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Spectral problems for operators with deviating arguments

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Abstract

The topic of this paper are direct and inverse spectral boundary problems of the Sturm-Liouville type with two deviating arguments, one delay and one advance. This type of problem was firstly introduced by M. Pikula, E. Čatrnja, I. Kalčo and A. Šarić at the 9th International Scientific Conference "Science and Higher Education in Function of Sustainable Development — SED 2016" and further developed M. Pikula, E. Čatrnja, I. Kalčo at the International Conference "Contemporary Problems of Mathematical Physics and Computational Mathematics" dedicated to the 110th anniversary of A. N. Tikhonov. In this paper we take both delays to have the same value and in its first part solve the direct boundary problem, construct the corresponding characteristic function and find the asymptotic behavior of eigenvalues. In the second part of the paper, we give the necessary and sufficient conditions for the existence of the solution of the inverse problem and give its solution by the method of Fourier coefficients.

Keywords: Inverse problem with delays, Fourier trigonometric coefficient, Volterra integral equation, boundary spectral problems

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1. Introduction

Direct and inverse spectral boundary problems of the Sturm-Liouville type are a field of differential equations to which many mathematicians gave their contribution. We consider [15] and [11] as good introductory books to this topic. A large contribution to this area gave M. Pikula with his associates in [8], [9], [13], where they consider Sturm-Liouville type differential equations with one and more delays of different type. The problem with a constant delay is also covered by [5]. We also must not forget a contribution to this area given by papers [3], [2], [1] and [12].

In this paper we consider the following boundary value problem on the interval $[0,\pi]$

$$-y''(x)+q_1(x)y(x-\tau)+q_2(x)y(x+\tau)=\lambda y(x),\ \lambda=z^2, \tag{1.1}$$

$$y'(0)-hy(0)=0, \tag{1.2}$$

$$y'(\pi)+Hy(\pi)=0, \tag{1.3}$$

$$y(x-\tau)=1,\ x\in[0,\tau], \tag{1.4}$$

$$y(x+\tau)=1,\ x\in(\pi-\tau,\pi]. \tag{1.5}$$

where $q_1,q_2\in L_2[0,\pi]$.

For τ we will assume that

$$\frac{\pi}{2}\le\tau<\pi \tag{1.6}$$

In the following the boundary value problem (1.1, 1.2, 1.3, 1.4, 1.5) will be denoted with $D^2y=z^2y$.

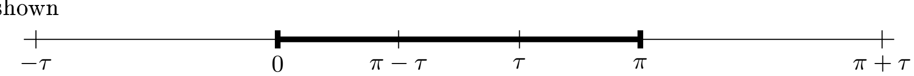
The first part of this paper is devoted to the obtaining of solution of the problem (1.1, 1.2, 1.4, 1.5), construction of the characteristic function and determination of the asymptotic behavior of eigenvalues. In the paper the operator $D^2=D^2(\tau,q_1,q_2,h,H)$ is the Sturm-Liouville type operator with deviating arguments. We will also assume that $q_2(x)\equiv 0$, $x\in[0,\pi-\tau]$.

2. Direct problem

2.1. Construction of solutions. Problem (1.1, 1.2) is equivalent to the integral Volterra equation

$$\begin{aligned} y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int_0^x q_1(t_1)\sin z(x-t_1)y(t_1-\tau,z)\,\mathrm{d}t_1+\\ +\frac{1}{z}\int_0^x q_2(t_1)\sin z(x-t_1)y(t_1+\tau,z)\,\mathrm{d}t_1. \end{aligned} \tag{2.1}$$

Let us find the solution of (2.1) by the steps method. Divide the interval $[0,\pi]$ as shown



On interval $[0,\pi-\tau]$ is $q_2(t_1)\equiv 0$, so (2.1) becomes

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int_0^x q_1(t_1)\sin z(x-t_1)y(t_1-\tau,z)\,\mathrm{d}t_1,\ x\in[0,\pi-\tau].$$

Using (1.4) we get the solution

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)\,\mathrm{d}t_1,\ x\in[0,\pi-\tau]. \tag{2.2}$$

For $x\in(\pi-\tau,\tau]$ using (1.5) we have

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)\,\mathrm{d}t_1+\frac{1}{z}\int\limits_{\pi-\tau}^x q_2(t_1)\sin z(x-t_1)\,\mathrm{d}t_1. \tag{2.3}$$

For $x\in(\tau,\pi]$ we have

$$\begin{aligned}y(x,z)&=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int\limits_0^\tau q_1(t_1)\sin z(x-t_1)\,\mathrm{d}t_1+\\&+\frac{1}{z}\int\limits_{\pi-\tau}^x q_2(t_1)\sin z(x-t_1)\,\mathrm{d}t_1+\frac{1}{z}\int\limits_\tau^x q_1(t_1)\sin z(x-t_1)y(t_1-\tau,z)\,\mathrm{d}t_1.\end{aligned} \tag{2.4}$$

From (2.3) follows

$$\begin{aligned}y(t_1-\tau,z)&=\cos z(t_1-\tau)+\frac{h}{z}\sin z(t_1-\tau)+\\&+\frac{1}{z}\int\limits_0^{t_1-\tau} q_1(t_2)\sin z(t_1-\tau-t_2)\,\mathrm{d}t_2\end{aligned} \tag{2.5}$$

Inserting (2.5) in (2.4) we obtain

$$\begin{aligned}y(x,z)&=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\int\limits_0^\tau q_1(t_1)\sin z(x-t_1)\,\mathrm{d}t_1+\\&+\frac{1}{z}\int\limits_{\pi-\tau}^x q_2(t_1)\sin z(x-t_1)\,\mathrm{d}t_1+\frac{1}{z}\int\limits_\tau^x q_1(t_1)\sin z(x-t_1)\cos z(t_1-\tau)\,\mathrm{d}t_1+\\&+\frac{h}{z^2}\int\limits_\tau^x q_1(t_1)\sin z(x-t_1)\sin z(t_1-\tau)\,\mathrm{d}t_1+\\&+\frac{1}{z^2}\int\limits_\tau^x q_1(t_1)\sin z(x-t_1)\int\limits_0^{t_1-\tau} q_1(t_2)\sin z(t_1-\tau-t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1.\end{aligned} \tag{2.6}$$

Let us introduce the following functions

$$a_s^{(1)}(\tilde{x},x,z)=\int\limits_0^{\tilde{x}} q_1(t_1)\sin z(x-t_1)\,\mathrm{d}t_1,$$

$$a_c^{(1)}(\tilde{x},x,z)=\int\limits_0^{\tilde{x}} q_1(t_1)\cos z(x-t_1)\,\mathrm{d}t_1,$$

$$a_s^{(2)}(x,z)=\int\limits_{\pi-\tau}^{x} q_2(t_1)\sin z(x-t_1)\,\mathrm{d}t_1,$$

$$a_c^{(2)}(x,z)=\int\limits_{\pi-\tau}^{x} q_2(t_1)\cos z(x-t_1)\,\mathrm{d}t_1,$$

$$a_{sc}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\sin z(x-t_1)\cos z(t_1-\tau)\,\mathrm{d}t_1,$$

$$a_{c^2}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\cos z(x-t_1)\cos z(t_1-\tau)\,\mathrm{d}t_1,$$

$$a_{s^2}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\sin z(x-t_1)\sin z(t_1-\tau)\,\mathrm{d}t_1,$$

$$a_{cs}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\cos z(x-t_1)\sin z(t_1-\tau)\,\mathrm{d}t_1,$$

$$a_{s^2}^{(1,1)}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\sin z(x-t_1)\int\limits_0^{t_1-\tau} q_1(t_2)\sin z(t_1-\tau-t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1,$$

$$a_{cs}^{(1,1)}(x,z)=\int\limits_{\tau}^{x} q_1(t_1)\cos z(x-t_1)\int\limits_0^{t_1-\tau} q_1(t_2)\sin z(t_1-\tau-t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1,$$

Now (2.6) we can write in the form

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}a_s^{(1)}(\tau,x,z)+\frac{1}{z}a_s^{(2)}(x,z)+\frac{1}{z}a_{sc}(x,z)+ \\ +\frac{h}{z^2}a_{s^2}(x,z)+\frac{1}{z^2}a_{s^2}^{(1,1)}(x,z). \tag{2.7}$$

Herewith we have proved the following.

2.1. Theorem. *If $q_2(x)\equiv 0$ for $x\in[0,\pi-\tau]$, then the solution of the problem (1.1, 1.2, 1.4, 1.5) is given by*

- (2.2) *for* $x\in(0,\pi-\tau]$,
- (2.3) *for* $x\in(\pi-\tau,\tau]$,
- (2.6) *for* $x\in(\tau,\pi]$.

2.2. Asymptotic behavior of eigenvalues. From (2.7) it follows

$$\frac{\mathrm{d}y(x,z)}{\mathrm{d}x} = -z\sin zx + h\cos zx + a_c^{(1)}(\tau,x,z) + a_c^{(2)}(x,z) + a_{c^2}(x,z) + \frac{h}{z}a_{cs}(x,z) + \frac{1}{z}a_{cs}^{(1,1)}(x,z). \tag{2.8}$$

Inserting $x=\pi$ in (2.7) and (2.8) and using (1.3) we obtain the characteristic function $F(z)$ in the form

$$\begin{aligned} F(z) = {} & \left(-z + \frac{hH}{z}\right)\sin\pi z + (h+H)\cos\pi z + a_c^{(1)}(\tau,z) + a_c^{(2)}(z) + \\ & + a_{c^2}(z) + \frac{H}{z}a_s^{(1)}(\tau,z) + \frac{H}{z}a_s^{(2)}(z) + \frac{H}{z}a_{sc}(z) + \frac{h}{z}a_{cs}(z) + \\ & + \frac{1}{z}a_{cs}^{(1,1)}(z) + \frac{hH}{z^2}a_{s^2}(z) + \frac{H}{z^2}a_{s^2}^{(1,1)}(z). \end{aligned} \tag{2.9}$$

Herewith we have proved the following.

2.2. Theorem. *The characteristic function of problem $D^2y = z^2y$ is a whole function of the exponential type and unity growth by z.*

Let us first write the function $F(z)$ in more convenient form. Introduce the following functions

$$\mathcal{J}_1^{(1)}(\tau) = \int_\tau^\pi q_1(t_1)\,\mathrm{d}t_1 = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}_1(\theta)\,\mathrm{d}\theta, \qquad \hat{q}_1(\theta) = q_1\left(\theta + \frac{\tau}{2}\right),$$

$$a_c^{(1)}(\tau,z) = 2\int_0^{\frac{\tau}{2}} q_1(2\theta)\cos z(\pi-2\theta)\,\mathrm{d}\theta = 2\check{a}_c^{(1)}(z),$$

$$a_a^{(1)}(\tau,z) = 2\int_0^{\frac{\tau}{2}} q_1(2\theta)\sin z(\pi-2\theta)\,\mathrm{d}\theta = 2\check{a}_s^{(1)}(z),$$

$$a_{c^2}(z) = \frac{\mathcal{J}_1^{(1)}(\tau)}{2}\cos z(\pi-\tau) + \frac{1}{2}\hat{a}_c(z),$$

$$\hat{a}_c^{(1)}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}_1(\theta)\cos z(\pi-2\theta)\,\mathrm{d}\theta,$$

$$a_{sc}(z) = \frac{\mathcal{J}_1^{(1)}(\tau)}{2}\sin z(\pi-\tau) + \frac{1}{2}\hat{a}_s(z),$$

$$\hat{a}_s^{(1)}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}_1(\theta)\sin z(\pi-2\theta)\,\mathrm{d}\theta,$$

$$a_{cs}(z) = \frac{\mathcal{J}_1^{(1)}(\tau)}{2}\sin z(\pi-\tau) - \frac{1}{2}\hat{a}_s(z),$$

$$a_{s^2}(z) = -\frac{\mathcal{J}_1^{(1)}(\tau)}{2}\cos z(\pi-\tau) + \frac{1}{2}\hat{a}_c(z),$$

$$a_s^{(2)} = 2\int\limits_{\pi-\tau}^{\frac{\pi}{2}} q_2(2\theta)\sin z(\pi-2\theta)\,\mathrm{d}\theta = 2\check{a}_s^{(2)}(z),$$

$$a_c^{(2)}(z) = 2\check{a}_s^{(2)}(z), \qquad \check{q}_2(\theta) = q_2(2\theta).$$

Let us define the function $Q_1(\theta)$ as follows

$$Q_1(\theta) = \begin{cases} 0 & ,\theta\in\left[0,\frac{\tau}{2}\right)\cup\left(\pi-\frac{\tau}{2},\pi\right], \\ q_1(2\theta-\tau)\int\limits_{2\theta}^{\pi} q_1(t_1)\,\mathrm{d}t_1 - \int\limits_{\theta+\frac{\tau}{2}}^{2\theta} q_1(2t_1-2\theta-\tau)q_1(t_1)\,\mathrm{d}t_1 & ,\theta\in\left[\frac{\tau}{2},\frac{\pi}{2}\right], \\ -\int\limits_{\theta+\frac{\tau}{2}}^{\pi} q_1(2t_1-2\theta-\tau)q_1(t_1)\,\mathrm{d}t_1 & ,\theta\in\left(\frac{\pi}{2},\pi-\frac{\tau}{2}\right]. \end{cases}$$

We now have

$$a_{cs}^{(1,1)}(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q_1(\theta)\sin z(\pi-2\theta)\,\mathrm{d}\theta = a_s^{(1,1)}(z),$$

$$a_{s^2}^{(1,1)}(z) = -\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q_1(\theta)\cos z(\pi-2\theta)\,\mathrm{d}\theta = -a_c^{(1,1)}(z).$$

The function (2.9) takes the form

$$\begin{aligned} F(z) = {} & \left(-z+\frac{hH}{z}\right)\sin\pi z + (h+H)\cos\pi z + 2\check{a}_c^{(1)}(z) + 2\check{a}_c^{(2)}(z) + \\ & + \frac{\mathcal{I}_1^{(1)}(\tau)}{2}\cos z(\pi-\tau) + \frac{1}{2}\hat{a}_c^{(1)}(z) + \frac{2H}{z}\check{a}_s^{(1)}(z) + \frac{2H}{z}\check{a}_s^{(2)}(z) + \\ & + \frac{h+H}{2z}\mathcal{I}_1^{(1)}(\tau)\sin z(\pi-\tau) + \frac{H-h}{z}\hat{a}_s(z) - \frac{hH\mathcal{I}_1^{(1)}}{2z^2}\cos z(\pi-\tau) + \\ & + \frac{hH}{2z^2}\hat{a}_c^{(1)}(z) + \frac{1}{z}a_s^{(1,1)}(z) - \frac{H}{z^2}a_c^{(1,1)}(z). \end{aligned} \tag{2.10}$$

Because $F(-z) = F(z)$, $\forall z\in\mathbb{C}$, it follows $F(z_n) = 0 \Rightarrow F(-z_n) = 0$.

It is known ([4] and [6]) that all complex eigenvalue are located in the complex plane inside of a certain circle with the center in point $z = n$. That means that all sufficiently large values by modulus are near real axes. This is in complete analogy with the classical Sturm-Liouville problems.

In [7], [10] and [14] we observed the asymptotic behavior of eigenvalues of differential operators with two constant delays. In the same way it can be proved that the following theorem holds.

2.3. Theorem. *If $q_1, q_2 \in L_2[0,\pi]$, $q_2(x)\equiv 0$, $x\in[0,\pi-\tau]$, then eigenvalues of the operator D^2 have following asymptotic behavior*

$$\begin{aligned} \lambda_n = {} & n^2 + \left(p_0 + p_1\cos n\tau + \frac{2}{\pi}a_{2n}^{(1)} + \frac{2}{\pi}\hat{a}_{2n} + \frac{2}{\pi}a_{2n}^{(2)}\right) + \\ & + \frac{1}{n}\left(r_1\sin n\tau + r_2\sin 2n\tau\right) + \left(\frac{1}{n}\right),\ n\to\infty, \end{aligned} \tag{2.11}$$

where $p_0 = \dfrac{2}{\pi}(h+H)$, $p_1 = \dfrac{\mathcal{I}_1(\tau)}{2}$, $r_1 = -\dfrac{\tau}{\pi^2}\mathcal{I}_1(\tau)(H+h)$, $r_2 = \dfrac{\pi-\tau}{4\pi^2}\mathcal{I}_1^2(\tau)$,
$a_{2n}^{(1)} = \int_0^{\frac{\tau}{2}} q_1(2\theta)\cos 2n\theta\, d\theta$, $\hat{a}_{2n} = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\cos 2n\theta\, d\theta$, $a_{2n}^{(2)} = \int_{\pi-\frac{\tau}{2}}^{\frac{\pi}{2}} q_2(2\theta)\cos 2n\theta\, d\theta$.

3. Inverse problem

3.1. Definition. Set $\Pi=\{\tau,h,H,q_1,q_2\}$ is called set of parameters of the operator D^2.

3.2. Definition. If $\lambda_{nj},\ n\in\mathbb{N}_0,\ j=1,2$ are eigenvalues of operator D^2 obtained for $H_j,\ j=1,2$, then the set $\Lambda=\{\lambda_{nj}, n\in\mathbb{N}_0, j=1,2\}$ is called spectral characteristic of the problem ((1.1, 1.2, 1.3, 1.4, 1.5)).

3.3. Definition. Solve the inverse problem for D^2 means to construct set Π from known Λ and known function q_2.

3.1. Determining numbers τ, h, H_1, H_2, $\mathfrak{I}_1(\tau)$. We start from assumption that

$$\begin{aligned}\lambda_n = n^2 &+ \left(p_0 + p_1\cos n\tau + \frac{2}{\pi}a_{2n}^{(1)} + \frac{2}{\pi}\hat{a}_{2n} + \frac{2}{\pi}a_{2n}^{(2)}\right) + \\ &+ \frac{1}{n}\left(r_1\sin n\tau + r_2\sin 2n\tau\right) + \left(\frac{1}{n}\right),\ n\to\infty,\end{aligned} \tag{3.1}$$

where $a_{2n}^{(1)}$, $\hat{a}_{2n}$, $a_{2n}^{(2)}$ converges to zero as $\frac{1}{n^\alpha}$, $0<\alpha<1$ and $\sum\limits_{n=1}^{\infty}(a_{2n}^{(j)})^2<\infty$, $\sum\limits_{n=1}^{\infty}(\hat{a}_{2n})^2<\infty$.

From Hadamard's theorem we have

$$F_j(z) = \pi\lambda_{0j}\prod_{n=1}^{\infty}\frac{\lambda_{nj}}{n^2}\left(1-\frac{z^2}{\lambda_{0j}}\right)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_{nj}}\right). \tag{3.2}$$

In the following we assume the identities

$$\begin{aligned}F_j(z) = &\left(-z+\frac{hH_j}{z}\right)\sin\pi z + (h+H_j)\cos\pi z + 2\check{a}_c^{(1)}(z) + 2\check{a}_c^{(2)}(z) + \\ &+ \frac{\mathfrak{I}_1^{(1)}(\tau)}{2}\cos z(\pi-\tau) + \frac{1}{2}\hat{a}_c^{(1)}(z) + \frac{2H_j}{z}\check{a}_s^{(1)}(z) + \frac{2H_j}{z}\check{a}_s^{(2)}(z) + \\ &+ \frac{h+H_j}{2z}\mathfrak{I}_1^{(1)}(\tau)\sin z(\pi-\tau) + \frac{H_j-h}{z}\hat{a}_s(z) - \\ &- \frac{hH_j\mathfrak{I}_1^{(1)}}{2z^2}\cos z(\pi-\tau) + \frac{hH_j}{2z^2}\hat{a}_c^{(1)}(z) + \frac{1}{z}a_s^{(1,1)}(z) - \frac{H_j}{z^2}a_c^{(1,1)}(z).\end{aligned} \tag{3.3}$$

From (3.3) we have

$$H_2 - H_1 = \lim_{n\to\infty}\left[F_2(2m) - F_1(2m)\right]. \tag{3.4}$$

If the sequence $\lambda_{nj}-n^2$ is not null sequence, then $\mathfrak{I}_1(\tau)\neq 0$ and we consider

$$\mu_{nj} = \frac{\lambda_{n+2,j,k}-(n+2)^2-\lambda_{n-2,j,k}+(n-2)^2}{\lambda_{n+1,j,k}-(n+1)^2-\lambda_{n-1,j,k}+(n-1)^2}.$$

It is easily shown that

$$\mu_{nj} = 2\cos\tau_1 + o(1),\ n\to\infty,\ j=1,2.$$

Herewith we have determined $\tau\in\left[\frac{\pi}{2},\pi\right)$ with

$$\tau = \arccos\frac{1}{2}\mu_j,\ \mu_j = \lim_{n\to\infty}\mu_{nj}. \tag{3.5}$$

Let $\tau\in\left[\frac{\pi}{2},\pi\right]$.

Let $n_k^{(1)}$ and $n_k^{(2)}$ are subsequences for which holds

$$\cos n_k^{(i)}\tau \neq 0,\ \ i=1,2$$

and

$$\left|\cos n_k^{(2)}\tau - \cos n_k^{(1)}\tau\right| \geq \delta > 0, \forall k.$$

From (3.1) it follows

$$p_{0j} = \lim_{k\to\infty} \frac{\left(\lambda_{n_k^{(2)},j} - \left(n_k^{(2)}\right)^2\right)\cos n_k^{(1)}\tau - \left(\lambda_{n_k^{(1)},j} - \left(n_k^{(1)}\right)^2\right)\cos n_k^{(2)}\tau}{\cos n_k^{(2)}\tau - \cos n_k^{(1)}\tau} \tag{3.6}$$

and

$$\mathcal{I}_1^{(1)}(\tau) = \frac{\pi}{2}\lim_{k\to\infty} \frac{\lambda_{n_l^{(i)},j} - \left(n_k^{(i)}\right)^2 - p_{0j}}{\cos n_l^{(i)}\tau},\ i=1,2,\ j=1,2. \tag{3.7}$$

From (3.3) we have

$$\begin{aligned} &F_j\left(2k+\frac{1}{2}\right)\left(2k+\frac{1}{2}\right) + \left(2k+\frac{1}{2}\right)^2 = \\ &\quad = \left(2k+\frac{1}{2}\right)\left[2\check{a}_c^{(1)}\left(2k+\frac{1}{2}\right) + 2\check{a}_c^{(2)}\left(2k+\frac{1}{2}\right) + \frac{1}{2}\hat{a}_c^{(1)}\left(2k+\frac{1}{2}\right)\right]. \end{aligned}$$

Finally,

$$h = \lim_{k\to\infty}\left\{\frac{2k+\frac{1}{2}}{H_2-H_1}\left[F_2\left(2k+\frac{1}{2}\right) + F_1\left(2k+\frac{1}{2}\right)\right] - \frac{\mathcal{I}_1(\tau)}{2}\cos\left(2k+\frac{1}{2}\right)\tau\right\}. \tag{3.8}$$

So we have proved

3.4. Theorem. *Spectral characteristics* Λ *uniquely determines numbers* τ, h, H_1, H_2 *and* $\mathcal{I}_1(\tau)$.

3.2. Determining potential q_1.

Let

$$\begin{aligned} A(z) &= \frac{1}{H_2-H_1}[H_2F_1(z) - H_1F_2(z)] + z\sin\pi z - h\cos\pi z, \\ B(z) &= \frac{z}{H_2-H_1}[F_2(z) - F_1(z)] - h\sin\pi z - z\cos\pi z. \end{aligned}$$

From (3.3) we have

$$\begin{aligned} A(z) &= 2\check{a}_c^{(1)}(z) + 2\check{a}_c^{(2)}(z) + \frac{\mathcal{I}_1^{(1)}}{2}\cos z(\pi-\tau) + \frac{1}{2}\hat{a}_c^{(1)}(z) + \\ &\quad + \frac{h\mathcal{I}_1^{(1)}(\tau)}{2z}\sin z(\pi-\tau) - \frac{h}{2z}\hat{a}_s^{(1)}(z) + \frac{1}{z}a_s^{(1,1)}(z), \end{aligned} \tag{3.9c}$$

$$\begin{aligned} B(z) &= 2\check{a}_s^{(1)}(z) + 2\check{a}_s^{(2)}(z) + \frac{\mathcal{I}_1^{(1)}}{2}\sin z(\pi-\tau) + \frac{1}{2}\hat{a}_s^{(1)}(z) - \\ &\quad - \frac{h\mathcal{I}_1^{(1)}(\tau)}{2z}\cos z(\pi-\tau) + \frac{h}{2z}\hat{a}_c^{(1)}(z) - \frac{1}{z}a_c^{(1,1)}(z). \end{aligned} \tag{3.9s}$$

In the following we will do integration by parts on $\frac{\hat{a}_s^{(1)}}{z}$, $\frac{a_s^{(1,1)}}{z}$, $\frac{\hat{a}_c^{(1)}}{z}$, $\frac{a_c^{(1,1)}}{z}$.

Let

$$(\mathcal{J}^{(1)}\hat{a}^{(1)})_c(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}_1(\theta_1)\,\mathrm{d}\theta_1 \right) \cos z(\pi - 2\theta)\,\mathrm{d}\theta,$$

$$(\mathcal{J}^{(1)}\hat{a}^{(1)})_s(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}_1(\theta_1)\,\mathrm{d}\theta_1 \right) \sin z(\pi - 2\theta)\,\mathrm{d}\theta,$$

$$(\mathcal{J}^{(1)}a^{(1,1)})_c(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} Q_1(\theta_1)\,\mathrm{d}\theta_1 \right) \cos z(\pi - 2\theta)\,\mathrm{d}\theta,$$

$$(\mathcal{J}^{(1)}a^{(1,1)})_s(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} Q_1(\theta_1)\,\mathrm{d}\theta_1 \right) \sin z(\pi - 2\theta)\,\mathrm{d}\theta.$$

Now we can write

$$\text{(3.10)} \quad \begin{cases} \dfrac{h\mathcal{J}_1^{(1)}(\tau)}{2z} \sin z(\pi-\tau) - \dfrac{h}{2z}\hat{a}_s^{(1)}(z) = \dfrac{h\mathcal{J}_1^{(1)}(\tau)}{z} \sin z(\pi-\tau) - h(\mathcal{J}^{(1)}\hat{a}^{(1)})_c(z), \\ -\dfrac{h\mathcal{J}_1^{(1)}(\tau)}{2z} \cos z(\pi-\tau) + \dfrac{h}{2z}\hat{a}_c^{(1)}(z) = -h(\mathcal{J}^{(1)}\hat{a}^{(1)})_s(z), \\ \dfrac{a_s^{(1,1)}(z)}{z} = -\left(\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q_1(\theta)\,\mathrm{d}\theta \right) \dfrac{\sin z(\pi-\tau)}{z} + 2\left(\mathcal{J}^{(1)}a^{(1,1)}\right)_c(z), \\ \dfrac{a_c^{(1,1)}(z)}{z} = -\left(\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q_1(\theta)\,\mathrm{d}\theta \right) \dfrac{\cos z(\pi-\tau)}{z} - 2\left(\mathcal{J}^{(1)}a^{(1,1)}\right)_s(z). \end{cases}$$

From (3.9s) we have $\lim\limits_{z\to\infty} \dfrac{1}{z} a_c^{(1,1)}(z) = 0$, so from [15] we have $\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q_1(\theta)\,\mathrm{d}\theta = 0$.

Using (3.10) we can write (3.9c) and (3.9s) in the following form

$$\text{(3.11c)} \quad \begin{aligned} &2\check{a}_c^{(1)}(z) + 2\check{a}_c^{(2)}(z) + \frac{1}{2}\hat{a}_c^{(1)}(z) - h(\mathcal{J}^{(1)}\hat{a}^{(1)})_c(z) + 2(\mathcal{J}^{(1)}a^{(1,1)})_c(z) = \\ &\qquad = A(z) - \frac{\mathcal{J}_1^{(1)}(\tau)}{2} \cos z(\pi-\tau) - \frac{h\mathcal{J}_1^{(1)}(\tau)}{z} \sin z(\pi-\tau) \end{aligned}$$

$$\text{(3.11s)} \quad \begin{aligned} &2\check{a}_s^{(1)}(z) + 2\check{a}_s^{(2)}(z) + \frac{1}{2}\hat{a}_s^{(1)}(z) - h(\mathcal{J}^{(1)}\hat{a}^{(1)})_s(z) + 2(\mathcal{J}^{(1)}a^{(1,1)})_s(z) = \\ &\qquad = B(z) - \frac{\mathcal{J}_1^{(1)}(\tau)}{2} \sin z(\pi-\tau). \end{aligned}$$

Identities (3.11c) and (3.11s) are equivalent to the system of equations obtained by inserting $z = m$, $m \in \mathbb{N}_0$. We have

$$2 \cdot \frac{2}{\pi} \int\limits_{0}^{\frac{\tau}{2}} \check{q}_1(\theta) \cos 2m\theta \, \mathrm{d}\theta + 2 \cdot \frac{2}{\pi} \int\limits_{\frac{\pi-\tau}{2}}^{\frac{\pi}{2}} \check{q}_2(\theta) \cos 2m\theta \, \mathrm{d}\theta +$$

$$+ \frac{1}{2} \cdot \frac{2}{\pi} \int\limits_{\frac{\tau}{2}} \pi - \frac{\tau}{2} \hat{q}_1(\theta) \cos 2m\theta \, \mathrm{d}\theta - h \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}_1(\theta_1) \, \mathrm{d}\theta_1 \right) \cos 2m\theta \, \mathrm{d}\theta +$$

$$+ 2 \cdot \frac{2}{\pi} \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} Q_1(\theta_1) \, \mathrm{d}\theta_1 \right) \cos 2m\theta \, \mathrm{d}\theta = A_{2m},$$

where

$$\text{(3.12c)} \quad A_{2m} = \left[(-1)^m A(m) - \frac{\mathcal{I}_1^{(1)}(\tau)}{2} \cos m\tau - \mathcal{I}_1^{(1)}(\tau) \frac{\sin m\tau}{m} \right] \cdot \frac{2}{\pi}$$

and

$$2 \cdot \frac{2}{\pi} \int\limits_{0}^{\frac{\tau}{2}} \check{q}_1(\theta) \sin 2m\theta \, \mathrm{d}\theta + 2 \cdot \frac{2}{\pi} \int\limits_{\frac{\pi-\tau}{2}}^{\frac{\pi}{2}} \check{q}_2(\theta) \sin 2m\theta \, \mathrm{d}\theta +$$

$$+ \frac{1}{2} \cdot \frac{2}{\pi} \int\limits_{\frac{\tau}{2}} \pi - \frac{\tau}{2} \hat{q}_1(\theta) \sin 2m\theta \, \mathrm{d}\theta - h \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}_1(\theta_1) \, \mathrm{d}\theta_1 \right) \sin 2m\theta \, \mathrm{d}\theta +$$

$$+ 2 \cdot \frac{2}{\pi} \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int\limits_{\frac{\tau}{2}}^{\theta} Q_1(\theta_1) \, \mathrm{d}\theta_1 \right) \sin 2m\theta \, \mathrm{d}\theta = B_{2m},$$

where

$$\text{(3.12s)} \quad B_{2m} = \left[(-1)^{m+1} B(m) - \frac{\mathcal{I}_1^{(1)}(\tau)}{2} \sin m\tau \right] \cdot \frac{2}{\pi}.$$

Let us extend the function $\check{q}_1(\theta)$ from the interval $\left[0, \frac{\tau}{2}\right]$ on the interval $\left(\frac{\tau}{2}, \pi\right]$ and the function $\check{q}_2(\theta)$ from the interval $\left[\frac{\pi-\tau}{2}, \frac{\tau}{2}\right]$ on the interval $\left[0, \frac{\pi-\tau}{2}\right) \cup \left(\frac{\tau}{2}, \pi\right]$ with zeros.

Also, let us extend the functions $\int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1) \, \mathrm{d}\theta_1$ and $\int\limits_{\frac{\tau}{2}}^{\theta} Q(\theta_1) \, \mathrm{d}\theta_1$ from the interval $\left[\frac{\tau}{2}, \pi - \frac{\tau}{2}\right]$ and $\left[0, \frac{\tau}{2}\right) \cup \left(\pi - \frac{\tau}{2}, \pi\right]$ with zeros.

From (3.1) and (3.2) easily follows $A_{2m} \to 0$ i $B_{2m} \to 0$ $(m \to \infty)$ and $\sum\limits_{m=1}^{\infty} A_{2m}^2 < \infty$, $\sum\limits_{m=1}^{\infty} B_{2m}^2 < \infty$.

Hence, sequences A_{2m} and B_{2m} are the Fourier coefficients of some function $f \in L_2[0, \pi]$.

Therefore from (3.12c) and (3.12s) we obtain the equation

$$\text{(3.13)} \quad \frac{1}{2} \hat{q}_1(\theta) + 2\check{q}_1(\theta) - 2\check{q}_2(\theta) = h \int\limits_{\frac{\tau}{2}}^{\theta} \hat{q}_1(\theta_1) \, \mathrm{d}\theta_1 - 2 \int\limits_{\frac{\tau}{2}}^{\theta} Q_1(\theta_1) \, \mathrm{d}\theta_1 + f(\theta)$$

Thus we have proved the following result.

3.5. Theorem. *In order to functions q_1 and q_2 be functions of operator D^2 it is necessary and sufficient that they satisfy the equation* (3.13).

Let

$$f_1(\theta) = \begin{cases} f(\theta), & \theta \in \left[0, \frac{\pi-\tau}{2}\right] \\ f(\theta) + 2\check{q}_2(\theta), & \theta \in \left(\frac{\pi-\tau}{2}, \pi\right]. \end{cases}$$

Now, for $\theta \in \left[0, \frac{\tau}{2}\right]$ from (3.13) follows

$$\check{q}_1(\theta) = \frac{1}{2} f_1(\theta). \tag{3.14}$$

In the following, we consider the three cases

(1) $\tau = \frac{2\pi}{3}$, i.e. $\pi - \tau = \frac{\tau}{2} = \frac{\pi}{3}$, $\pi - \frac{\tau}{2} = \frac{2\pi}{3}$,
(2) $\tau \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$, i.e. $\frac{\tau}{2} < \pi - \tau$,
(3) $\tau \in \left(\frac{2\pi}{3}, \pi\right)$, i.e. $\pi - \tau < \frac{\tau}{2}$.

For $\theta \in \left(\frac{\tau}{2}, \pi - \frac{\tau}{2}\right]$ equation (3.13) for $\tau \in \left[\frac{\pi}{2}, \pi\right)$ is already solved. Hence, let us solve the equation for $\theta \in \left(\frac{\tau}{2}, \frac{\pi}{2}\right]$. From (2.2) we have

$$Q_1(\theta_1) = q_1(2\theta_1 - \tau) \int_{2\theta_1}^{\pi} q_1(t_1)\,\mathrm{d}t_1 - \int_{\theta_1 + \frac{\tau}{2}}^{2\theta_1} q_1(2t_1 - 2\theta_1 - \tau) q_1(t_1)\,\mathrm{d}t_1.$$

Because $q_1(2\theta_1 - \tau) = \check{q}_1\left(\theta_1 - \frac{\tau}{2}\right)$, $q_1(2t_1 - 2\theta_1 - \tau) = \check{q}_1\left(t_1 - \theta - \frac{\tau}{2}\right)$ and $q_1(t_1) = \hat{q}_1\left(t_1 - \frac{\tau}{2}\right)$ equation (3.13) takes the form

$$\begin{aligned} \hat{q}_1(\theta) = 2f_1(\theta) + \int_{\frac{\tau}{2}}^{\theta} \Bigg[& 2h\hat{q}_1(\theta) - 4\check{q}_1\left(\theta_1 - \frac{\tau}{2}\right) \int_{2\theta_1 - \frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}_1(t_1)\,\mathrm{d}t_1 + \\ & + 4 \int_{\theta_1}^{2\theta_1 - \frac{\tau}{2}} \check{q}_1(t_1 - \theta_1)\hat{q}_1(t_1)\,\mathrm{d}t_1 \Bigg] \mathrm{d}\theta. \end{aligned} \tag{3.15}$$

The function $\check{q}_1$ is defined on the interval $\left[0, \frac{\tau}{2}\right]$ and since $\theta_1 - \frac{\tau}{2} \in \left[0, \frac{\pi}{2} - \frac{\tau}{2}\right] \subset \left[0, \frac{\tau}{2}\right]$ and $t_1 - \theta_1 \in \left[0, \frac{\pi}{2} - \frac{\tau}{2}\right] \subset \left[0, \frac{\tau}{2}\right]$ the equation (3.15) is linear Volterra equation by function $\hat{q}$. Suppose that $\hat{q}_1^*$ is solution of (3.15).

Thus we have proved the result.

3.6. Theorem. *The equation* (3.14) *has one and only one solution $\hat{q}_1(x)$, $x \in [0, \pi]$ and we have the relation*

$$q_1(x) = \begin{cases} \check{q}_1\left(\frac{x}{2}\right), & x \in [0, \tau] \\ \hat{q}_1^*\left(x - \frac{\tau}{2}\right), & x \in (\tau, \pi]. \end{cases}$$

3.7. Corollary. *In order to set $\Pi = \{\tau, h, H_1, H_2, q_1\}$ by given function $\hat{q}_2$ be the set of parameters of the operator D^2 with the spectral characteristic $\Lambda = \{\lambda_{nj}, n \in \mathbb{N}_0, j = 1, 2\}$ it is necessary and sufficient to q_1 be the solution of the equation* (3.13) *and τ determined by* (3.5)*, $\mathcal{I}_1(\tau)$ by* (3.7)*, h by* (3.8) *and $h + H_J$ by* (3.6).

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