Hacettepe Journal of Mathematics and Statistics
Volume 47 (5) (2018), 1128 – 1143

Qualitative study of a higher order rational difference equation

Abdul Khaliq*† and E. M. Elsayed‡

Abstract

In this paper we study the behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_n x_{n-l}}{\beta x_{n-m} + \gamma x_{n-l}}, \quad n = 0, 1, ...,$$

where the initial conditions x_{-r}, $x_{-r+1}, ...,$ x_0 are arbitrary non zero real numbers where $r = max\{l, m\}$ is a non-negative integer and α, β and γ are constants. Also, we obtain the solutions of some special cases of this equation. At the end we present some numerical examples to support our theoretical discussion.

Keywords: difference equation, stability, boundedness, global attractivity.

Mathematics Subject Classification (2010): 39A10, 39A11, 39A99, 34C99

Received : 03.01.2017 *Accepted* : 23.06.2017 *Doi* : 10.15672/HJMS.2017.512

1. Introduction

In this paper we deal with the behavior of the solutions of the following difference equation

$$x_{n+1} = \frac{\alpha x_n x_{n-l}}{\beta x_{n-m} + \gamma x_{n-l}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions x_{-r}, $x_{-r+1}, ...,$ x_0 are arbitrary non zero real numbers and where $r = max\{l, m\}$ is a non-negative integer and α, β and γ are constants. Also, we obtain the solutions of some special cases of this equation. We trust that nonlinear rational difference equations are of supreme importance in their specific righteous, and

*Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan.
Email : khaliqsyed@gmail.com

†Corresponding Author.

‡Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia and Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
Email : emmelsayed@yahoo.com.

in addition that results about such equations deal with patterns for the development of the basic theory of the global behavior of nonlinear difference equations.

The dynamical characteristics of population system have been modelled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations.

In practice, one can formulate a discrete model directly from experiments and observations. Sometimes, for numerical purposes one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points [39]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is desirable that a differential equation, when derived from a difference equation, preserves the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous-time models and it will preserve the considered realities; such discrete-time models can be called 'dynamically consistent' with the continuous-time models.

Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations.

El-Metwally et al. [14] investigated the asymptotic behavior of the population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},$$

where α is the immigration rate and β is the population growth rate.

The generalized Beverton-Holt stock recruitment model has investigated in [5]:

$$x_{n+1} = a x_n + \frac{b x_{n-1}}{1 + c x_{n-1} + d x_n}.$$

In recent time nonlinear difference equations have fascinated the minds of many researchers. In fact, we have endorsed a swift growth of concern in these types of equations in the earlier decade. Maybe, the desire was rooted from the evidence that these type of equations have various applications not only in the field of mathematics but also in relevant sciences, notably in biological sciences, engineering, ecology, discrete time systems, economics, physics and so on. We trust that this line of research will continue to appeal to the thoughts of more researchers in coming years as more compelling and captivating results are obtained and conveyed in recent analysis. The problem of finding the closed-form solutions of nonlinear difference equations have become a tendency over this research topic. As a matter of fact, numerous papers negotiate with the problem of solving nonlinear difference equations in any way possible, see, for instance [7]–[15]. Apparently, finding the solution form of these types of equations is, in general, a very challenging task. Nevertheless, various methods were offered recently to reduce complicated nonlinear difference equations into linear forms which have already known solution forms. For instance, through transforming into linear types, a large class of nonlinear difference equations were resolved in closed-forms (see, e.g., [15]–[29]).

Many researchers have investigated the behavior of the solution of difference equations for example: Cinar [6] has obtained the solutions of the following difference equation

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

In [12] Elabbasy et al. studied the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} - cx_{n-q}}.$$

Elsayed and Khaliq [28] investigated the boundedness, global stability and existence of periodic solutions of the following difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}.$$

Amleh et al. [3] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{a + bx_{n-1}}{A + Bx_{n-2}}.$$

Yan et al. [51] studied the global attractivity for the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-1}}.$$

Dehghan et al. [8] found the invariant intervals, global stability, the character of semi-cycles, and the boundedness of the equation

$$x_{n+1} = \frac{\alpha y_{n-2}}{\beta + \gamma y_n^k y_{n-1}^k y_{n-2}^k}.$$

Yang [51] studied the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2} + x_{n-3} + a}{x_{n-1} + x_{n-2}x_{n-3} + a}.$$

See also [1]-[5], [30]-[45]. Other related results on rational difference equations can be found in refs. [46]–[56].

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [37].

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of f.

1.1. Definition. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is a locally stable solution of Eq.(1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is a global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).
(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$(1.3) \qquad y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(x_n, x_{n-1}, ..., x_{n-k})}{\partial x_{n-i}}\Big|_{x_n = x_{n-1} = ... = x_{n-k} = \overline{x}} y_{n-i}$$

Theorem A [37]: *Assume that* $p, q \in R$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark. Theorem A can be easily extended to a general linear equation of the form

$$(1.4) \qquad x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq.(1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$(1.5) \qquad x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [38]: *Let* $[p, q]$ *be an interval of real numbers and assume that*

$$g : [p, q]^3 \to [p, q]$$

is a continuous function satisfying the following properties :

(a) $g(x, y, z)$ *is non-decreasing in* x *and* y *in* $[p, q]$ *for each* $z \in [p, q]$, *and is non-increasing in* $z \in [p, q]$ *for each* x *and* y *in* $[p, q]$;

(b) If $(m, M) \in [p, q] \times [p, q]$ *is a solution of the system*

$$M = g(M, M, m) \quad \text{and} \quad m = g(m, m, M)$$

then $m = M$.

Then Eq.(1.5) has a unique equilibrium $\overline{x} \in [p, q]$ *and every solution of Eq.(1.5) converges to* $\overline{x}$.

2. Dynamics and behaviour of solutions of eq.(1.1)

In this section we study some qualitative behavior of Eq. (1.1) such that local stability, global attractor of the equilibrium point and boundedness character of solutions of Eq.(1.1) when the initial conditions $x_{-r}, x_{-r+1}, ..., x_0$ *and the constants* $\alpha,\ \beta$ *and* γ *are arbitrary positive real numbers.*

2.1. Local stability of eq.(1.1). *In this section we investigate the local stability character of the solutions of Eq.(1.1). It has a unique equilibrium point and is given by the equation*

$$\overline{x}=\frac{\alpha\overline{x}^2}{\beta\overline{x}+\gamma\overline{x}},$$

or,

$$\overline{x}^2(\beta+\gamma)=\alpha\overline{x}^2,$$

if $(\beta+\gamma)\neq\alpha$*, then the unique equilibrium point is* $\overline{x}=0$.
Let $f:(0,\infty)^3\longrightarrow(0,\infty)$ *be a function defined by*

$$f(u,w,t)=\frac{\alpha uw}{\beta t+\gamma w}. \tag{2.1}$$

Therefore it follows that

$$f_u(u,w,t)=\frac{\alpha w}{(\beta t+\gamma w)},\qquad f_w(u,w,t)=\frac{\alpha\beta ut}{(\beta t+\gamma w)^2},$$
$$\text{and}\quad f_t(u,w,t)=\frac{-\alpha\beta uw}{(\beta t+\gamma w)^2},$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x})=\frac{\alpha}{(\beta+\gamma)},\qquad f_w(\overline{x},\overline{x},\overline{x})=\frac{\alpha\beta}{(\beta+\gamma)^2},$$
$$\text{and}\quad f_t(\overline{x},\overline{x},\overline{x})=\frac{-\alpha\beta}{(\beta+\gamma)^2}.$$

The linearized equation of Eq.(1.1) about $\overline{x}$ *is*

$$y_{n+1}-\frac{\alpha}{(\beta+\gamma)}y_n-\frac{\alpha\beta}{(\beta+\gamma)^2}y_{n-l}+\frac{\alpha\beta}{(\beta+\gamma)^2}y_{n-m}=0. \tag{2.2}$$

2.1. Theorem. *Assume that*

$$\frac{3\beta+\gamma}{(\beta+\gamma)^2}<\frac{1}{\alpha}.$$

Then the equilibrium point of Eq.(1.1) is locally asymptotically stable.

Proof. It follows by Theorem A that Eq.(2.2) is asymptotically stable if

$$\left|\frac{\alpha}{(\beta+\gamma)}\right|+\left|\frac{\alpha\beta}{(\beta+\gamma)^2}\right|+\left|\frac{\alpha\beta}{(\beta+\gamma)^2}\right|<1,$$

or,

$$\alpha(\beta+\gamma)+\alpha\beta+\alpha\beta<(\beta+\gamma)^2.$$

Thus

$$\frac{3\beta+\gamma}{(\beta+\gamma)^2}<\frac{1}{\alpha}.$$

The proof is complete. □

2.2. Global attractor of the equilibrium point of eq.(1.1). *In this section we investigate the global attractivity character of solutions of Eq.(1.1).*

2.2. Theorem. *The equilibrium point $\overline{x}$ of Eq.(1.1) is global attractor if $\alpha \neq \gamma$.*

Proof. Let p, q be real numbers and assume that $g : [p, q]^3 \longrightarrow [p, q]$ is a function defined by $g(u, w, t) = \dfrac{\alpha u w}{\beta t + \gamma w}$. Then we can easily see that the function $g(u, w, t)$ increasing in u, w and decreasing in t.

Suppose that (m, M) is a solution of the system

$$M = g(M, M, m) \quad and \quad m = g(m, m, M).$$

Then from Eq.(1.1), we see that

$$M = \frac{\alpha M^2}{\beta m + \gamma M}, \quad m = \frac{\alpha m^2}{\beta M + \gamma m},$$

or,

$$Mm\beta = M^2(\alpha - \gamma), \quad Mmb = m^2(\alpha - \gamma).$$

Then subtracting we obtain

$$(\alpha - \gamma)(M^2 - m^2) = 0, \quad \alpha \neq \gamma.$$

Thus

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is complete. □

2.3. Boundedness of solutions of Eq.(1.1). *In this section we study the boundedness of solutions of Eq.(1.1).*

2.3. Theorem. *Every solution of Eq.(1.1) is bounded if $\dfrac{\alpha}{\gamma} < 1$.*

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ where $r = max\{l, m\}$ be a solution of Eq.(1.1). It follows from Eq.(1.1) that

$$x_{n+1} = \frac{\alpha x_n x_{n-l}}{\beta x_{n-m} + \gamma x_{n-l}} \leq \frac{\alpha x_n x_{n-l}}{\gamma x_{n-l}} = (\frac{\alpha}{\gamma}) x_n.$$

Then when $\dfrac{\alpha}{\gamma} < 1$, we see that

$$x_{n+1} \leq x_n \quad \text{for all} \quad n \geq 0.$$

Then the subsequences $\{x_n\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-r+1},\ x_{-r+1},\ x_{-r+2}, ...,\ x_0\}$. □

3. Special cases of eq.(1.1)

Our goal in this section is to obtain the form of the solutions of some special cases of Eq.(1.1) when the initial conditions $x_{-r},\ x_{-r+1}, ...,\ x_0$ are arbitrary non zero real numbers and the constants $\alpha,\ \beta$ and γ are integer numbers.

3.1. First Case. *In this section we study the following special case of Eq.(1.1)*

$$x_{n+1}=\frac{x_n x_{n-4}}{x_{n-4}+x_{n-3}}, \qquad n=0,1,..., \tag{3.1}$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ *are arbitrary non zero real numbers.*

3.1. Theorem. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Eq.(3.1). Then for* $n=0,1,2,...$

$$\begin{aligned}
x_{4n-4} &= \frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-1}d+F_{n-2}e)},\\
x_{4n-3} &= \frac{abcde}{(F_{n}a+F_{n-1}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-1}d+F_{n-2}e)},\\
x_{4n-2} &= \frac{abcde}{(F_{n}a+F_{n-1}b)(F_{n}b+F_{n-1}c)(F_{n-1}c+F_{n-2}d)(F_{n-1}d+F_{n-2}e)},\\
x_{4n-1} &= \frac{abcde}{(F_{n}a+F_{n-1}b)(F_{n}b+F_{n-1}c)(F_{n}c+F_{n-1}d)(F_{n-1}d+F_{n-2}e)},
\end{aligned}$$

where $x_{-4}=a,\ x_{-3}=b,\ x_{-2}=c,\ x_{-1}=d,\ x_0=e.\ \{F_m\}_{m=0}^{\infty}=\{1,1,2,3,5,8.....\}$, $F_{-1}=0,\ F_{-2}=1.$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1,\ n-2$. That is:

$$\begin{aligned}
x_{4n-9} &= \frac{abcde}{(F_{n-2}a+F_{n-3}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-3}d+F_{n-4}e)},\\
x_{4n-8} &= \frac{abcde}{(F_{n-2}a+F_{n-3}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-2}d+F_{n-3}e)},\\
x_{4n-7} &= \frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-2}d+F_{n-3}e)},\\
x_{4n-6} &= \frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-2}c+F_{n-3}d)(F_{n-2}d+F_{n-3}e)},\\
x_{4n-5} &= \frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-2}d+F_{n-3}e)},
\end{aligned}$$

Now, it follows from Eq.(3.1) that

$$\begin{aligned}
x_{4n-4} &= \frac{x_{4n-5}x_{4n-9}}{x_{4n-9}+x_{4n-8}} \\
&= \frac{\left(\frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-2}d+F_{n-3}e)}\right)\times\left(\frac{abcde}{(F_{n-2}a+F_{n-3}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-3}d+F_{n-4}e)}\right)}{\left(\frac{abcde}{(F_{n-2}a+F_{n-3}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-3}d+F_{n-4}e)}+\frac{abcde}{(F_{n-2}a+F_{n-3}b)(F_{n-2}b+F_{n-3}c)(F_{n-2}c+F_{n-3}d)(F_{n-2}d+F_{n-3}e)}\right)} \\
&= \frac{\frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-2}d+F_{n-3}e)}\frac{1}{(F_{n-3}d+F_{n-4}e)}}{\left(\frac{1}{(F_{n-3}d+F_{n-4}e)}+\frac{1}{(F_{n-2}d+F_{n-3}e)}\right)} \\
&= \frac{\frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)}}{(F_{n-2}d+F_{n-3}e)(F_{n-3}d+F_{n-4}e)\left(\frac{1}{(F_{n-3}d+F_{n-4}e)}+\frac{1}{(F_{n-2}d+F_{n-3}e)}\right)} \\
&= \frac{\frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)}}{(F_{n-2}d+F_{n-3}e+F_{n-3}d+F_{n-4}e)} \\
&= \frac{abcde}{(F_{n-1}a+F_{n-2}b)(F_{n-1}b+F_{n-2}c)(F_{n-1}c+F_{n-2}d)(F_{n-1}d+F_{n-2}e)}.
\end{aligned}$$

Also, we can prove the other relations. □

3.2. Second Case. *In this section we give a specific form of the solutions of the difference equation*

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n - x_{n-4}}, \qquad n = 0, 1, ..., \tag{3.2}$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ *are arbitrary non zero real numbers with* $4x_0 \neq x_{-4},\ x_{-4}\ \neq x_0,\ 2x_0 \neq x_{-4},\ 3x_0 \neq x_{-4}$.

3.2. Theorem. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Eq.(3.2). Then for* $n = 0, 1, 2, ...$

$$x_{4n-4} = \frac{e^n}{a^{n-1}},\ x_{4n-3} = \frac{be^n}{(e-a)^n},\ x_{4n-2} = \frac{ce^n}{a^n},\ x_{4n-1} = \frac{de^n}{(e-a)^n},$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c,\ x_{-1} = d,\ x_0 = e$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is,

$$\begin{aligned}
x_{4n-9} &= \frac{de^{n-2}}{(e-a)^{n-2}}, \quad x_{4n-8} = \frac{e^{n-1}}{a^{n-2}}, \quad x_{4n-7} = \frac{be^{n-1}}{(e-a)^{n-1}}, \\
x_{4n-6} &= \frac{ce^{n-1}}{a^{n-1}}, \qquad x_{4n-5} = \frac{de^{n-1}}{(e-a)^{n-1}}.
\end{aligned}$$

From Eq.(3.2), it follows that

$$x_{4n-4} = \frac{x_{4n-5}x_{4n-8}}{x_{4n-5}-x_{4n-9}} = \frac{\left(\frac{de^{n-1}}{(e-a)^{n-1}}\right)\left(\frac{e^{n-1}}{a^{n-2}}\right)}{\left(\frac{de^{n-1}}{(e-a)^{n-1}}\right)-\left(\frac{de^{n-2}}{(e-a)^{n-2}}\right)}$$
$$= \frac{e^{n-1}\left(\frac{e^{n-1}}{a^{n-2}}\right)}{e^{n-1}-e^{n-2}(e-a)} = \frac{\left(\frac{e^{n}}{a^{n-2}}\right)}{e-(e-a)} = \frac{e^n}{a^{n-1}}.$$

The remaining relations can be found in similar way, therefore left to reader. Thus the proof is completed. □

3.3. Third Case. *In this section we obtain the solution of the following special case of Eq.(1.1)*

$$(3.3) \qquad x_{n+1} = \frac{x_n x_{n-3}}{x_{n-3}+x_{n-4}}, \qquad n = 0, 1, ...,$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ *are arbitrary non zero real numbers with* $x_0 \neq x_{-1},\ \ x_{-1} \neq x_{-2},\ x_{-2} \neq x_{-3}$.

3.3. Theorem. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Eq.(3.3). Then for* $n = 0, 1, 2, ...$

$$x_{4n-3} = \frac{b^{n+1}c^n d^n e^n}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)},$$
$$x_{4n-2} = \frac{b^n c^{n+1} d^n e^n}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+(i+1)c)(c+id)(d+ie)}$$
$$x_{4n-1} = \frac{b^n c^n d^{n+1} e^n}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+ie)},$$
$$x_{4n} = \frac{b^n c^n d^n e^{n+1}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+(i+1)e)}.$$

Where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c,\ x_{-1} = d,\ x_0 = e$ *and* $\prod_{i=0}^{-1} A_i = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is;

$$x_{4n-7} = \frac{b^n c^{n-1} d^{n-1} e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+ic)(c+id)(d+ie)},$$
$$x_{4n-6} = \frac{b^{n-1} c^n d^{n-1} e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+(i+1)c)(c+id)(d+ie)},$$
$$x_{4n-5} = \frac{b^{n-1} c^{n-1} d^n e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+ie)},$$

$$x_{4n-4} = \frac{b^{n-1}c^{n-1}d^{n-1}e^{n}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+(i+1)e)},$$

$$x_{4n-8} = \frac{b^{n-2}c^{n-2}d^{n-2}e^{n-1}}{\prod_{i=0}^{n-3}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+(i+1)e)},$$

$$x_{4n-9} = \frac{b^{n-2}c^{n-2}d^{n-1}e^{n-2}}{\prod_{i=0}^{n-3}(a+(i+1)b)(b+(i+1)c)(c+(i+1)d)(d+ie)}.$$

Now, it follows from Eq.(3.3) that

$$\begin{aligned}
x_{4n-2} &= \frac{x_{4n-3}x_{4n-6}}{x_{4n-6}+x_{4n-7}} \\
&= \frac{\left(\frac{b^{n+1}c^{n}d^{n}e^{n}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)}\right)\left(\frac{b^{n-1}c^{n}d^{n-1}e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+(i+1)c)(c+id)(d+ie)}\right)}{\left(\frac{b^{n-1}c^{n}d^{n-1}e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+(i+1)c)(c+id)(d+ie)}\right)+\left(\frac{b^{n}c^{n-1}d^{n-1}e^{n-1}}{\prod_{i=0}^{n-2}(a+(i+1)b)(b+ic)(c+id)(d+ie)}\right)} \\
&= \frac{\left(\frac{b^{n+1}c^{n}d^{n}e^{n}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)}\right)\left(\frac{c}{\prod_{i=0}^{n-2}(b+(i+1)c)}\right)}{\left(\frac{c}{\prod_{i=0}^{n-2}(b+(i+1)c)}\right)+\left(\frac{b}{\prod_{i=0}^{n-2}(b+ic)}\right)} \\
&= \frac{\left(\frac{b^{n+1}c^{n+1}d^{n}e^{n}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)}\right)}{c+\left(\frac{b\prod_{i=0}^{n-2}(b+(i+1)c)}{\prod_{i=0}^{n-2}(b+ic)}\right)} \\
&= \frac{\left(\frac{b^{n+1}c^{n+1}d^{n}e^{n}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)}\right)}{c+(b+(n-1)c)} \\
&= \frac{b^{n+1}c^{n+1}d^{n}e^{n}}{(b+(n)c)\prod_{i=0}^{n-1}(a+(i+1)b)(b+ic)(c+id)(d+ie)}.
\end{aligned}$$

Since $(b+(n)c)\prod_{i=0}^{n-1}(b+ic) = b\prod_{i=0}^{n-1}(b+(i+1)c)$. Therefore

$$x_{4n-2} = \frac{b^{n}c^{n+1}d^{n}e^{n}}{\prod_{i=0}^{n-1}(a+(i+1)b)(b+(i+1)c)(c+id)(d+ie)}.$$

Other relations can be found similarly. The proof is completed. □

3.4. Fourth Case. *In this section we obtain the solution of the following special case of Eq.(1.1)*

$$(3.4) \qquad x_{n+1} = \frac{x_n x_{n-3}}{x_n + x_{n-3}}, \qquad n = 0, 1, ...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ *are arbitrary non zero real numbers with* $x_0 \neq -x_{-3}$.

3.4. Theorem. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of Eq.(3.4). Then* $x_1 = \dfrac{ad}{(a+d)}$. *For* $n = 1, 2, ...$

$$x_{n+1} = \frac{abcd}{g_n abc + g_{n-3} abd + g_{n-2} acd + g_{n-1} bcd},$$

where $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c,\ x_0 = d,\ \{g_m\}_{m=0}^{\infty} = \{1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, ...\}$, *i.e.* $g_{m+1} = g_m + g_{m-3},\ m \geq 0,\ g_{-3} = 0,\ g_{-2} = 0,\ g_{-1} = 1$ *and* $g_0 = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1, n-4$. That is;

$$x_{n-3} = \frac{abcd}{g_{n-4} abc + g_{n-7} abd + g_{n-6} acd + g_{n-5} bcd},$$

$$x_n = \frac{abcd}{g_{n-1} abc + g_{n-4} abd + g_{n-3} acd + g_{n-2} bcd}.$$

Now it follows from Eq.(3.4) that

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n + x_{n-3}}$$

$$= \frac{\left(\frac{abcd}{g_{n-1}abc+g_{n-4}abd+g_{n-3}acd+g_{n-2}bcd}\right)\left(\frac{abcd}{g_{n-4}abc+g_{n-7}abd+g_{n-6}acd+g_{n-5}bcd}\right)}{\frac{abcd}{g_{n-1}abc+g_{n-4}abd+g_{n-3}acd+g_{n-2}bcd} + \frac{abcd}{g_{n-4}abc+g_{n-7}abd+g_{n-6}acd+g_{n-5}bcd}}$$

$$= \frac{abcd}{g_{n-1}abc + g_{n-4}abd + g_{n-3}acd + g_{n-2}bcd + g_{n-4}abc + g_{n-7}abd + g_{n-6}acd + g_{n-5}bcd}$$

$$= \frac{abcd}{g_n abc + g_{n-3} abd + g_{n-2} acd + g_{n-1} bcd}.$$

Hence the proof is completed. □

4. Numerical examples

In order to explain and support the results of the previous discussion we present several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to special cases of Eq.(1.1).

In this section, to observe numerical results clearly, we present graphs of solutions that were carried out using MATLAB. We choose different values for the parameters α, β *and* γ. *It should be noted that* $x_0,\ x_{-1},\ x_{-2},\ x_{-3}$ *are also different initial values.*

4.1. Example. Consider the difference equation (1.1) when $m = 3$, $l = 4$, $\alpha = 0.9$, $\beta = 0.3$, $\gamma = 0.6$, with $x_{-4} = 7$, $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = 1$, $x_0 = 0.9$. [See Fig. 1].

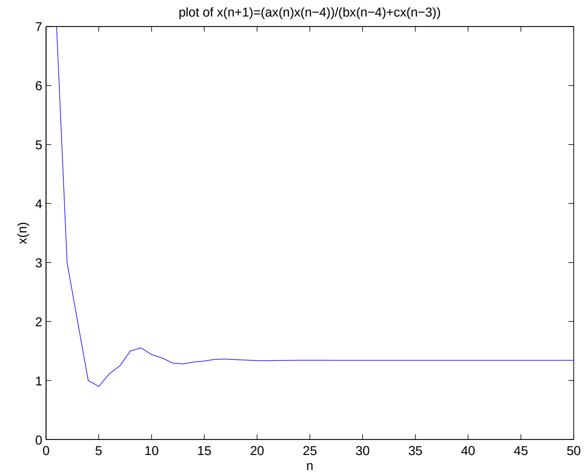

Figure 1. This figure shows the stability of the solutions of Eq. (1.1) when $m = 3$, $l = 4$, $\alpha = 0.9$, $\beta = 0.3$, $\gamma = 0.6$, $x_{-4} = 7$, $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = 1$, $x_0 = 0.9$.

4.2. Example. Put $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 2$, $x_0 = 9$, $m = 3$, $l = 2$, $\alpha = 1.1$, $\beta = 0.9$, $\gamma = 0.2$ in the difference equation (1.1). See Fig. 2.

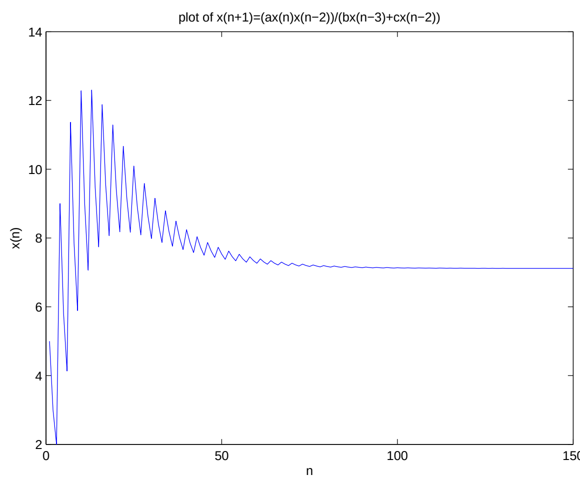

Figure 2. This figure shows the behavior for Eq. (1.1) with $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 2$, $x_0 = 9$, $m = 3$, $l = 2$, $\alpha = 1.1$, $\beta = 0.9$, $\gamma = 0.2$

4.3. Example. Consider Eq.(1.1), where $m = 3$, $l = 2$, $\alpha = 1.2$, $\beta = 0.54$, $\gamma = 0.65$, $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 2$, $x_0 = 9$. [See Fig. 3].

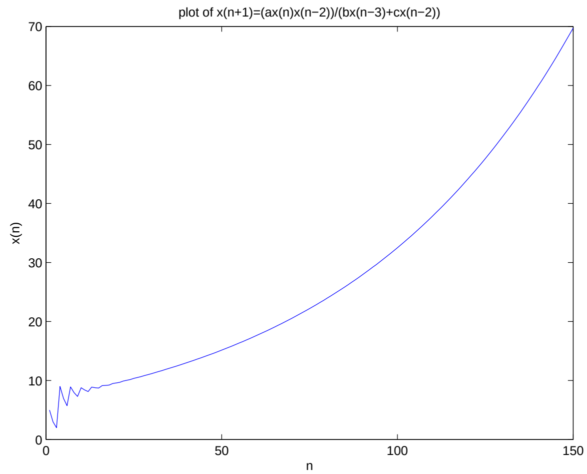

Figure 3. This figure shows the unboundedness solutions for Eq.(1.1) since $m = 3$, $l = 2$, $\alpha = 1.2$, $\beta = 0.54$, $\gamma = 0.65$, $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 2$, $x_0 = 9$.

4.4. Example. Consider Eq. (3.1), when $x_{-4} = 0.5$, $x_{-3} = 0.3$, $x_{-2} = 0.2$, $x_{-1} = 0.9$, $x_0 = 0.7$. [See Fig. 4].

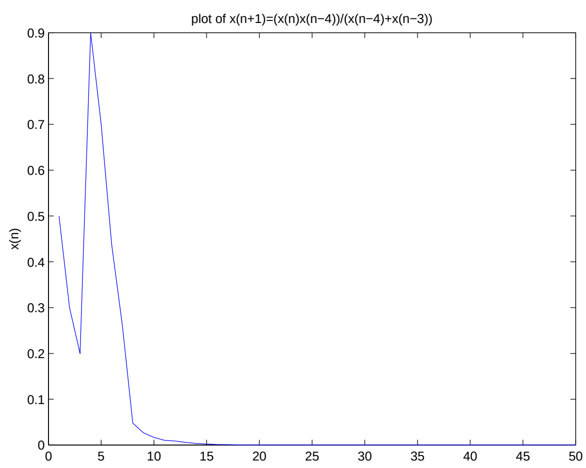

Figure 4. This figures shows the solutions of $x_{n+1} = \dfrac{x_n x_{n-4}}{x_{n-4} + x_{n-3}}$, *when* $x_{-4} = .5$, $x_{-3} = 0.3$, $x_{-2} = 0.2$, $x_{-1} = 0.9$, $x_0 = 0.7$.

4.5. Example. See Fig. 5 when we consider the difference equation (3.2) with $x_{-4} = 5$, $x_{-3} = 3$, $x_{-2} = 0.2$, $x_{-1} = 9$, $x_0 = 7$.

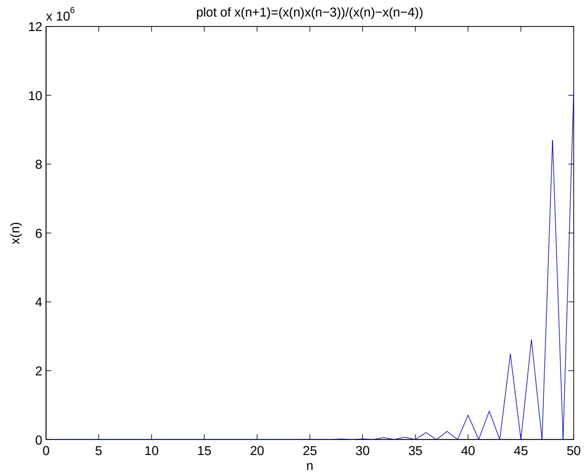

Figure 5. This figure shows the solution of Eq.(3.2) with $x_{-4} = 5$, $x_{-3} = 3$, $x_{-2} = 0.2$, $x_{-1} = 9$, $x_0 = 7$.

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