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Averaged modulus of smoothness and two-sided monotone approximation in Orlicz spaces

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Abstract

The paper deals with basic properties of averaged modulus of smoothness in Orlicz spaces L^*_φ. Some direct and inverse two-sided approximation problems in L^*_φ are proved. In the last section, some inequalities concerning monotone two sided approximation by trigonometric polynomials in L^*_φ are considered.

Keywords: Orlicz spaces, Averaged modulus of smoothness.

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1. Introduction

The problems of approximation by trigonometric or algebraic polynomials in classical Orlicz spaces were investigated by several mathematicians. In 1966, Tsyganok [26] obtained the Jackson type inequality of trigonometric approximation. In 1965, Kokilashvili [17] obtained inverse theorems of trigonometric approximation. In 1966, Ponomarenko [20] proved some direct theorem of trigonometric approximation by summation means of Fourier series. In 1968, Cohen [9] proved some direct theorem of trigonometric approximation by its partial sum of Fourier series. In Orlicz spaces when the generating Young function satisfying quasiconvexity condition similar problems were investigated by Akgün, Israfilov, Jafarov, Koç, Ramazanov and others [1, 2, 3, 5, 4, 11, 14, 15, 12, 13, 16, 21].

On the other hand, monotone approximation of functions by trigonometric polynomials [23] and Jackson type theorems for monotone approximation of functions by trigonometric polynomials in the classical Lebesgue spaces L_p [25] were proved by Shadrin. Ganelius [10], Babenko and Ligun [8], Sadrin [23] proved theorems about one sided approximation by trigonometric polynomials for functions in L_p-metric.

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In this paper firstly we give basic properties of averaged modulus of smoothness in Orlicz spaces L_{φ}^{*}. Then we prove some direct and inverse two-sided approximation problems in Orlicz spaces L_{φ}^{*}. Finally we study monotone two sided approximation by trigonometric polynomials in Orlicz spaces L_{φ}^{*}.

Firstly we give basic definitions and notations.

We can consider a right continuous, monotone increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$; $\lim_{t\to\infty} \varphi(t) = \infty$ and $\varphi(t) > 0$ whenever $t > 0$; then the function defined by

$$N(x) = \int_0^{|x|} \varphi(t)dt$$

is called N-function [18]. The class of increasing N-functions will be denoted by Φ. When φ is an N-function [18] we always denote by $\psi(u)$ the mutually complementary N-function of φ. Everywhere in this work we suppose that φ is an N-function. The class of real-valued functions which denoted by L_{φ} defined on $I := [a,b] \subset \mathbb{R}$ such that;

$$\rho\left(u;\varphi\right) := \int_I \varphi\left[\left|u\left(x\right)\right|\right] dx < \infty$$

are called Orlicz classes. The class of measurable functions f defined on I such that the product $f\left(x\right) g(x)$ is integrable over (a,b) for every measurable function $g \in L_{\psi}$, will be denoted by $L_{\varphi}^{*}\left(I\right)$ which is called Orlicz space. We put

$$\|f\|_{L_{\varphi}^{*}(I)} := \sup_g \left|\int_I f(x)g(x)dx\right|$$

where the supremum being taken with respect to all g with $\rho\left(g;\psi\right) \leq 1$. When $I = \mathbb{T} := [0,2\pi]$ we set $L_{\varphi}^{*} := L_{\varphi}^{*}\left(I\right)$ and $\|f\|_{L_{\varphi}^{*}} := \|f\|_{L_{\varphi}^{*}(I)}$.

1.1. Definition. [25] Let $M\left[a,b\right]$ be the set of bounded and measurable functions on interval $[a,b]$ and $M := M\left[0,2\pi\right]$. Let φ is an N-function, $f \in M \cap L_{\varphi}^{*}$ and $x \in \mathbb{T}$. Suppose that sequence $\left\{t_n^{\pm}\right\}_1^{\infty}$ of trigonometric polynomials satisfy the monotonicity condition*:*

$$t_1^{+} \geq t_2^{+} \geq ... \geq t_n^{+} \geq ... \geq f \geq ... \geq t_n^{-} \geq ... \geq t_2^{-} \geq t_1^{-}.$$

The quantity

$$\widehat{E}_n(f)_{\varphi} := \inf\left\{\left\|t_n^{+} - t_n^{-}\right\|_{L_{\varphi}^{*}} : t_n^{\pm} \in T_n, t_n^{+} \geq f \geq t_n^{-}\right\}$$

is called the best two sided monotone approximation of the function $f \in M \cap L_{\varphi}^{*}$ by polynomials from $\mathcal{T}_n$, which is consist of all real trigonometric polynomials of degree at most n.

1.2. Definition. If $f \in M$ we can define

$$\mathcal{T}_n^{-}(f) := \left\{t \in \mathcal{T}_n : t(x) \leq f(x) \text{ for every } x \in \mathbb{R}\right\},$$

$$\mathcal{T}_n^{+}(f) := \left\{T \in \mathcal{T}_n : f(x) \leq T(x) \text{ for every } x \in \mathbb{R}\right\}.$$

In case φ is an N-function and $f \in M \cap L_{\varphi}^{*}$ we set

$$E_n^{-}(f)_{\varphi} := \inf_{t\in\mathcal{T}_n^{-}(f)} \|f - t\|_{L_{\varphi}^{*}}, \quad E_n^{+}(f)_{\varphi} := \inf_{T\in\mathcal{T}_n^{+}(f)} \|T - f\|_{L_{\varphi}^{*}}.$$

The quantities $E_n^{-}(f)_{\varphi}$ and $E_n^{+}(f)_{\varphi}$ are, respectively, called the best lower(upper) one-sided approximation errors for $f \in M \cap L_{\varphi}^{*}$.

$$\widetilde{E}_n(f)_{\varphi} := \inf\left\{\|T - t\|_{L_{\varphi}^{*}} :\ t, T \in \mathcal{T}_n,\ t(x) \leq f(x) \leq T(x) \text{ for every } x \in \mathbb{R}\right\}.$$

be the error of two-sided approximation for $f \in M \cap L_{\varphi}^{*}$. Similarly, the best trigonometric approximation error for $f \in L_{\varphi}^{*}$ is defined as usual by

$$E_n(f)_{\varphi} := \inf_{S \in \mathcal{T}_n} \|f - S\|_{L_{\varphi}^{*}}.$$

We note that

$$E_n(f)_{\varphi} \leq E_n^{\pm}(f)_{\varphi} \leq \widetilde{E}_n(f)_{\varphi}, \widehat{E}_n(f)_{\varphi} \quad [19].$$

Let φ be a N-function and for arbitrary $r = 0, 1, 2, ...$, there exists an r-times continuously differentiable function $f \in M$, such that

$$\limsup_{n \to \infty} \frac{\widetilde{E}_n(f)_{\varphi}}{E_n(f)_{\varphi}} = \infty.$$

This gives us the question of the estimation of the value of $\widetilde{E}_n(f)_{\varphi}$ [24].

2. The averaged modulus of smoothness

2.1. Definition. For $h \geq 0$, $k \in \mathbb{N}$, the expression

$$\Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} f(x + mh), \ \Delta_h f(x) = \Delta_h^1 f(x)$$

is called k-th difference of the function f with step h at a point x, where

$$\binom{k}{m} = \frac{k!}{m!(k-m)!}$$

is the binomial coefficients.

2.2. Definition. We define the modulus of continuity of the function $f \in M[a, b]$ by

$$(2.1) \qquad \omega(f; \delta) = \sup \left\{ \left| f(x) - f(x') \right| : \left| x - x' \right| \leq \delta; \ x, x' \in [a, b] \right\}, \ \delta \in [0, b - a].$$

2.3. Definition. The modulus of smoothness of a function $f \in M[a, b]$ of order k is the following function, $\delta \in [0, (b-a)/k]$

$$(2.2) \qquad \omega_k(f; \delta) = \sup \left\{ \left| \Delta_h^k f(x) \right| : |h| \leq \delta; \ x, x + kh \in [a, b] \right\}.$$

2.4. Definition. Let $C[a, b]$ be the set of continuous functions on interval $[a, b]$. The local modulus of smoothness of function of f of order $k \in \mathbb{N}$ at a point $x \in [a, b]$ is the following function, $\delta \in \left[0, \frac{b-a}{k}\right]$:

$$\omega_k(f, x; \delta) = \sup \left\{ \left| \Delta_h^k f(t) \right| : t, t + kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2} \right] \cap [a, b] \right\}.$$

We set

$$\omega_k(f; \delta) = \|\omega_k(f, .; \delta)\|_{C[a,b]}.$$

2.5. Definition. Let φ is an N-function, $h \geq 0$ and

$$I_h := \begin{cases} [a, b - h] & : 0 \leq h \leq b - a, \\ \varnothing & : h > b - a, \\ [0, 2\pi] & : I = \mathbb{T}. \end{cases}$$

The integral modulus of the function $f \in M[a, b] \cap L_{\varphi}^{*}(I)$ of order $k \in \mathbb{N}$ is the following function of $\delta \in \left[0, \frac{b-a}{k}\right]$:

$$\omega_k(f; \delta)_{\varphi} = \sup_{0 \leq h \leq \delta} \sup_{g} \left\{ \int_{I_{kh}} \left| \Delta_h^k f(x) \right| |g(x)| \, dx : g \in L_{\psi}, \ \rho(g, \psi) \leq 1 \right\}.$$

2.6. Definition. When φ is an N-function, the averaged modulus of smoothness of the function $f \in M[a,b] \cap L^*_\varphi(I)$ of order $k \in \mathbb{N}$ is the following function of $\delta \in \left[0, \frac{b-a}{k}\right]$:

$$\tau_k(f;\delta)_\varphi = \|\omega_k(f,.;\delta)\|_{L^*_\varphi(I)}$$

$$= \sup_g \left\{ \int_I |\omega_k(f,x,\delta)| \, |g(x)| \, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}.$$

2.7. Lemma. *In Orlicz spaces $L^*_\varphi(I)$ the averaged modulus of smoothness $\tau_k(f;\cdot)_\varphi$ has the following properties: If φ is an N-function, $f,g \in L^*_\varphi(I)$, $k,n \in \mathbb{N}$, $0 < \delta' \le \delta''$ and $\delta, \lambda > 0$, then*

(1.) $\tau_k(f;\delta')_\varphi \le \tau_k(f;\delta'')_\varphi$, $\delta' \le \delta''$,

(2.) $\tau_k(f+g;\delta)_\varphi \le \tau_k(f;\delta)_\varphi + \tau_k(g;\delta)_\varphi$,

(3.) $\tau_k(f;\delta)_\varphi \le 2\tau_{k-1}(f;\frac{k}{k-1}\delta)_\varphi$,

(4.) $\tau_k(f;\delta)_\varphi \le \delta\tau_{k-1}(f';\frac{k}{k-1}\delta)_\varphi$,

(5.) $\tau_k(f;n\delta)_\varphi \le (2n)^{k+1}\tau_k(f;\delta)_\varphi$,

(5.)' $\tau_k(f;\lambda\delta)_\varphi \le (2(\lambda+1))^{k+1}\tau_k(f;\delta)_\varphi$,

(6.) $\tau_k(f;\delta)_\varphi \le \delta \|f'\|_{L_\varphi(I)}$,

(6.)' $\tau_k(f;\delta)_\varphi \le c(k)\delta^k \left\|f^{(k)}\right\|_{L_\varphi(I)}$,

(7.) If f is bounded variation on $[a,b]$, then $\tau_k(f;\delta)_\varphi \le \delta V_a^b f$ where $V_a^b f$ is the total variation of f on $[a,b]$.

Proof. (1.) Let $\delta_i \in [0, b-a]$ $(i = 1,2)$ and $\delta_1 \ge \delta_2$. Using

$$\omega_k(f,x;\delta_2) \le \omega_k(f,x;\delta_1) \quad \text{and}$$

$$\tau_k(f;\delta_i)_\varphi = \sup_g \left\{ \int_I |\omega_k(f,x;\delta_i)| \, |g(x)| \, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}$$

we get (1.)

(2.) By the properties

$$\left|\Delta_h^k(f+g)\right| \le \left|\Delta_h^k(f)\right| + \left|\Delta_h^k(g)\right|,$$

$$\omega_k(f+g,\cdot;\delta) \le \omega_k(f,\cdot;\delta) + \omega_k(g,\cdot;\delta) \quad \text{and}$$

$$\tau_k(f+g;\delta)_\varphi = \sup_g \left\{ \int_I |\omega_k(f+g,x;\delta)| \, |g(x)| \, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}$$

one can find

$$\|\omega_k(f+g,\cdot;\delta)\|_{L^*_\varphi(I)} \le \|\omega_k(f,\cdot;\delta)\|_{L^*_\varphi(I)} + \|\omega_k(g,\cdot;\delta)\|_{L^*_\varphi(I)}.$$

This gives (2.)

(3.) By $\Delta_h^k f(\cdot) = \Delta_h^{k-1} f(\cdot + h) - \Delta_h^{k-1} f(\cdot)$ we have

$$\omega_k(f,x;\delta) = \sup \left\{ \left|\Delta_h^k f(t)\right| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b] \right\}$$

$$\le \quad \sup \left\{ \left|\Delta_h^{k-1} f(t+h)\right| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b] \right\} +$$

$$+ \sup \left\{ \left|\Delta_h^{k-1} f(t)\right| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b] \right\}.$$

Last two terms can be majorized by $\tau_{k-1}(f;\frac{k}{k-1}\delta)_\varphi$ and hence (3.) follows.

(4.) Since [22]

$$(2.3)\qquad \Delta_h^k f(t) = \int_0^h \Delta_h^{k-1} f'(t+u)du,\ h>0$$

we obtain

$$\sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap[a,b]\right\}$$

$$(2.4)\qquad \leq \sup\left\{\int_0^h \left|\Delta_h^{k-1} f'(t+u)\right| du : t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap[a,b]\right\}.$$

If $t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap[a,b]$ and $h>0$, then the points $t+u$, $t+u+(k-1)h$ in the same interval for $0 \leq u \leq h$. Then $\left|\Delta_h^{k-1} f'(t+u)\right| \leq \omega_{k-1}(f', x; \delta')$ with $\delta' = \frac{k}{k-1}\delta$. Continuing from (2.4)

$$\omega_k(f,x;\delta) \leq \delta\omega_{k-1}\left(f', x; \frac{k}{k-1}\delta\right), \quad x \in [a,b].$$

If we take the Orlicz norm of both sides of the last inequality we obtain (4.).

(5.) From [22, p.9] the identity

$$\Delta_{n,h}^k f(t) = \sum_{i=0}^{(n-1)k} A_i^{n,k} \Delta_h^k f(t+ih)$$

where $A_i^{n,k}$ are defined by

$$(1+t+...+t^{n-1})^k = \sum_{i=0}^{(n-1)k} A_i^{n,k} t^i$$

the inequality

$$(2.5)\qquad \omega_k(f,x;n\delta) \leq \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{*\ j=1}^{2n-1} \omega_k\left(f, x-(n-j)\frac{k\delta}{2}; \delta\right)$$

holds where

$$(2.6)\qquad \sum_{i=0}^{(n-1)k} A_i^{n,k} = n^k$$

and the only terms to appear in the sum $\sum_{*}$ are those for which $x-(n-j)\frac{k\delta}{2} \in [a,b]$. Now taking the Orlicz norm of both sides of (2.5), and by using equation (2.6) we obtain

$$(2.7)\qquad \tau_k(f;n\delta)_\varphi \leq (2n)^k(2n-1)\tau_k(f;\delta)_\varphi.$$

(5.)$'$ Let $\lambda>0$. Then $\exists n_0 \in \mathbb{N} : n_0 - 1 \leq \lambda < n_0$. Hence $(n_0-1)\delta \leq \lambda\delta < n_0\delta$ for $\delta > 0$ and $n_0 \leq \lambda+1$.

$$\begin{aligned}\tau_k(f;\lambda\delta)_\varphi &\overset{(1.)}{\leq} \tau_k(f;n_0\delta)_\varphi \overset{(2.7)}{\leq} (2n_0)^k(2n_0-1)\tau_k(f;\delta)_\varphi \\ &= (2(\lambda+1))^k(2(\lambda+1)-1)\tau_k(f;\delta)_\varphi \leq (2(\lambda+1))^k(2(\lambda+1))\tau_k(f;\delta)_\varphi \\ &= (2(\lambda+1))^{k+1}\tau_k(f;\delta)_\varphi\end{aligned}$$

as desired.

(6.) Let us extend f outside the interval $[a,b]$ by setting $f(x)=f(a)$, $x<a$ and $f(x)=f(b)$, $x>b$. Then for every $x\in[a,b]$ we have

$$\begin{aligned}\omega(f,x;\delta)&=\sup\left\{\left|f(t')-f(t'')\right| : t',t''\in\left[x-\frac{\delta}{2},x+\frac{\delta}{2}\right]\right\}\\ &= \sup\left\{\left|\int_{t'}^{t''}f'(t)dt\right| : t',t''\in\left[x-\frac{\delta}{2},x+\frac{\delta}{2}\right]\right\}\\ &\le \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}\left|f'(t)\right|dt=\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left|f'(x+t)\right|dt.\end{aligned}$$

From this inequality, taking the Orlicz norm, we obtain

$$\tau_1(f;\delta)_\varphi=\|\omega(f,\cdot;\delta)\|_{L^*_\varphi(I)}\le\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left\|f'(\cdot+t)\right\|_{L^*_\varphi(I)}dt=\delta\left\|f'\right\|_{L^*_\varphi(I)}.$$

More generally, if the function f has a bounded derivative (of order k), from properties (4.) and (6.) we obtain the following property of $\tau_1(f;\delta)_\varphi$.

(6.)$'$ Since

$$\tau_k(f;\delta)\overset{(4.)}{\le}\delta\tau_{k-1}\left(f';\frac{k}{k-1}\delta\right)_\varphi$$

we can write

$$\begin{aligned}\tau_k(f;\delta)&\le\delta\frac{k}{k-1}\delta\tau_{k-2}\left(f'';\left(\frac{k}{k-1}\right)^2\delta\right)_\varphi\\ &\le\delta^2\left(\frac{k}{k-1}\right)^2\delta\tau_{k-3}\left(f''';\left(\frac{k}{k-1}\right)^3\delta\right)_\varphi\le\dots\\ &\le\delta^{k-1}\left(\frac{k}{k-1}\right)^{k-1}\tau_1\left(f^{(k-1)};\left(\frac{k}{k-1}\right)^{k-1}\delta\right)_\varphi\\ &\overset{(6.)}{\le}\delta^k\left(\frac{k}{k-1}\right)^k\left\|f^{(k)}\right\|_{L^*_\varphi(I)}=c_k\delta^k\left\|f^{(k)}\right\|_{L^*_\varphi(I)}.\end{aligned}$$

(7.) Let $f(x)=f(a)$, $x<a$ and $f(x)=f(b)$, $x>b$. Then

$$\omega(f,\cdot;\delta)\le V_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}f(\cdot).$$

Therefore

$$\begin{aligned}\tau_1(f;\delta)_\varphi&\le\|\omega(f,\cdot;\delta)\|_{L^*_\varphi(I)}\le\left\|V_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}f\right\|_{L^*_\varphi(I)}\\ &=\sup_g\left\{\int_I V_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}f(x)\left|g(x)\right|dx;\ g\in L_\psi,\ \rho(g,\psi)\le1\right\}\\ &\le\left(\delta V_a^bf\right)\sup_g\left\{\int_I\left|g(x)\right|dx;\ g\in L_\psi,\ \rho(g,\psi)\le1\right\}\le\delta V_a^bf.\end{aligned}$$

The proof of Lemma 2.7 is completed. □

3. Main results

3.1. Theorem. *Let φ be an N-function and $f \in M[a,b] \cap L^*_\varphi(I)$. For any $\delta > 0$, inequalities*

$$c_1 \omega_k (f;\delta)_\varphi \le \tau_k (f;\delta)_\varphi \le c_2 \omega_k (f;\delta)(b-a) \tag{3.1}$$

holds, where the constants c_1 depend only on φ, k and c_2 depend only on φ.

Proof. We set for $h > 0$

$$A := \sup_g \left\{ \int_{I_{kh}} \left| \Delta_h^k f(x) \right| |g(x)|\, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}.$$

Then

$$A \le \sup_g \left\{ \int\limits_a^{b-kh} \left| \omega_k \left(f, x + \frac{kh}{2}, \delta \right) \right| |g(x)|\, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}$$

$$\le \sup_g \left\{ \int\limits_{a+\frac{kh}{2}}^{b-\frac{kh}{2}} |\omega_k (f, x, \delta)| \, |g(x)|\, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}.$$

From Definition 2.6, the last expression

$$\begin{aligned} &\le \quad \|\omega_k(f,.,\delta)\|_{L^*_\varphi(I)} \le \sup_{0\le h \le \delta} \|\omega_k(f,.,h)\|_{L^*_\varphi(I)} \\ &= \quad \sup_{0\le h\le\delta} \tau_k (f;h)_\varphi \le \tau_k(f;\delta)_\varphi . \end{aligned}$$

Now $\omega_k(f;\delta)_\varphi = \sup\limits_{0\le h\le\delta} A$ gives the left hand side of (3.1). For the proof of the right hand side of (3.1)

$$\tau_k (f;\delta)_\varphi = \sup_g \left\{ \int_I |\omega_k (f,x,\delta)| \, |g(x)|\, dx;\ g \in L_\psi,\ \rho(g,\psi) \le 1 \right\}$$

$$\le \|\omega_k(f,.,\delta)\|_{C[a,b]} \|1\|_{L^*_\varphi(I)} .$$

Then, from Young inequality, we find $\|1\|_{L^*_\varphi(I)} \le C_\varphi (b-a)$. Hence from Definition 2.6

$$\tau_k (f;\delta)_\varphi \le c_2 \omega_k (f;\delta)(b-a).$$

□

3.2. Theorem. *Let φ be a N-function, $k \in \mathbb{N}$ and $f \in L^*_\varphi \cap M$. Then there is a constant $c > 0$, dependent only on k and φ, such that the inequality*

$$\widetilde{E}_n(f)_\varphi \le c_{k,\varphi} \tau_k (f, \frac{1}{n})_\varphi$$

holds for $n \in \mathbb{N}$.

Proof. We know from [24, Lemma 5] that there exist trigonometric polynomials $t_n^+, t_n^- \in \mathcal{T}_n$ with the property

$$t_n^+ \ge f \ge t_n^-$$

and

$$t_n^+(x) - t_n^-(x) \le 16 \int_0^\pi \omega_k(f,x,2t) I_{r,m}(t) dt \tag{3.2}$$

where $k, r, m \in \mathbb{N}$, $n = r(m-1)$, $I_{r,m}(t) = \gamma_{r,m}\left[\frac{\sin mt/2}{m\sin t/2}\right]^{2r}$, and $(1/\gamma_{r,m}) = \int_{\mathbb{T}}\left[\frac{\sin mt/2}{m\sin t/2}\right]^{2r} dt$.

Taking Orlicz norm and changing the order of integration we obtain

$$\widetilde{E}_n(f)_\varphi \leq 16\int_0^\pi \tau_k(f,2t)_\varphi I_{r,m}(t)dt. \tag{3.3}$$

For any $i \leq 2(r-1)$

$$\int_0^\pi I_{r,m}(t)t^i dt$$

is equivalent to m^{-i} ([24, p.180]). Choosing r such that $k \leq 2r-3$ and $m = \lfloor \frac{n}{r} \rfloor + 1$ with regard to property $(5.)'$, we have

$$\int_0^\pi \tau_k(f,2t)_\varphi I_{r,m}(t)dt \leq c_{k,\varphi}\int_0^\pi (2mt+2)^{k+1}\tau_k(f,\frac{1}{m})_\varphi I_{r,m}(t)dt$$

$$\leq C_{k,\varphi}\tau_k(f,\frac{1}{m})_\varphi \leq c_{k,\varphi}\tau_k(f,\frac{r}{n})_\varphi \leq C_{k,\varphi}\tau_k(f,\frac{1}{n})_\varphi. \tag{3.4}$$

(3.2), (3.3) and (3.4) gives

$$\widetilde{E}_n(f)_\varphi \leq C_{k,\varphi}\tau_k(f,\frac{1}{n})_\varphi.$$

□

3.3. Theorem. *Let $k \in \mathbb{N}$. If φ is a N-function and $f \in L_\varphi^* \cap M$, then*

$$\tau_k\left(f,\frac{1}{n}\right)_\varphi \leq \frac{c_{k,\varphi}}{n^k}\sum_{v=0}^{n}(v+1)^{k-1}E(v,f,\varphi) \tag{3.5}$$

holds for $n \in \mathbb{N}$, where
$E(v,f,\varphi)\left\{= E_n(f)_\varphi \text{ or } = E_n^{\pm}(f)_\varphi \text{ or} = \widetilde{E}_n(f)_\varphi \text{ or } =\widehat{E}_n(f)_\varphi\right\}$ and constant $c_{k,\varphi} > 0$ dependent only on k and φ.

Proof. It is enough to prove (3.5) for $E(v,f,\varphi) = E_n(f)_\varphi$. Let $n \in \mathbb{N}$ and let the trigonometric polynomial $T_n \in \mathfrak{T}_n$ be such that $E_n(f)_\varphi = \|f - T_n\|_{L_\varphi^*}$. For $\delta > 0$,

$$\tau_k(f;\delta)_\varphi \leq \tau_k(f - T_n;\delta)_\varphi + \tau_k(T_n;\delta)_\varphi$$

$$\leq c_k\left[\|f-T_n\|_{L_\varphi^*}\right] + \tau_k(T_n;\delta)_\varphi = c_k E_n(f)_\varphi + \tau_k(T_n;\delta)_\varphi. \tag{3.6}$$

We set $n = 2^{v_0}$. Then from (3.6) we obtain

$$\tau_k(f;\delta)_\varphi \leq \sum_{i=1}^{v_0}\left[\tau_k(f;\delta)_\varphi + \tau_k(T_{2^i} - T_{2^{i-1}};\delta)_\varphi\right]$$

$$+\tau_k(f;\delta)_\varphi + \tau_k(T_1 - T_0;\delta)_\varphi + 2^k(k\delta n+1)E_n(f)_\varphi. \tag{3.7}$$

From property $(6'.)$

$$\tau_k(T_n;\delta)_\varphi \leq k\delta^k\left\|(T_{2^i} - T_{2^{i-1}})^{(k)}\right\|_{L_\varphi^*}$$

$$\leq k\delta^k 2^{ik}\|T_{2^i} - T_{2^{i-1}}\|_{L_\varphi^*} \leq k\delta^k 2^{ik}\left[\|f - T_{2^i}\|_{L_\varphi^*} + \|f - T_{2^{i-1}}\|_{L_\varphi^*}\right]$$

$$\leq k\delta^k 2^{ik}\left[\|f - T_{2^i}\|_{L_\varphi^*} + \|f - T_{2^{i-1}}\|_{L_\varphi^*}\right] \leq 2k\delta^k 2^{ik}E_{2^{i-1}}(f)_\varphi. \tag{3.8}$$

From (3.7) and (3.8)

$$\tau_k(f;\delta)_\varphi \le 4k\delta^k \sum_{i=1}^{v_0}\left[2^{ik}E_{2^{i-1}}(f)_\varphi + 2k\delta^k E_0(f)_\varphi + 2^k(kn\delta+1)E_n(f)_\varphi\right]$$

$$\le 4^{k+1}k\delta^k \sum_{v=0}^{n}\left[(v+1)^{k-1}E_v(f)_\varphi + 2^k(kn\delta+1)E_n(f)_\varphi\right] \tag{3.9}$$

Let $\delta = \frac{1}{n}$. From (3.9)

$$\tau_k(f;\delta)_\varphi \le 4^{k+1}kn^{-k}\sum_{v=0}^{n}\left[(v+1)^{k-1}E_v(f)_\varphi + 2^k(k+1)E_n(f)_\varphi\right]$$

$$\le 2^{3k+1}n^{-k}\sum_{v=0}^{n}(v+1)^{k-1}E_v(f)_\varphi .$$

If $2^{v_0} \le n < 2^{v_0+1}$,

$$\tau_k(f;\frac{1}{n})_\varphi \le \tau_k(f;\frac{1}{2^{v_0}})_\varphi \le 2^{3k+1}n^{-v_0k}\sum_{v=0}^{n}(v+1)^{k-1}E_v(f)_\varphi$$

$$\le 2^{4k+1}n^{-k}\sum_{v=0}^{n}(v+1)^{k-1}E_v(f)_\varphi = \frac{c}{n^k}\sum_{v=0}^{n}(v+1)^{k-1}E_v(f)_\varphi .$$

$\square$

From last two theorems we have the following two corollaries.

3.4. Corollary. *Let $k \in \mathbb{N}$. If φ is a N-function, $f \in L_\varphi^* \cap M$, and*

$$\widetilde{E}_n(f)_\varphi = \mathcal{O}\left(n^{-\sigma}\right),\ \sigma > 0,\ n \in \mathbb{N},$$

then

$$\tau_k(f;\delta)_\varphi = \begin{cases} \mathcal{O}(\delta^\sigma) & ; k > \sigma, \\ \mathcal{O}(\delta^\sigma\left|\log(1/\delta)\right|) & ; k = \sigma, \\ \mathcal{O}(\delta^\alpha) & ; k < \sigma, \end{cases}$$

hold.

3.5. Definition. Let $k \in \mathbb{N}$ and φ be an N-function. For $0 < \sigma < k$ we set $Lip\sigma(k,\varphi) := \left\{f \in L_\varphi^* \cap M : \tau_k(f;\delta)_\varphi = \mathcal{O}(\delta^\sigma),\quad \delta > 0\right\}$.

3.6. Corollary. *Let $k \in \mathbb{N}$, φ be an N-function, $0 < \sigma < k$ and let $f \in L_\varphi^* \cap M$. Then the following conditions are equivalent:*

(*a*) $f \in Lip\sigma(k,\varphi)$.
(*b*) $\widetilde{E}_n(f)_\varphi = \mathcal{O}\left(n^{-\sigma}\right),\quad n \in \mathbb{N}$.

Our monotone approximation estimate is given in the following.

3.7. Theorem. *Suppose that $r \in \mathbb{N}$, φ be an N-function and $f, f^{(r)} \in L_\varphi^* \cap M$. Then there exists sequences $\left\{t_n^+\right\}_1^\infty$, $\left\{t_n^-\right\}_1^\infty$, $t_n^\pm \in \mathcal{T}_n$, such that*

$$t_1^+ \ge t_2^+ \ge ... \ge t_n^+ \ge ... \ge f \ge ... \ge t_n^- \ge ... \ge t_2^- \ge t_1^-,$$

$$\left\|t_n^+ - t_n^-\right\|_{L_\varphi^*} \le c_{r,\varphi}n^{-r}\left\|f^{(r)}\right\|_{L_\varphi^*}$$

for $n \in \mathbb{N}$, where $c_{r,\varphi}$ is an constant dependent only on r and φ.

To proof this theorem we need the following lemmas. Let

$$f * g := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt$$

denote the convolution of f and g.

3.8. Lemma. [23] *Suppose* $D_1(u) = u - \pi$, $u \in [0, 2\pi)$, $D_1(4+2\pi) = D_1(u)$. *Then there exist sequences* $\left\{U_n^+\right\}_1^\infty$, $\left\{U_n^-\right\}_1^\infty$, $U_n^\pm \in \mathfrak{T}_n$, *such that*

$$U_1^+ \geq U_2^+ \geq ... \geq U_n^+ \geq ... \geq D_1 \geq ... \geq U_n^- \geq ... \geq U_2^- \geq U_1^-,$$

$$\left\|U_n^+ - U_n^-\right\|_1 \leq \frac{c}{n},$$

for $n \in \mathbb{N}$.

3.9. Lemma. *Suppose that* f *is absolutely continuous on* $\mathbb{T}$, $f' \in L_\varphi^* \cap M$ *and there exist a sequence* $\{T_n\}_1^\infty$, $T_n \in \mathfrak{T}_n$ *such that* $T_1 \geq T_2 \geq ... \geq T_n \geq ... \geq f'$. *Then there exist sequences* $\left\{R_n^+\right\}_1^\infty$, $\left\{R_n^-\right\}_1^\infty$, $R_n^\pm \in \mathfrak{T}_n$, *so that*

$$R_1^+ \geq R_2^+ \geq ... \geq R_n^+ \geq ... \geq f \geq ... \geq R_n^- \geq ... \geq R_2^- \geq R_1^-, \text{ and}$$

$$\left\|R_n^+ - R_n^-\right\|_{L_\varphi^*} \leq \frac{c}{n}\left\|T_n - f'\right\|_{L_\varphi^*}, \quad n \in \mathbb{N}.$$

Proof of Lemma 3.9. Let $T_n(x) = a_0 + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx)$ and $\widetilde{T}_n(x) = T_n(x) - a_0$, $h(x) = f(x) - \int_0^x \widetilde{T}_n(t)dt$. Then from [10]

$$h(x) = A - D_1 * h' = A - D_1 * (f' - \widetilde{T}_n) = A + D_1 * (\widetilde{T}_n - f'),$$

where $A = A(f)$ is a constant. By using U_n^+ and U_n^- of Lemma 3.8 we put

$$Q_n^+ = A + U_n^+ * (T_n - f'),\ Q_n^- = A + U_n^- * (T_n - f'),$$

$$R_n^+(x) = \int_0^x \widetilde{T}_n(t)dt + Q_n^+(x),\ R_n^-(x) = \int_0^x \widetilde{T}_n(t)dt + Q_n^-(x).$$

By using the fact that $a_0 \int_{\mathbb{T}} D_1(u)du = 0$, we have

$$\begin{aligned} R_n^+ - R_{n+1}^+ &= \left(R_n^+ - f\right) - \left(R_{n+1}^+ - f\right) \\ &= \left(U_n^+ - D_1\right) * (T_n - f') - \left(U_{n+1}^+ - D_1\right) * (T_{n+1} - f') \\ &= \left(U_n^+ - U_{n+1}^+\right) * (T_n - f') + \left(U_{n+1}^+ - D_1\right) * (T_n - T_{n+1}) \geq 0. \end{aligned}$$

Monotonicity of $\left\{R_n^-\right\}_1^\infty$ can be established analogously. Finally

$$\begin{aligned} \left\|R_n^+ - R_{n+1}^+\right\|_{L_\varphi^*} &= \left\|\left(U_n^+ - U_n^-\right) * (T_n - f')\right\|_{L_\varphi^*} \\ &\leq \frac{1}{2\pi}\left\|U_n^+ - U_n^-\right\|_1 \left\|T_n - f')\right\|_{L_\varphi^*} \leq \frac{c}{n}\left\|T_n - f')\right\|_{L_\varphi^*}. \end{aligned}$$

□

Proof. For $f, f' \in L_\varphi^*$ we set $f'_+(x) = \max\{0, f'(x)\}$ and $f'_-(x) = \max\{0, -f'(x)\}$. Then $f = A - D_1 * f' = A - D_1 * f'_+ + D_1 * f'_-$. Putting $t_n^+ = A - U_n^- * f'_+ + U_n^+ * f'_-$, $t_n^- = A - U_n^+ * f'_+ + U_n^- * f'_-$, we can show by exactly the same argument as in Lemma 3.9 that

$$t_1^+ \geq t_2^+ \geq ... \geq t_n^+ \geq ... \geq f \geq ... \geq t_n^- \geq ... \geq t_2^- \geq t_1^-,$$

and we have

$$\begin{aligned}\left\|t_n^+ - t_n^-\right\|_{L_\varphi^*} &\le \left\|\left(U_n^+ - U_n^-\right) * \left(f'_+ - f'_-\right)\right\|_{L_\varphi^*} \le \frac{1}{2\pi}\left\|\left(U_n^+ - U_n^-\right)\right\|_1 \left\|f'\right\|_{L_\varphi^*} \\ &\le \frac{c}{n}\left\|f'\right\|_{L_\varphi^*}.\end{aligned}$$

Applying Lemma 3.9 $(r-1)$ times, the proof of the Theorem 3.7 is obtained. □

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