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Combinatorial sums and binomial identities associated with the Beta-type polynomials

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Abstract

In this paper, we first provide some functional equations of the generating functions for beta-type polynomials. Using these equations, we derive various identities of the beta-type polynomials and the Bernstein basis functions. We then obtain some novel combinatorial identities involving binomial coefficients and combinatorial sums. We also derive some generalizations of the combinatorics identities which are related to the Gould's identities and sum of binomial coefficients. Next, we present some remarks, comments, and formulas including the combinatorial identities, the Catalan numbers, and the harmonic numbers. Moreover, by applying the classical Young inequality, we derive a combinatorial inequality related to beta polynomials and combinatorial sums. We also give another inequality for the Catalan numbers.

Keywords: Combinatorial sums; Binomial identities; Generating functions; Functional equations; Beta polynomials; Beta function; Gamma function; Bernstein basis functions; Catalan numbers; Harmonic numbers; Young inequality.

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1. Introduction

Combinatorial sums and binomial identities have been appeared in mathematical physics, engineering, and different branches of mathematics, such as combinatorics, probability, number theory, analysis of algorithms, and graph theory. These identities have been established by various techniques, some of which are generating functions, inverse relations, integral representations, special functions, and special series (hypergeometric functions, gamma function, beta function and the other special functions) (cf. [3]-[28]). In this paper, our aim is to illustrate some aspects of combinatorial sums and binomial identities. Our approach is related to beta-type polynomials and Bernstein polynomials

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identities, which have been intensively used in both mathematics and engineering. Our results will be useful in these fields because polynomials have become a crucial tool to represent different systems, simulate various processes, and compute the ultimate decision results.

Using generating functions, many kind of the special polynomials have been discovered. In 2015, the author [22] and [24] constructed generating functions for beta-type polynomials, then derived many properties of them. These polynomials are used in combinatorics, in probability, in number theory, in mathematical analysis, and in algebra (*cf.* [22], [24], [21], [3]); thus, they have variety of important applications.

The following generating functions for the functions, $\mathfrak{M}_{k,n}(x)$ is given by

$$\mathfrak{h}_k(t,x) = \left(\frac{x}{1+x}\right)^k e^{t(1+x)} = \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^n}{n!}, \tag{1.1}$$

where $k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and for all $t \in \mathbb{C}$, set of complex numbers (*cf.* [24]). There exists one generating function for each value of k.

By using (1.1), we have

$$\mathfrak{M}_{k,n}(x) = x^k(1+x)^{n-k}, \tag{1.2}$$

where $n, k \in \mathbb{N}_0$ (*cf.* [24]). From (1.1), if $n \geq k$, we have the beta polynomials:

$$\mathfrak{M}_{k,n}(x) = \mathfrak{B}_{k,n}(x) = x^k\,(x+1)^{n-k} \tag{1.3}$$

and if $n < k$, we define a rational function, $\mathfrak{b}_{k,n}(x)$ as follows:

$$\mathfrak{M}_{k,n}(x) = \mathfrak{b}_{k,n}(x) = \frac{x^k}{(1+x)^{k-n}},$$

where $n \in \{0, 1, 2, \cdots, k-1\}$. Therefore, for $n, k \in \mathbb{N}_0$, we have

$$\mathfrak{M}_{k,n}(x) = \mathfrak{B}_{k,n}(x) + \mathfrak{b}_{k,n}(x)$$

(*cf.* [24]).

The beta polynomials are defined by means of the following generating functions:

$$\mathfrak{F}_k(t,x) = \mathfrak{h}_k(t,x) - \sum_{n=0}^{k-1} \mathfrak{b}_{k,n}(x)\frac{t^n}{n!} = \sum_{n=k}^{\infty} \mathfrak{B}_{k,n}(x)\frac{t^n}{n!},$$

where $t \in \mathbb{C}$ (*cf.* [24]). If $k < 0$ or $k > n$, then $\mathfrak{B}_{k,n}(x) = 0$ (*cf.* [3], [22]).

1.1. Integral of the beta polynomials and combinatorial identities. In [24], integral formulas for the beta polynomials by using beta function and gamma function were given. These integral formulae are used in the next section for proving combinatorial identities and combinatorial sums.

The Beta function $B(\alpha, \beta)$ is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = B(\beta,\alpha), \tag{1.4}$$

where $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\beta) > 0$ (*cf.* [27, p. 9, Eq-(60)], [19]). The following formula show the relation between beta and gamma functions:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Replacing α by $n \in \mathbb{Z}^+$ and β by $m \in \mathbb{Z}^+$ in the above equation, we get

$$B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!} \tag{1.5}$$

(*cf.* [27, p. 9, Eq-(62)], [19]).

If we integrate of the beta polynomials from -1 to 0 by applying the result (1.4), we have

$$\int_{-1}^{0} \mathfrak{B}_{k,n}(x)dx = (-1)^k B(k+1, n-k+1), \tag{1.6}$$

or equivalently,

$$\int_{-1}^{0} \mathfrak{B}_{k,n}(x)dx = \frac{(-1)^k}{(n+1)\left(\begin{array}{c} n \\ k \end{array}\right)} \tag{1.7}$$

(*cf.* [24]). By combining (1.6) and (1.7), the following theorem can be obtained:

1.1. Theorem ([24]). *The following identity holds true:*

$$\sum_{j=0}^{n-k} (-1)^{n-j} \left(\begin{array}{c} n-k \\ j \end{array}\right) \frac{1}{n+1-j} = \frac{(-1)^k}{(n+1)\left(\begin{array}{c} n \\ k \end{array}\right)}. \tag{1.8}$$

If we replace k by n and n by $2n$ in (1.8), we reach the following result.

1.2. Corollary ([24]). *The following identity holds true:*

$$\sum_{j=0}^{n} (-1)^{j} \left(\begin{array}{c} n \\ j \end{array}\right) \frac{1}{2n+1-j} = \frac{(-1)^n}{(2n+1)\left(\begin{array}{c} 2n \\ n \end{array}\right)}.$$

By these short introduction, we provided fundamental results that are necessary for our derivation in the following sections. Thus, this paper is outlined as follows. In Section 2, by using functional equations of the generating functions, we derive some identities related to the beta-type polynomials. In Section 3, we obtain various combinatorial sums and identities by applying the Riemann integral to identities related to the beta-type polynomials. In Section 4, combinatorial identities, which are derived in Section 3, are used to obtain identities and relations related to finite sums and the Catalan numbers. We also give some remarks and relations on the generating functions for the Catalan numbers. In Section 5, we give further remarks and observation on combinatorial sums, binomial identities, Harmonic numbers, and the Bernstein basis functions. We also give some combinatorial sums and identities. In Section 6, by using the classical Young inequality, we present a combinatorial inequality including beta-type polynomials. We also show another inequality for the Catalan numbers.

2. Identity for beta-type polynomials

In this section, by using functional equations of the generating functions, we drive some new identities related to the beta-type polynomials. By using (1.1), we derive the following functional equation:

$$(1+x)^k \mathfrak{h}_k(t,x) = x^k e^t e^{tx}.$$

Combining (1.1) with the above function yields

$$(1+x)^k \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x) \frac{t^n}{n!} = x^k \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}.$$

By using Cauchy product from the above equation, we obtain

$$(1+x)^k \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) x^{j+k} \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

2.1. Theorem. *The following identity holds true:*

$$(1+x)^k \mathfrak{M}_{k,n}(x) = \sum_{j=0}^{n} \binom{n}{j} x^{j+k}. \tag{2.1}$$

Now, by rearranging (1.1), we derive the following functional equation:

$$\mathfrak{h}_k(\frac{t}{2}, x)\mathfrak{h}_k(\frac{t}{2}, x) = \mathfrak{h}_{2k}(t, x).$$

By combining the above function with (1.1), we get

$$\sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x) \frac{t^n}{2^n n!} \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x) \frac{t^n}{2^n n!} = \sum_{n=0}^{\infty} \mathfrak{M}_{2k,n}(x) \frac{t^n}{n!}.$$

Applying the Cauchy product in the above equation, we derive

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{\mathfrak{M}_{k,j}(x)\mathfrak{M}_{k,n-j}(x)}{2^n} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{M}_{2k,n}(x) \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

2.2. Theorem. *The following identity holds true:*

$$\mathfrak{M}_{2k,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{\mathfrak{M}_{k,j}(x)\mathfrak{M}_{k,n-j}(x)}{2^n}. \tag{2.2}$$

Next, by using (1.1), we derive the following functional equation:

$$(1+x)^k \, \mathfrak{h}_k(t, x) e^{-t} = x^k e^{tx}. \tag{2.3}$$

By combining (2.3) with (1.1), we get

$$(1+x)^k \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} x^{k+n} \frac{t^n}{n!}.$$

Therefore, using the Cauchy product yields

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1+x)^k \, \mathfrak{M}_{k,j}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} x^{k+n} \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

2.3. Theorem. *The following identity holds true:*

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1+x)^k \, \mathfrak{M}_{k,j}(x) = x^{k+n}. \tag{2.4}$$

Again by using (1.1), we derive the following functional equation:

$$\prod_{j=1}^{v} \mathfrak{h}_{k_j}(t, x) = \mathfrak{h}_{k_1+k_2+\cdots+k_v}(vt, x) \tag{2.5}$$

By combining (2.5) with (1.1), we obtain

$$\sum_{n=0}^{\infty} v^n \mathfrak{M}_{k_1+k_2+\cdots+k_v,n}(x) \frac{t^n}{n!} = \prod_{j=1}^{v} \sum_{n_1=0}^{\infty} \mathfrak{M}_{k_{1j},n_{j1}}(x) \frac{t^{n_j}}{n_j!}$$

after some elementary algebraic calculations, we have

$$\sum_{n=0}^{\infty} v^n \mathfrak{M}_{k_1+k_2+\cdots+k_v,n}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_v=n} \prod_{j=1}^{v} \frac{\mathfrak{M}_{k_j,n_j}(x)}{n_j!}\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

2.4. Theorem. *Let v and n be positive integers. Then we have*

$$v^n \mathfrak{M}_{k_1+k_2+\cdots+k_v,n}(x) = \sum_{n_1+n_2+\cdots+n_v=n} \binom{n}{n_1, n_2, \cdots, n_v} \prod_{j=1}^{v} \mathfrak{M}_{k_j,n_j}(x) \tag{2.6}$$

Substituting $v = 2$ into (2.6), we have the following result:

2.5. Corollary. *The following identity holds true:*

$$2^n \mathfrak{M}_{k_1+k_2,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{M}_{k_1,j}(x)\mathfrak{M}_{k_2,n-j}(x). \tag{2.7}$$

In order to derive some combinatorial sums in next section, we also need the following theorems. The proofs of which are given in [24].

2.6. Theorem ([24]). *The following identity holds true:*

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \mathfrak{M}_{0,j}(x) = x^n. \tag{2.8}$$

2.7. Theorem ([24]). *The following identity holds true:*

$$(1+x)^k \mathfrak{M}_{k,n}(x) = \sum_{j=0}^{n} \binom{n}{j} x^{n+k-j}. \tag{2.9}$$

2.8. Theorem ([24]). *Let $n \neq 0$. Then we have*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \mathfrak{M}_{k,j}(x)\mathfrak{M}_{k,n-j}(x) = 0. \tag{2.10}$$

3. Combinatorial sums and identities

By applying the Riemann integral and the beta functions to identities including the beta polynomials and $\mathfrak{M}_{k,n}(x)$, we derive various combinatorial sums and identities.

Integrating both sides of (2.8) from -1 to 0 and using (1.7), we arrive at the following theorem:

3.1. Theorem. *The following identity holds true:*

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} = 1. \tag{3.1}$$

3.2. Remark. Equation (3.1) can also be written as

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{1}{j+1} = \frac{(-1)^n}{n+1}.$$

Another proof of which is a special case of the following Gould's equation [10, P. 5, Eq-(1. 37)]):

$$\sum_{j=0}^{n} \binom{n}{j} \frac{x^j}{j+1} = \frac{(x+1)^{n+1} - 1}{x(n+1)}.$$

Notice here that letting $x=-1$ into the above identity yields (3.1), that is

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{1}{j+1}=\frac{1}{n+1}.$$

3.3. Theorem. *Let n and k be positive integer with $n\geq k$. Then we have*

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{n-j+k+1}=\frac{1}{(n+k+1)\binom{n+k}{k}}. \tag{3.2}$$

Proof. Integrating both sides of (2.9) from -1 to 0 and using (1.7), we get

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{n-j+k+1}=\frac{k!n!}{(n+k+1)!}.$$

If substitute the following well-known identity

$$\binom{n+k}{k}=\binom{n+k}{n}=\frac{(n+k)!}{n!k!}$$

in the above equation, we arrive at the desired result. □

Replacing k by n in (3.2) yields the following result.

3.4. Corollary.

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{2n-j+1}=\frac{1}{(2n+1)\binom{2n}{n}}.$$

Integrating both sides of (2.1) from -1 to 0 and using (1.7), we arrive at the following theorem:

3.5. Theorem.

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{1}{j+k+1}=\frac{1}{(n+k+1)\binom{n+k}{k}}. \tag{3.3}$$

If we replace k by n in (3.3), we obtain the following result:

3.6. Corollary.

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{1}{j+n+1}=\frac{1}{(2n+1)\binom{2n}{n}}.$$

Combining (3.2) with (3.3) yields the following corollary:

3.7. Corollary. *Let n and k be positive integer with $n\geq k$. Then we have*

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{n-j+k+1}=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{1}{j+k+1}.$$

Integrating both sides of (2.10) from -1 to 0 and using (1.7), we arrive at the following theorem:

3.8. Theorem. *Let n be a positive integer. Then we have*

$$\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}=0. \tag{3.4}$$

3.9. Remark. For alternative proof of (3.4), see [10, P. 4 Eq-(1, 25)].

3.10. Theorem.

$$\sum_{l=0}^{n}(-1)^l \frac{1}{\binom{n}{l}} = (1+(-1)^n)\frac{n+1}{n+2}. \tag{3.5}$$

Proof. Adding both sides of (1.2) from $k=0$ to n, we obtain

$$\sum_{k=0}^{n}\mathfrak{M}_{k,n}(x) = (1+x)^{n+1} - x^{n+1}. \tag{3.6}$$

Integrating both sides of (3.6) from -1 to 0 and using (1.7), we arrive at (3.5). □

3.11. Remark. An alternative proof of (3.5) can be found in [11, Vol. 3, Eq-(5.13)]. Also, by using the below equation given by Gould [10, P. 18, Eq-(1)]

$$\sum_{l=0}^{n}(-1)^l \frac{1}{\binom{x}{l}} = \left(1 + \frac{(-1)^n}{\binom{x+1}{n+1}}\right)\frac{x+1}{x+2} \tag{3.7}$$

and setting $x=n$, one can also easily arrive at equation (3.6).

Substituting $x=2n$ into (3.7) yields the following result:

3.12. Corollary.

$$\sum_{l=0}^{n}(-1)^l \frac{2}{\binom{2n}{l}} - \frac{(-1)^n}{\binom{2n}{n}} = \frac{2n+1}{n+1}.$$

Integrating both sides of (2.4) from -1 to 0 and using (1.7), we arrive at the following theorem:

3.13. Theorem. *Let n and k be positive integer with $n \geq k$. Then we have*

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B(k+1,j+1) = \frac{(-1)^n}{n+k+1},$$

where $B(u,v)$ denotes the beta function.

4. Remarks on the Catalan numbers

It is well-known that the Catalan numbers form a sequence of integers, and there are many applications of these numbers. For instance, these numbers have been used in the solutions of many counting problems. Also, they have been related to the Riordan arrays, the Riordan group, and the ballot problem (*cf.* [13], [2], [8], [12]; and the references cited therein).The Catalan numbers are defined by means of the following generating functions (*cf.* [2, Theorem 3.2], [8], [12, P. 344], [13], [14]):

$$g(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} C_n z^n,$$

where

$$0 < |z| \leq \frac{1}{4}; g(0) = C_0 = 1; C_n = \frac{1}{n+1}\binom{2n}{n}.$$

By using combinatorial identities in Section 3, we give relationships between finite sums and the Catalan numbers C_n. In [25], the Catalan numbers of order v are defined by

$$g^v(z) = \sum_{n=0}^{\infty} \mathfrak{C}_n^{(v)} z^n, \tag{4.1}$$

where $\mathfrak{C}_n^{(v)}$ denotes the Catalan numbers of order $v \in \mathbb{N}$. Setting $v = 2$ in the above equation, we get following relation (*cf.* [12, P. 343, Eq-(7. 65)], [8], [25]):

$$\mathfrak{C}_n^{(2)} = \sum_{k=0}^{n} C_k C_{n-k}.$$

By (4.1), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{C}_n^{(v)} z^n = \sum_{n=0}^{\infty} C_n z^n \sum_{n=0}^{\infty} C_n z^n \cdots \sum_{n=0}^{\infty} C_n z^n.$$

It can be seen that there are v factors on the right-hand side of the above equation. Hence, using the Cauchy product in the above equation yields

$$\sum_{n=0}^{\infty} \mathfrak{C}_n^{(v)} z^n = \sum_{n=0}^{\infty} \sum_{j_1,j_2,\cdots,j_v=0}^{j_1+j_2+\cdots+j_v=n} C_{j_1} C_{j_2} \cdots C_{n-j_1-j_2-\cdots-j_{v-1}-j_v} z^n,$$

where

$$\sum_{j_1,j_2,\cdots,j_v=0}^{j_1+j_2+\cdots+j_v=n} = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_v=0}^{n-j_1-j_2-\cdots-j_{v-1}}.$$

By comparing the coefficients of z^n on the both sides of the above equation, we obtain the following theorem:

4.1. Theorem. *Let v be an positive integer. Then we have*

$$\mathfrak{C}_n^{(v)} = \sum_{j_1,j_2,\cdots,j_v=0}^{j_1+j_2+\cdots+j_v=n} C_{j_1} C_{j_2} \cdots C_{j_v}.$$

4.2. Remark. Proof of the above theorem can be done by the mathematical induction method. This theorem was also proved by Graham, Knuth and Patashnik [12, P. 343, Eq-(7. 65)].

From Corollary 3.4, the following result can be derived:

4.3. Corollary.

$$\frac{1}{C_n} = (n+1)(2n+1) \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{1}{2n-j+1}.$$

From Corollary 3.6, we obtain the following result:

4.4. Corollary.

$$\frac{1}{C_n} = (n+1)(2n+1) \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \frac{1}{n+j+1}.$$

5. Remarks on the harmonic numbers and finite sums

In this section, by using Gould's identities [10], related to finite sums, we derive below combinatorial sums. In order to obtain these sums, we need the harmonic numbers defined by

$$H_n = \sum_{k=0}^{n} \frac{1}{k}$$

(*cf.* [5], [27], [10]). The following theorem gives us a relation between combinatorial sum and the harmonic numbers.

5.1. Theorem. *Let n be a positive integer. Then we have*

$$H_{n+1} = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{k+1}, \tag{5.1}$$

where H_n denotes the harmonic numbers.

Proof. In [10, P. 5, Eq-(1.37)], Gould gave the following identity:

$$(n+1)\sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{k+1} = \frac{(x+1)^{n+1}-1}{x}. \tag{5.2}$$

Integrating both sides of of the above equation from -1 to 0, we get

$$(n+1)\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)^2} = \sum_{k=0}^{n} \int_{-1}^{0} (x+1)^k dx.$$

After some elementary calculations, we arrive at the desired result. □

A different proof of the following theorem was also given by Chu ([5]).

Integrating both sides of (5.2) from 0 to 1, we arrive at the following theorem:

5.2. Theorem. *Let n be a positive integer. Then we have*

$$H_{n+1} = \sum_{k=0}^{n} \frac{2^{k+1}}{k+1} - \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{1}{k+1}. \tag{5.3}$$

Note also that the proof of this theorem is same as that of (5.1).

Combining (5.1) with (5.3), we get the following result:

5.3. Corollary.

$$\sum_{k=0}^{n} \left(1+(-1)^k\right) \binom{n+1}{k+1} \frac{1}{k+1} = \sum_{k=0}^{n} \frac{2^{k+1}}{k+1}.$$

Integrating both sides of the following equation from 0 to 1

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \frac{x^{2k}}{2k+1} = \frac{(x+1)^{n+1}-(1-x)^{n+1}}{2(n+1)x}$$

where $[x]$ denotes the greatest integer function [10, P. 6, Eq-(1.36)], we obtain:

5.4. Corollary.

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1} \frac{1}{2k+1} = \sum_{k=0}^{n} \frac{2^k}{k+1}.$$

5.5. Remark. The Bernstein basis functions $B_k^n(x)$ can be defined by

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{5.4}$$

where $k = 0, 1, \ldots, n$ (cf. [9], [20]). By (5.4), we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} B_k^n(x) = (1-x)^{n+1} - (-x)^{n+1} \tag{5.5}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} B_k^n(x) = \frac{x^{n+1} - (1-x)^{n+1}}{2x-1}. \tag{5.6}$$

Integrating both sides of (5.5) from 0 to 1 gives another proof of (3.5). Now, integrating both sides of (5.6) from 0 to 1, we arrive at the following well-known combinatorial sum:

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=0}^{n+1} \frac{2^k}{k}.$$

(*cf.* [1, 4, 6, 7, 10, 16, 23, 26, 29]).

6. Combinatorial inequalities

The classical Young inequality for two scalars is the v-weighted arithmetic-geometric mean inequality, which is a fundamental relation between two nonnegative real numbers. This inequality says that if $a, b > 0$ and $0 < v < 1$, then

$$a^v b^{1-v} \leq va + (1-v)b \tag{6.1}$$

with equality if and only if $a = b$ (*cf.* [17], [18], [15]). Substituting $v = \frac{1}{2}$ into (6.1), we have the following well-known inequality, which is known as arithmetic-geometric mean inequality,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Note that (6.1) can also be written as follows:

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \tag{6.2}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. The above well-known inequality also known as Young inequality (*cf.* [17], [18], [15]).

Let $x \geq 0$ and substitute $a = x^k$ and $b = (1+x)^{n-k}$ into (6.2), we get the following theorem:

6.1. Theorem. *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\mathfrak{M}_{k,n}(x) \leq \frac{1}{p} x^{kp} + \frac{1}{q} (1+x)^{q(n-k)} \tag{6.3}$$

Integrating both sides of (6.3) from 0 to 1, we arrive at the following theorem:

6.2. Theorem. *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} \frac{1}{k+j+1} \leq \frac{1}{p(pk+1)} + \frac{2^{q(n-k)+1} - 1}{q(n-k)+1}.$$

Since

$$\sqrt{n}\left(\begin{array}{c} 2n \\ n \end{array}\right) \geq 2^{2n-1},$$

(*cf.* [30]), we have the following result:

6.3. Corollary.

$$C_n \geq \frac{4^n}{2\,(n+1)\,\sqrt{n}}.$$

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