As f is continuous, $f^{-1}(V)$ is open in X. We show that $f^{-1}(A) \subseteq \overline{f^{-1}(V)}^o$ by showing that $f^{-1}(\overline{V}^o) \subseteq \overline{f^{-1}(V)}^o$. For every $x \in f^{-1}(\overline{V}^o)$, $f(x) \in \overline{V}^o$ and so there exists an open set U in Y such that $f(x) \in U \subseteq \overline{V}$. Now; for every $y \in f^{-1}(U)$ and for every open subset W of X such that $y \in W$, $f(y) \in f(W)$ which is open in Y as f is an open map and as $f(y) \in \bar{V}$, $V \cap f(W) \neq \varnothing$. Hence there exists $z \in V \cap f(W)$ and so $f^{-1}(z) \in f^{-1}(V) \cap W$. That is $f^{-1}(z) \in \overline{f^{-1}(V)}$. Therefore by Lemma 1.1 (b), $f^{-1}(A)$ is $\alpha-$open. □

2.3. Example. Consider $X = \mathbb{R}$ with the usual topology $\mathcal{T}_u$. Then clearly $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ are continuous and open by the Open Mapping Theorem. Hence by Lemma 1.1 (b), $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ are $\alpha-$irresolute maps and consequently $(X, \alpha O(\mathbb{R}))$ is an αTVS.

Another sufficient condition for an αTVS to be a TVS will be given next, but first we need the following definition and lemma.

2.4. Definition. [14] Let $(X,\mathfrak{T})$ be a topological space. A subset $S \subseteq X$ is *locally closed* if $S = U \cap F$ where U is open and F is closed.

2.5. Lemma. [14] *Let $(X,\mathfrak{T})$ be a topological space. A subset $S \subseteq X$ is open if and only if it is α-open and locally closed.*

2.6. Corollary. *Let $(X, \alpha O(X))$ be an αTVS in which every α-open set is locally closed. Then $(X, \alpha O(X))$ is a TVS.*

Note that every TVS equipped with the discrete topology is an αTVS, but in general a TVS need not be an αTVS, for example $(\mathbb{R}, \mathfrak{T}_u)$ is a TVS which is not an αTVS. In [14], a subset A of a topological space $(X, \mathfrak{T})$ is an $A-$set if $A = U \cap F$ where U is open and $F = \overline{F^o}$.

Next we recall the following result.

2.7. Lemma. [14] *For a map $f : X \to Y$, the following are equivalent:*

(a) f is continuous.

(b) f is precontinuous and LC-continuous.

(c) f is precontinuous and A–continuous.

The following main result follows from Lemma 1.1.

2.8. Theorem. *Let $(X, \mathfrak{T})$ be a TVS such that the maps S_X and M_X are open. Then $(X, \alpha O(X))$ is an αTVS if S_X and M_X satisfy any of the three equivalent statements in Lemma 2.7.*

3. Properties of αTVSs

Recall that if $(X, \alpha O(X))$ is an αTVS, then by an α–neighborhood (neighborhood) of an element $x \in X$ we mean any subset of X that includes an α–open set (open set). The set of all $\alpha-$neighborhoods of $x \in X$ will be denoted by $N_x(X)$. In particular, $N_0(X)$ denotes the set of all $\alpha-$neighborhoods of 0 (the zero element of X). The following result is immediate from the fact that every open set is $\alpha-$open and that the set of all α-open sets on a space X is a topology.

3.1. Lemma. *If S is any α-open subset of a topological space $(X, \mathfrak{T})$ and $W \subseteq X$ is open, then $S \bigcap W$ is α–open.*

A fundamental result in which four basic neighborhood properties is given next.

3.2. Theorem. *Let $(X, \alpha O(X))$ be an αTVS. Then*

(a) If $U \in N_x(X)$ is an α-neighborhood of a point $x \in X$, then $x \in U$.

(b) If $U \in N_x(X)$ is an α-neighborhood of a point $x \in X$ and V a neighborhood of x, then $U \cap V$ is an α-neighborhood of x.

(c) If $U \in N_x(X)$ is an α-neighborhood of a point $x \in X$, then there exists an $\alpha-$neighborhood $V \in N_x(X)$ of x such that $U \in N_y(X)$ is an α-neighborhood of y, for every $y \in V$.

(d) If $U \in N_x(X)$ is an α-neighborhood of a point $x \in X$ and $U \subseteq V$, then $V \in N_x(X)$ is an α-neighborhood of x.

Proof. Only the proof of Part (b) is given. The proofs of other parts are similar and hence omitted. If $U \in N_x(X)$ and V is a neighborhood of x, then there exists an α–open set S and an open set W such that $x \in S \subseteq U$ and $x \in W \subseteq V$. Then $x \in S \bigcap W \subseteq U \bigcap V$ and by Lemma 3.1, $S \bigcap W$ is α–open. Therefore , $(U \bigcap V) \in N_x(X)$ is an α–neighborhood of x. □

We have shown in Part (b) in the preceding theorem that the intersection of an α–neighborhood of $x \in X$ with a neighborhood of x is an α–neighborhood of x. We recall that a property $\mathfrak{A}$ of an αTVS X is called a *α-topological property* if every α-homeomorphic αTVS Y to X also achieves the property $\mathfrak{A}$. It follows from the next Lemma that having an α-neighborhood is an α-topological property. The proof of the Lemma is straightforward.

3.3. Lemma. *Let $(X, \alpha O(X))$ and $(Y, \alpha O(Y))$ be $\alpha TVSs$ and let $f : X \to Y$ be an $\alpha-$homeomorphism. A subset U of X is an $\alpha-$neighborhood of $x \in X$ if and only if $f(U)$ is an $\alpha-$neighborhood of $f(x)$.*

Proof. If U is an α-neighborhood of $x \in X$, then there exists an $\alpha-$open V such that $x \in V \subseteq U$. Thus $f(x) \in f(V) \subseteq f(U)$ and since f is pre α-open, $f(V)$ is α-open in Y. Thus $f(U)$ is an α-neighborhood of $f(x)$.

Conversely, if $f(U)$ is an α-neighborhood of $f(x)$, then there exists an α-open W in Y such that $f(x) \in W \subseteq f(U)$. Hence $x \in f^{-1}(W) \subseteq U$ and as f is $\alpha-$irresolute, $f^{-1}(W)$ is α-open. Therefore, U is an α-neighborhood of x. □

The gist of the following theorem is that the α-topological structure of an αTVS at the point $x \in X$ is determined by $\alpha-$neighborhoods of 0.

3.4. Theorem. *Let $(X, \alpha O(X))$ be an αTVS and $y \in X$. Then*

(a) For $U \subseteq X$, $U \in N_0(X)$ if and only if $y + U \in N_y(X)$.

(b) If $U \in N_0(X)$, then $tU \in N_0(X)$ for all scalars $t \in \mathbb{R} \setminus \{0\}$.

Proof. To prove Part (a), note that by assumption, the map S_X is $\alpha-$irresolute. Define the map $f_y : X \to X$ by $f_y(x) = x + y$. Then since $f_y(x) = S_X(x, y)$, f_y is $\alpha-$irresolute and as $f_y^{-1}(x) = S_X(x, -y)$, f_y^{-1} is also $\alpha-$irresolute. Therefore, f_y is an $\alpha-$homeomorphism. The proof of this part is completed by applying Lemma 3.3.

A similar argument using the map $g_t : X \to X$ defined by $g_t(x) = tx$ can be used to establish Part (b). □

Recall that a subset A of a vector space X is called *balanced* if $tA \subseteq A$ for $|t| \leq 1$; and *absorbing* if for every $x \in X$, there exists $\epsilon > 0$ such that $tx \in A$ for $|t| < \epsilon$.

3.5. Theorem. *Let $(X, \alpha O(X))$ be an αTVS. Then every $U \in N_0(X)$ is absorbing.*

Proof. Let $U \in N_0(X)$. Then there exists an $\alpha-$open set $U_1 \in N_0(X)$ such that $U_1 \subseteq U$. By assumption, the scalar map $M_X : \mathbb{R} \times X \to X$ is $\alpha-$irresolute. Therefore, there exist $\alpha-$open sets $V_1 \in N_0(\mathbb{R})$ and $V_2 \in N_0(X)$ such that $M_X(V_1 \times V_2) \subseteq U_1$. The set V_1 contains an open interval of the form $(-\varepsilon, \varepsilon)$ for an $\varepsilon > 0$, and thus $tx \in U_1$ for all $t \in (-\varepsilon, \varepsilon)$ and for all $x \in V_2$. This shows that U_1 is absorbing. □

An αTVS is called *regular* (*$\alpha-$regular*) if each $\alpha-$neighborhood of the origin contains a closed ($\alpha-$closed) neighborhood of the origin. Clearly, every regular αTVS is $\alpha-$regular, while the converse holds if every $\alpha-$open set in X is locally closed (an intersection of an open set with a closed set).

Next, we characterize the $\alpha-$closure of subsets of an αTVS. For that, we need the following result whose proof is an easy consequence of the definition.

3.6. Lemma. *Let $(X, \alpha O(X))$ be an αTVS, $x \in X$ and $A \subseteq X$. Then $x \in \alpha CL(A)$ if and only if A intersects every $U \in N_x(X)$.*

3.7. Theorem. *Let $(X, \alpha O(X))$ be an αTVS. Then*

(a) For every $U \in N_0(X)$, there exists a balanced $V \in N_0(X)$ such that $V \subseteq U$.

(b) If $A \subseteq X$, then $\alpha CL(A) = \bigcap_{U \in N_0(X)} (A + U)$.

(c) If $A \subseteq X$, then $\alpha CL(A) \subseteq A + U$, for all $U \in N_0(X)$.

Proof. For the proof of Part (a), note that by assumption, the map $M_X : \mathbb{R} \times X \to X$ is $\alpha-$irresolute, and hence for every $U \in N_0(X)$ there exists $V \in N_0(\mathbb{R} \times X)$ such that $M_X(V) \subseteq U$. Thus, there exists $\varepsilon > 0$ such that $V = V_1 \times V_2$, $(-\varepsilon, \varepsilon) \subseteq V_1 \in N_0(\mathbb{R})$ and $V_2 \in N_0(X)$. Define $W := \bigcup_{|t|<\varepsilon} tV_2$ and note that by Part (b) of Theorem 3.4, $tV_2 \in N_0(X)$ for $t \neq 0$ and $tV_2 \subseteq U$ for all $|t| < \varepsilon$. It remains to show that W is balanced. If $|s| \leq 1$ then $sW = \bigcup_{|t|<\varepsilon} (st)V_2$. Since $|st| < \varepsilon|s| < \varepsilon$, it follows that $sW = \cup_{|r|<\varepsilon} r\, V_2 \subseteq W$, with $r = s\, t$. Thus, W is balanced.

To prove Part (b) , let $x \in \alpha CL(A)$. If $U \in N_0(X)$, then by Part (a) there exists a balanced neighborhood $V \in N_0(X)$ such that $V \subseteq U$. Therefore, $x + V \in N_x(X)$ and by Lemma 3.6, $(x + V) \cap A \neq \emptyset$, which implies that $x \in A - V$. But since V is balanced, $A - V = A + V$ and hence $x \in A + V \subseteq A + U$. This shows that $\alpha CL(A) \subseteq \bigcap_{U \in N_0(X)}(A + U)$. For the other direction, we prove the contrapositive: If $x \notin \alpha CL(A)$, then there exists a balanced $U \in N_0(X)$ such that $(x + U) \cap A = \emptyset$. Consequently, $x \notin A - U = A + U$. This completes the proof of Part (b).

The proof of Part (c) follows immediately from Part (b). □

4. $\alpha-$regular SαTVSs

We define *strong $\alpha-$irresolute–topological vector space* (SαTVS) $(X, \alpha O(X))$ to be an αTVS in which the addition map $M_X : X \times X \to X$ is inverse α–continuous. Note that every SαTVS is an αTVS. But, the converse need not be true, for example $(\mathbb{R}, \mathcal{T}_\ell)$ where $\mathcal{T}_\ell$ denotes the the left–ray topology is an αTVS, but as the addition map is not continuous, it is not an SαTVS. Thus, the notion of SαTVS is stronger than the notion of αTVS.

The following Lemma is needed to prove our main result of this section in Theorem 4.2.

4.1. Lemma. *If $(X, \alpha O(X))$ is an $S\alpha TVS$, then for every $U \in N_0(X)$, there exists $V \in N_0(X)$ such that $V + V \subseteq U$.*

Proof. Let $U \in N_0(X)$. Since the map $S_X : X \times X \to X$ is inverse α–continuous, there exist open sets V_1 and V_2 in $N_0(X)$ such that $S_X(V_1, V_2) \subseteq U$, that is $V_1 + V_2 \subseteq U$. Let $V := (V_1 \bigcap V_2)$. The set V is open, and hence α-open with the property that $V \subseteq V_1$ and $V \subseteq V_2$, thus, we have $V + V \subseteq V_1 + V_2 \subset U$. □

4.2. Theorem. *Let $(X, \alpha O(X))$ be an $S\alpha TVS$. Then for every $U \in N_0(X)$, there exists an $\alpha-$closed balanced $V \in N_0(X)$ such that $V \subseteq U$. Hence every $S\alpha TVS$ is $\alpha-$regular.*

Proof. Let $U \in N_0(X)$. By Lemma 4.1, there exists $V \in N_0(X)$ such that $V + V \subseteq U$. By Part (a) of Theorem 3.7, there exists a balanced neighborhood $W \in N_0(X)$ such that $W \subseteq V$. But by Part (c) of Theorem 3.7, $\alpha CL(W) \subseteq W + V$. Finally, $\alpha CL(W) \subseteq W + V \subseteq V + V \subseteq U$. This shows that U contains the α-closed neighborhood $\alpha CL(W) \in N_0(X)$. Thus X is α–regular. □

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