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Weightable quasi-metrics related to fuzzy sets

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Abstract

We show that the definition of a fuzzy set is directly related to the existence of a weightable quasi-metric on a universe. This relationship is also explored in terms of functional equations coming either from the membership function of a fuzzy set or from the disymmetry function of a quasi-metric.

Keywords: Fuzzy sets; weightable quasi-metrics; functional equations; representable total preorders.

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1. Introduction

The classical definition of a fuzzy set on a universe gives rise to additional structures. Thus, the concept of a fuzzy set can actually be regarded from various alternative points of view, as functional equations (see [6, 5]), nested families of sets and nested topologies (see [1, 7, 20]) or representable total preorders (see [12, 6]) among others. In addition, in [8] the concept of a weightable quasi-metric was shown to be closely related to some classical functional equations, of which some were also intimately linked to the definition of a fuzzy set of a universe as proved in [6, 5]. However, to complete the panorama, a *direct* link between the definition of a fuzzy set of a universe and the possibility of endowing that universe with a suitable weightable quasi-metric had not been explored yet. This will be our task to be developed throughout the present note.

As regards quasi-metrics, as already commented in [8], we may observe that in the last years, (weightable) quasi-metric spaces have proven to be useful in modeling many processes that arise in Theoretical Computer Science and that involve some situation of *asymmetry*. The aforementioned usefulness is due to the fact that quasi-metric spaces lack the symmetry and the Hausdorffness enjoyed by metric spaces. This fact allows to introduce techniques of measuring that, contrarily to the metric ones, reflect the asymmetry inherent to the computational process. Recent applications of (weightable) quasi-metrics to Complexity Analysis of Algorithms, Denotational Semantics and Program Correctness can be found e.g in [15, 16, 17, 18] and [14].

Inspired in part by its utility in Theoretical Computer Science, we focus our attention on the definition of (weightable) quasi-metric ([10, 13]). In [8] we already analyzed some functional equation that appears associated in a natural way. Indeed, weightable quasi-metrics were characterized as the quasi-metrics that satisfy the functional equation of circuit invariance. Furthermore, the disymmetry function of a weightable quasi-metric satisfies Sincov's functional equation and induces a representable total preorder. But Sincov's functional equation has been also proved in [6] to be closely related to the classical definition of a fuzzy set of a universe. We may then conclude, that a weightable quasi-metric could be then directly retrieved from the mere definition of a fuzzy set, by means of an associated real-valued bivariate function that satisfies Sincov's functional equation. This allows us to establish a link between apparently disparate notions, namely: i) fuzzy sets of a universe, ii) weightable quasi-metrics, iii) real-valued bivariate functions that satisfy Sincov's functional equation and iv) representable total preorders and their corresponding order-preserving real-valued utility functions.

The structure of the present note goes as follows: After the Introduction and the necessary definitions introduced in the subsequent Section 2 (Preliminaries), in Section 3 we analyze the relationship between fuzzy sets and weightable quasi-metrics on a universe. Some miscellaneous examples, looking for possible applications, are discussed in Section 4. A final Section 5 of concluding remarks, pointing out some open problems arising in this literature, will close the paper.

2. Previous concepts

Preliminaries on fuzzy sets.

2.1. Definition. Let U be a nonempty set, also called a *universe*. We call *crisp* subset X of U to any application of the form $\nu_X : U \to \{0,1\}$, so that $t \in X \Leftrightarrow \nu_X(t) = 1$, and consequently $t \notin X \Leftrightarrow \nu_X(t) = 0$. This is denoted $X \subseteq U$. The map ν_X is said to be the *characteristic* function of the subset X.

Generalizing this last notion of characteristic function, the standard definition of a fuzzy set is established now in a formal way.

2.2. Definition. ([22]) Let U be a universe. A *fuzzy set* X of U is defined as the graph of a function $\mu_X : U \to [0,1]$. The map μ_X is said to be the *membership function* (or *indicator degree*) of X. Notice that X is then a (crisp) subset of the Cartesian product $U \times [0,1]$.

The *support* of X is the crisp subset $Supp(X) = \{t \in U : \mu_X(t) \neq 0\} \subseteq U$, whereas the *kernel* of X is the crisp subset $Ker(X) = \{t \in U : \mu_X(t) = 1\} \subseteq U$. The fuzzy set X is said to be *normal* provided that it has nonempty kernel, and it is said to be *quasi-normal* when $\sup\{\mu_X(t) : t \in U\} = 1$.

2.3. Remark. A quasi-normal fuzzy subset X of a universe U may fail to be normal. In other words, despite 1 being the supremum of the set $\sup\{\mu_X(t) : t \in U\}$, it could still happen that this supremum is not attained at any point of the universe U. As a clear example, consider $U = (0,1)$ and the fuzzy subset X defined by $\mu_X(t) = t$ for every $t \in U$.

Given $\alpha \in [0,1]$, the crisp subset of U defined by $U_\alpha = \{t \in U : \mu_X(t) \geq \alpha\}$ is said to be the *α-cut* of the fuzzy set X. The family of α-cuts of a fuzzy set is obviously nested.

2.4. Remark. Suitable nested families of subsets of a given universe can be put in correspondence with the α-cuts of a fuzzy set, as analyzed in [1]. The term crisp is usually understood in contraposition to the term fuzzy. A fuzzy set X of U is identified to a crisp subset of U provided that its membership function μ_X takes values in $\{0,1\}$. The main difference between these concepts, namely crisp vs. fuzzy, is that unlike the first one, the second one allows a *graduation* in the membership of an element to a set X, so that $\mu_X(t) = \alpha$ means that the element $t \in U$ belongs to X with the *grade* $\alpha \in [0,1]$. If $\mu_X(t) = 0$ we interpret that t does not belong to X.

Preliminaries on quasi-metrics.

2.5. Definition. Let U be a nonempty set (also called a universe). By a *quasi-metric* on U we mean a function $d : U \times U \to \mathbb{R}$ such that for all $x, y, z \in U$ the following conditions hold:

(i) $d(x,y) \geq 0$;
(ii) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$;
(iii) $d(x,y) + d(y,z) \geq d(x,z)$.

Of course a metric on a set U is a quasi-metric d on U satisfying, in addition, the following condition for all $x, y \in U$:

(iv) $d(x,y) = d(y,x)$.

By a *quasi-metric space* we mean a pair (U,d) such that U is a universe and d is a quasi-metric on U.

Given a quasi-metric d on U, and an ordered pair $(x,y) \in U \times U$, the real number $F(x,y) = d(x,y) - d(y,x)$ is said to be the *disymmetry of the pair* (x,y). The function $F : U \times U \to \mathbb{R}$ defined by $F(x,y) = d(x,y) - d(y,x) \quad (x, y \in U)$ is said to be the *disymmetry function* associated to the quasi-metric d on the given universe U.

2.6. Definition. ([10, 14]) Let U be a universe. A quasi-metric d on U, as well as the associated quasi-metric space (U,d), are said to be *weightable* if there exists a function $w : U \to \mathbb{R}$ such that $d(x,y) + w(x) = d(y,x) + w(y)$ holds for every $x, y \in U$. The function w is called a *weighting function* for d.

In the particular case in which there is at least one weighting function that only takes non-negative values ($w(U) \subseteq [0,+\infty)$) we say that the quasi-metric d is *positively weightable*.

2.7. Example. Let $d : \mathbb{R}\times\mathbb{R} \to [0, +\infty)$ be the function given by $d(x, y) = \max\{y-x, 0\}$ for all $x, y \in \mathbb{R}$. It is clear that $(\mathbb{R}, d)$ is a quasi-metric space which is weightable with weighting function $w : \mathbb{R} \to \mathbb{R}$ given by $w(x) = x$ for all $x \in \mathbb{R}$.

2.8. Example. Next we provide a more informal and miscellaneous example.

Let us assume that the prices $P(x, y)$ that you pay to travel with a certain airline depend on the departure airport x as well as on the arrival airport y, in a way such that $P(x, y) \geq 0$ and $P(x, y) > 0 \Leftrightarrow x \neq y$. Moreover, any travel with a stop can never be cheaper than the corresponding direct flight, so that $P(x, z) \leq P(x, y) + P(y, z)$, for every $x, y, z \in U$, where U stands for the set of all the airports covered by this airline. Finally, the price $P(x, y)$ could be slightly different from the price $P(y, x)$, depending for instance, on local taxes. Under all this hypotheses, the function $P : U \times U \to [0, +\infty)$ is a quasi-metric. Suppose now that, apart from paying $P(x, y)$ for your boarding ticket in order to fly from city x to city y, you are also obliged to pay an amount $w(x)$ as an airport tax (to leave), that only depends on the departure airport. Assume also that the system of prices and taxes accomplishes that the total amount $P(x, y) + w(x)$ that you pay when you travel from a city x to a city y is equal to the money $P(y, x) + w(y)$ that you pay when coming back from y to x. In other words: $P(x, y)+w(x) = P(y, x)+w(y)$, for every $x, y \in U$. In this case, it is clear that the function P constitutes an example of a positively weightable quasi-metric, in the sense of Definition 2.6.

Preliminaries on functional equations and representable total preorders.

2.9. Definition. Let U be a universe. A bivariate function $F : U \times U \longrightarrow \mathbb{R}$ is said to satisfy the *Sincov's functional equation on the universe* U if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in U$.

The following result is well-known ([19]):

2.10. Proposition. *Let U be a universe. A bivariate function $F : U \times U \longrightarrow \mathbb{R}$ satisfies the Sincov's functional equation if and only if $F(x, y) = g(y) - g(x)$ $(x, y \in U)$, for some function $g : U \to \mathbb{R}$ that only depends on one single variable.*

Having this fact in mind, we say that a function $g : U \to \mathbb{R}$ *generates* F if $F(x, y) = g(y) - g(x)$, for every $x, y \in U$.

It is easy to prove that a function $g : U \to \mathbb{R}$ that generates F is unique up to an additive constant. Also, if U denotes a universe and $F : U \times U \longrightarrow \mathbb{R}$ is a function that satisfies the Sincov's functional equation, it holds true that if we fix an element $a \in U$ and a real number $k \in \mathbb{R}$, then there exists a unique function $g : U \to \mathbb{R}$ such that g generates F and $g(a) = k$

Let us recall now the notion of a total preorder.

2.11. Definition. Let U denote a universe. A *preorder* $\precsim$ on U is a binary relation on U which is reflexive and transitive.

An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set U is a preorder such that if $a, b \in U$ then $[a \precsim b]$ or $[b \precsim a]$ holds true. A total order is also called a *linear order*.

If $\precsim$ is a preorder on U, then the associated *asymmetric* relation is denoted by $\prec$, whereas $\sim$ will stand for the associated *equivalence* relation. These relations are respectively defined by $[a \prec b \Leftrightarrow (a \precsim b) \wedge \neg(b \precsim a)]$ and $[a \sim b \Leftrightarrow (a \precsim b) \wedge (b \precsim a)]$. Moreover, the binary relation $\precsim_d$ defined by $a \precsim_d b \Leftrightarrow b \precsim a$ for every $a, b \in U$, which is also a preorder on U, is said to be the *dual* preorder associated to $\precsim$.

The asymmetric part of a linear order is said to be a *strict linear order*.

2.12. Definition. Let U be a universe. Let $\precsim$ be a total preorder defined on U. The preorder $\precsim$ is said to be *representable* if there exists a function $u : U \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$, for every $x, y \in U$. The order-preserving function u involved is said to be a *utility function* (also known as an *isotony* or an *order isomorphism*) for the preorder $\precsim$ on U.

2.13. Remark. For further information concerning characterizations of the representability of total preorders through utility functions, see e.g. the first three chapters of [3].

Sincov's functional equations are closely related to the representability of total preorders defined on a universe U, as the next well-known result shows. (See e.g. Theorem 1 in [2]).

2.14. Proposition. *Let $\precsim$ be a total preorder defined on a universe U. Then $\precsim$ is representable if and only if there exists a bivariate function $F : U \times U \to \mathbb{R}$ such that F satisfies the Sincov's functional equation and, in addition, $x \precsim y \Leftrightarrow F(x, y) \leq 0$ holds for all $x, y \in U$.*

Weightable quasi-metrics and functional equations.

As proved in [8], the definition of a weightable quasi-metric gives rise to the consideration of several functional equations.

2.15. Theorem. *Let (U, d) be a quasi-metric space. The following statements are equivalent:*

(i) *The quasi-metric d is weightable.*

(ii) *The quasi-metric d satisfies the functional equation of the 3-circuit, namely $d(x, y) + d(y, z) + d(z, x) = d(x, z) + d(z, y) + d(y, x)$, for every $x, y, z \in U$.*

(iii) *For every $n \geq 3$, $n \in \mathbb{N}$, the quasi-metric d satisfies the functional equation of the n-circuit, namely $d(x_1, x_2)+d(x_2, x_3)+\ldots+d(x_{n-1}, x_n)+d(x_n, x_1) = d(x_1, x_n)+ d(x_n, x_{n-1}) + \ldots + d(x_3, x_2) + d(x_2, x_1)$, for every $x_1, x_2, x_3, \ldots, x_n \in U$. (Here $\mathbb{N}$ stands for the set of positive integer numbers).*

(iv) *For some $k \geq 3$, $k \in \mathbb{N}$, the quasi-metric d satisfies the functional equation of the k-circuit, namely $d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{k-1}, x_k) + d(x_k, x_1) = d(x_1, x_k)+d(x_k, x_{k-1})+\ldots+d(x_3, x_2)+d(x_2, x_1)$, for every $x_1, x_2, x_3, \ldots, x_k \in U$.*

v) *The disymmetry function F associated to d satisfies Sincov's functional equation $F(x, y) + F(y, z) = F(x, z)$, for every $x, y, z \in U$.*

Proof. See Theorem 3.2 and Theorem 3.5 in [8]. □

Now we consider an example that is related to periodical processes.

2.16. Example. Let $n > 1$ and $U = \{x_1, \ldots, x_n\}$. Define $d : U \times U \to \{0, \ldots, n-1\}$ as follows $d(x_i, x_j) = j - i$ if $i \leq j$ and $d(x_i, x_j) = n + j - i$ if $i > j$. It is straightforward to see that d is a quasi-metric on U. Moreover, being $x_i, x_j \in U$ we observe that $d(x_i, x_j) - d(x_j, x_i) = j - i - (n + i - j) = 2j - 2i - n$ if $i < j$; $d(x_i, x_j) - d(x_j, x_i) = 0$ if $i = j$; and finally $d(x_i, x_j) - d(x_j, x_i) = (n - i + j) - i + j = n - 2i + 2j$ if $i > j$.

If $n \geq 3$ this quasi-metric d is *not* weightable, by part (iii) of Theorem 2.15. In fact, $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) = 1 + 1 + (n - 2) = n$, whereas $d(x_1, x_3) + d(x_3, x_2) + d(x_2, x_1) = 2 + (n - 1) + (n - 1) = 2n$.

As aforesaid, this example is typical in periodical processes. Suppose for instance that we have a process in which the elements $\{x_1, \ldots, x_n\}$ appear periodically as the hours in a clock or the seasons in a year. (For instance, this happens when I insert a compact disk in the CD-slot of my car: the CD has 20 songs, but after hearing the last one, the system restarts again in the first soundtrack). Thus $d(x_i, x_j)$ measures the number of elements encountered from an occurrence of x_i to the next occurrence of x_j $(i, j = 1, \ldots, n)$.

3. Weightable quasi-metrics vs. fuzzy sets

A fuzzy set defined from a weightable quasi-metric on a universe.

Let us see how a weightable quasi-metric on a given universe generates a fuzzy set on it.

3.1. Lemma. *Let U be a universe. Let $F : U \times U \to [-1, 1]$ be a function that satisfies Sincov's functional equation. Then there exists a fuzzy subset X of U, such that $F(x, y) = \mu_X(y) - \mu_X(x)$, for all $x, y \in X$, where μ_X denotes the membership function of X.*

Proof. Fix an element $a \in U$. Let $k = \inf\{F(a, t) : t \in U\}$. Define $G(x) = F(a, x) - k$, for every $t \in U$. By definition of k we have that $G(x) \geq 0$ for all $t \in U$. Moreover, fixed $x \in X$ we notice that $G(x) = F(a, x) - k = F(a, x) - \inf\{F(a, t) : t \in U\} = \sup\{F(a, x) - F(a, t) : t \in U\} = \sup\{F(x, t) : t \in U\}$, because $F(a, x) + F(x, t) = F(a, t)$ by hypothesis. Since $F(U) \subseteq [-1, 1]$, it follows that $G(x) \leq 1$ for every $x \in U$. Hence $G(U) \subseteq [0, 1]$, so that G is the indicator of a certain fuzzy subset X of U. To conclude, notice that $F(x, y) = G(y) - G(x)$, for every $x, y \in X$. □

3.2. Proposition. *Let U denote a universe, and let d be a weightable quasi-metric on X such that $d(x, y) \leq 1$ for every $x, y \in X$. Then, there exists a fuzzy set X of the universe U such that $d(x, y) - d(y, x) = \mu_X(y) - \mu_X(x)$ holds for all $x, y \in U$.*

Proof. Notice that the disymmetry function $F : X \times X \to \mathbb{R}$ given by $F(x, y) = d(x, y) - d(y, x)$ $(x, y \in U)$ satisfies Sincov's functional equation by Theorem 2.15. In addition, F takes values in the interval $[-1, 1]$. Furthermore, by Lemma 3.1, there is a fuzzy set X of the universe U such that $d(x, y) - d(y, x) = F(x, y) = \mu_X(y) - \mu_X(x)$ holds for all $x, y \in U$. where μ_X denotes the membership function of the fuzzy set X. □

3.3. Remark. By the way, under the statement of Proposition 3.2 the fuzzy set X is not unique, in general. An easy example is the following: Let d denote the discrete metric on U, that is $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$ $(x, y \in U$. Then $d(x, y) - d(y, x) = 0$ for every $x, y \in U$. Consider the fuzzy subsets X, T of U, respectively defined by $\mu_X(t) = 1$ and $\mu_T(t) = 0.5$, for every $\in U$. Obviously $d(x, y) - d(y, x) = 0 = \mu_X(y) - \mu_X(t) = \mu_T(y) - \mu_T(x)$ holds for every $x, y \in U$, so that both fuzzy subsets X and T agree with the statement of Proposition 3.2.

The collection of all the fuzzy sets that share the property in the statement of Proposition 3.2, can actually be characterized through the following Proposition 3.4.

3.4. Proposition. *Let U stand for a universe. Let $F : U \times U \to [-1, 1]$ be a bivariate function that satisfies Sincov's functional equation. Let $f : U \to [0, 1]$ be defined as $f(x) = \sup\{F(t, x) : t \in U\}$. Let $a = \inf\{f(t) : t \in U\}$ and $b = \sup\{f(t) : t \in U\}$. Then X is a fuzzy set of U such that $F(x, y) = \mu_X(y) - \mu_X(x)(x, y \in X)$, where μ_X denotes the membership function of X, if and only if there exists a constant $k \in [-a, 1 - b]$ such that $\mu_X = f(x) + k$ holds true for every $x \in U$.*

Proof. See Proposition 3.3 in [6]. □

To force the *uniqueness* of the fuzzy subset induced by a Sincov's functional equation $F : U \times U \to [-1, 1]$ on a universe U, some additional condition is compulsory. One such condition is quasi-normality of the fuzzy set considered, as next Corollary 3.5 states.

3.5. Corollary. *Let U be a universe. Let $F : U \times U \to [-1, 1]$ be a function that satisfies Sincov's functional equation. Then there exists a unique quasi-normal fuzzy subset X of U, such that $F(x, y) = \mu_X(y) - \mu_X(x)$, for all $x, y \in X$, where μ_X denotes the indicator of X.*

Proof. First, let us prove the existence of a quasi-normal fuzzy subset X of U, such that X agrees with the statement of Proposition 3.2. To do so, take a fuzzy subset X such that its membership function μ_X satisfies that $F(x,y) = \mu_X(y) - \mu_X(x)$, for every $x, y \in X$. (The existence of X is guaranteed by Proposition 3.2). Let $q = \sup\{\mu_X(t) : t \in U\}$. Define now the fuzzy subset T of U, by means of the indicator μ_T given by $\mu_T(t) = \mu_X(t) + 1 - q$, for every $t \in U$. It is plain that t is quasi-normal and $F(x,y) = \mu_X(y) - \mu_X(x) = \mu_T(y) - \mu_T(x)$ for all $x, y \in U$.

By Proposition 3.4, if X_1 and X_2 are two fuzzy subsets of U such that $F(x,y) = \mu_{X_1}(y) - \mu_{X_1}(x) = \mu_{X_2}(y) - \mu_{X_2}(x)$ holds for all $x, y \in U$, then $\mu_{X_1} - \mu_{X_2}$ is a constant, say $k \in [-1,1]$. Therefore $\sup\{\mu_{X_1}(t) : t \in U\} = k + \sup\{\mu_{X_2}(t) : t \in U\}$. Consequently, if X_1 and X_2 are both quasi-normal, it follows that $k = 0$. Thus μ_{X_1} and μ_{X_2} coincide, so that X_1 and X_2 are indeed the same fuzzy set of the universe U. (See also Corollary 3.6 in [6]). □

3.6. Remark. The condition $d(x,y) \leq 1$ that appears in the statement of Proposition 3.2 is restrictive. However, if d is a *bounded* quasi-metric defined on a universe U, we still can obtain a fuzzy set from d, as follows: Let $K > 0$ be a bound for d, that is, $d(x,y) \leq K$ holds true for every $x, y \in X$. Consider now the quasi-metric $d' : X \times X \to [0,1]$ given by $d'(x,y) = \frac{d(x,y)}{K}$ $(x, y \in X)$. Obviously, d' satisfies the conditions of the statement of Proposition 3.2, so that we can get a fuzzy set X, of the universe U, such that $\frac{d(x,y)-d(y,x)}{K} = d'(x,y) - d'(y,x) = \mu_X(y) - \mu_X(x)$ holds true for all $x, y \in U$, where μ_X stands for the membership function of the fuzzy set X. Observe that, in particular, if U is compact with respect to a topology τ such that $d : X \times X \to [0, +\infty)$ is continuous as regards the product topology $\tau \times \tau$ on $X \times X$ and the usual topology on the real line, then d is bounded and the argument given above applies. A typical situation of this kind appears when the universe U is finite. Thus, any quasi-metric on a finite universe U gives rise to a fuzzy set of U.

As a matter of fact, looking at the proof of Proposition 3.2, we may realize that the condition $d(x,y) \leq 1$ $(x, y \in U)$ could be replaced by the weaker one $|d(x,y) - d(y,x)| \leq 1$ $(x, y \in U)$. In the same direction, the condition of d being bounded could still be replaced by a weaker condition, namely, the boundedness of the disymmetry function associated to the quasi-metric d.

Weightable quasi-metrics defined from fuzzy sets.

First we see how a weightable quasi-metric can be obtained in a natural way from the indicator function of a fuzzy set on a universe, provided that the support of the fuzzy set is the whole universe.

3.7. Theorem. *Let U be a universe. Let $\mu_X : U \to (0,1]$ define a fuzzy set X of the universe U. Let $F : U \times U \to [-1,1]$ be such that $F(x,y) = \mu_X(y) - \mu_X(x)$, for every $x, y \in U$. Then there exists a positively weightable quasi-metric $d : U \times U \to [0, +\infty)$ whose disymmetry function is F and so that μ_X is a weighting function for d.*

Proof. Observe that since $\mu_X(t) \neq 0$ $(t \in U)$, the fuzzy set X accomplishes that $Supp(X) = U$. Given $x, y \in U$, we define $d(x,y) = 0$ if $x = y$ and $d(x,y) = \mu_X(y)$ otherwise. Notice that $d(x,y) \geq 0$ holds true by definition of d. To check the triangle inequality, given $x, y, z \in U$ we distinguish the following cases:

Case 1: $x = y$. In this case we have that $0 = d(x,y) \leq d(x,z) + d(z,y)$ because, by definition of d, we have that $d(x,z) \geq 0$ and $d(z,y) \geq 0$.

Case 2: $x \neq y$; $y = z$. In this case we have that $d(x,y) = d(x,y) + d(y,y)$ because $d(y,y) = 0$ by definition of d. Thus $d(x,y) = d(x,z) + d(z,y)$ since z and y coincide.

Case 3: $x \neq y$; $x = z$. In this case the proof runs as in Case 2.

Case 4: $x \neq y$; $y \neq z$. In this case we have that $\mu_X(y) = d(x,y) \leq \mu(y) + d(x,z) = d(z,y) + d(x,z) = d(x,z) + d(z,y)$ since $d(x,z) \geq 0$ and $d(x,y) = d(z,y) = \mu_X(y)$ by definition of d.

Notice now that $d(x,y) = d(y,x) = 0$ immediately implies $x = y$ because, by hypothesis, $\mu_X(t) \neq 0$ for every $t \in U$.

Therefore d is a quasi-metric on the universe U.

It is straightforward to check that F is the disymmetry function associated to d, so that μ_X is a weighting function for d. Hence d is positively weightable. □

3.8. Remark. In fact, the condition of whole support that appears in Theorem 3.7 can still be replaced by the less restrictive one of asking the cardinality of the set $\{t \in U : \mu_X(t) = 0\}$ to be at most one. A particular case of this situation occurs when the membership function μ_X of the fuzzy set X is injective. The steps in the proof of Theorem 3.7 are still valid here since $d(x,y) = d(y,x) = 0$ still forces the equality $x = y$.

3.9. Remark. Notice that, in general, the quasi-metric satisfying the conditions in the statement of Theorem 3.7 *is not unique*. A trivial situation appears when X is actually a crisp set, so that $\mu_X(t) = 1$ for every $t \in U$. In that situation, any metric d on U is ad hoc. Notice also that whenever the set X is crisp, the quasi-metric d is actually the *trivial metric* on X, so that $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$ hold for every $x, y \in X$. The trivial metric d induces the *discrete* topology on X. However, when the quasi-metric d is different from the trivial metric, it also defines a non-trivial topology on the given universe U. (Compare this last fact to other parallel results analyzed in [1]).

Now we see how to define a weightable quasi-metric from any fuzzy set of a universe, even if the support of the fuzzy set does not coincide with the universe. To do so, we furnish here a suitable modification of the proof of Theorem 3.7.

3.10. Theorem. *Let U be a universe. Let μ_X be the membership function of a fuzzy set X of the universe U. Let $F : U \times U \to [-1,1]$ be such that $F(x,y) = \mu_X(y) - \mu_X(x)$, for every $x, y \in U$. Then there exists a positively weightable quasi-metric $d : U \times U \to [0,+\infty)$ whose disymmetry function is F and so that μ_X is a weighting function for d.*

Proof. Given $x, y \in U$, we define $d(x,y) = 0$ if $x = y$ and $d(x,y) = \mu_X(y) + 1$ otherwise. Notice that $d(x,y) \geq 0$ holds true by definition of d. Observe also that $d(x,y) = 0$ immediately implies $x = y$. To check the triangle inequality, given $x, y, z \in U$ we distinguish the following cases:

Case 1: $x = y$. In this case we have that $0 = d(x,y) \leq d(x,z) + d(z,y)$ because, by definition of d, we have that $d(x,z) \geq 0$ and $d(z,y) \geq 0$.

Case 2: $x \neq y$; $y = z$. In this case we have that $d(x,y) = d(x,y) + d(y,y)$ because $d(y,y) = 0$ by definition of d. Thus $d(x,y) = d(x,z) + d(z,y)$ since z and y coincide.

Case 3: $x \neq y$; $x = z$. This case is similar to Case 2.

Case 4: $x \neq y$; $y \neq z$. In this case we have that $d(x,y) = \mu_X(y) + 1 \leq \mu_X(z) + 1 + \mu_X(y) + 1 = d(x,z) + d(z,y)$.

Therefore d is a quasi-metric on the universe U.

It is straightforward to check that F is the disymmetry function associated to d, so that μ_X is a weighting function for d. Hence d is positively weightable. □

3.11. Remark. Needless to say, we could have directly proved this more general Theorem 3.10, ignoring the previous Theorem 3.7. Nevertheless, we have decided to include both of them, because the construction of the quasi-metric in the proof of Theorem 3.10 is a bit more artificial than that in Theorem 3.7, much more natural (see e.g. Example 4.1 and Remark 4.2 below). In addition, the quasi-metric built in Theorem 3.7 also has some topological implications as briefly commented in Remark 3.9.

4. Miscellaneous examples

Let us analyze some practical situations in which the results introduced along the previous sections may play a role.

4.1. Example. Some candidates compete in a contest that consists in doing an exam. A correct answer to all the questions posed in the exam is considered as "perfect". Thus, the score that a person obtains in the contest, from 0% till 100% of correct answers, gives us an idea of the perfection of the candidate. Obviously, that quality can be assigned this way a number between 0 and 1, so that the whole set of scores that the candidates have got in the exam can be understood as a fuzzy set of the universe of candidates. Suppose also that all candidates have some skills, so that none of them is assigned the score 0. Under these hypotheses, when comparing candidates x and y, and in order to say who is the best one, we may assign to the ordered pair (x, y) the value $d(x, y) = d(y, x) = 0$ if $x = y$ and, whenever $x \neq y$, then $d(x, y)$ is the score obtained for the candidate y. Intuitively, $d(x, y)$ gives to candidate x an information about how good has been y in the contest. We may then observe that d is actually a weightable quasi-metric on the set of candidates, following the ideas of the proof of Theorem 3.7 above. And candidate x is at least as good as candidate y if and only if $d(x, y) \leq d(y, x)$, or equivalently if and only if $d(y, x) - d(x, y) \geq 0$.

4.2. Remark. The ideas underlying in the previous example are *general*, in the following sense. Suppose that we have got a total preorder $\precsim$ defined on a finite universe U. Given an element $x \in U$ we may assign to x the percentile $p(x)$ such that x is at least as good as the $p(x)$ percent of the cardinality (i.e., number of elements) of U. Obviously, the function $\mu : U \to (0, 1]$ given by $\mu(t) = \frac{p(t)}{100}$ defines a fuzzy set, with whole support, of the universe U. Again, as in the proof of Theorem 3.7, the function $d : U \times U \to [0, 1]$ given by $d(a, b) = d(b, a) = 0 \Leftrightarrow a = b$ and $d(a, b) = \mu(b)$ if $a \neq b \in U$ is a quasi-metric. Moreover, $a \precsim b \Leftrightarrow d(b, a) \leq d(a, b)$ holds true for every $a, b \in U$.

4.3. Remark. The previous Example 4.1 and Remark 4.2 have been given for a situation in which the universe U is finite. However, when U is infinite and it is endowed with a *representable* total preorder $\precsim$, we may consider without loss of generality a utility function u defined on U and taking values in the unit interval $(0, 1)$, so that $x \precsim y \Leftrightarrow u(x) \leq u(y) \quad (x, y \in U)$. To do so, we can combine a given utility function $v : U \to \mathbb{R}$ representing $\precsim$ with the function $h : \mathbb{R} \to (0, 1)$ given by $h(t) = \frac{arctg(t)}{\pi} + \frac{1}{2}$ that is strictly increasing, so that the composition $u = h \circ v$ is also a utility function for $\precsim$. Once more, u can be considered as the membership function of a fuzzy set of U. And the bivariate map $d : U \times U \to [0, 1]$ given by $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ and $d(x, y) = u(y)$ if $x \neq y \quad (x, y \in U)$ is indeed a quasi-metric.

4.4. Example. Let $U = \{x_1, \ldots, x_n\}$ a set of stochastic independent basic events such that each event x_i has a probability $p_i \neq 0$ $(i = 1, \ldots, n)$ and $p_1 + \ldots + p_n = 1$. Obviously, these probabilities give rise to a fuzzy set X of U whose membership function μ_X is given by the probabilities, that is $\mu_X(x_i) = p_i \quad (i = 1, \ldots, n)$. The novelty here is that we can combine independent events to get more sophisticated events in the algebra generated by U. (For instance, the probability of the event "x_j or x_k" is $p_j + p_k$ and the probability of the event "x_i will not happen" is $1 - p_i$). In other words, from the fuzzy set X on the universe U we may generate another fuzzy set on a new universe U' such that U' is the algebra generated from U by taking unions and complements. Since U is finite, that algebra is the power set of U. Of course, we may pass to consider the corresponding weightable quasi-metric associated to the fuzzy set X following the ideas introduced in the previous section (see, in particular, Remark 3.8), and we could still extend that quasi-metric to a new one, also weightable, defined on the power set of U.

4.5. Example. A shipping company operates in different coastal cities. The price $P(x, y)$ of a travel in a ferryboat from the city x to the city y ($\neq x$) is composed of two summands, namely an amount of money $a(x, y)$ that is directly proportional to the distance $d(x, y)$ between x and y, and a tax $b(y)$ paid to be allowed to debark in the second harbor, in the city y. That is $P(x, y) = 0$ if $x = y$ and $P(x, y) = k \cdot d(x, y) + b(y)$ otherwise, where $k > 0$ is a strictly positive constant. Let U be the set of cities covered by the shipping company. The function of prices $P : U \times U \to [0, +\infty)$ is, by its own construction, a positively weightable quasi-metric. Moreover, this quasi-metric is obviously bounded, since U is finite. So we may obtain a fuzzy set from P following the ideas introduced in Remark 3.6.

4.6. Example. (Problem of a travel in several stages.)

Usually, when transporting goods for delivery, trucks have to travel a great distance and make the travel in several stages. The cost of transporting goods from a city A to a city Z following a route ($A \to x_1 \to x_2 \to \cdots \to x_{n-1} \to Z$) depends on the chosen route. The cost associated to a route is the sum of the costs of every stage ($x_i \to x_{i+1}$), and the cost of any stage is given by adding two amounts: the first one (where oil and tolls are included) is proportional to the miles made in the stage (it is always a mathematical distance, $d(x_i, x_{i+1})$, whereas the second summand ($w(x_{i+1}) \geq 0$) depends only on the arriving place (staying the night, special conditions for the goods, or unload and load of goods, for example). It is a logistical problem to find the best route (the one with minimal cost) between two cities.

This problem can be represented with a weighted graph with loops (or reflexive graph). The cities are the nodes, the branches show the 1-stage possible routes and have a weight representing the first summand of the cost of the stage, and the loops at the nodes, also with a weight, represent the costs of the stop. A very easy modification of Dijkstra algorithm ([11]) provides the best route between two any nodes and also the minimal cost ($C(A, Z)$) of transporting goods between two points A and Z. This minimal cost can be calculated easily (by means of an algorithm) although it cannot be easily expressed with a mathematical formula. Besides, this function of total costs fails to be a metric (except in trivial cases). Instead, it is a weightable quasi-metric, whose disymmetry function has a much easier expression, namely $C(A, Z) - C(Z, A) = w(Z) - w(A)$. Of course, from this quasi-metric that is obviously bounded, we may obtain a fuzzy set, once more in the light of Remark 3.6.

5. Final remarks

As already commented in the Introduction, the mere definition of a fuzzy set as a function from a universe into the unit interval carries many additional structures, of different (but complementary) mathematical kind. Thus we may consider Sincov's functional equations, or nested topologies generated by α-cuts, or representable total preorders. Throughout this paper we have analyzed the relationship between fuzzy sets of a universe and weightable quasi-metrics, from a direct approach.

Some open question appears. For instance, in Theorem 3.7 we obtain a weightable quasi-metric from a fuzzy set on a universe. But the so obtained quasi-metric is not unique, in general. Therefore, we could search for some additional condition on the weightable quasi-metric in order to get uniqueness, in the spirit of Corollary 3.5.

As an unexplored possible application, that comes to our mind in a natural way, from the relationship between weightable quasi-metrics and fuzzy sets, we may think about *aggregation* of fuzzy sets vs. combination of weightable quasi-metrics into a new one. To put an example, if two fuzzy sets X_1 and X_2 on the same universe U are aggregated into o new fuzzy set X of U by means of a suitable operator, then we may consider that the quasi-metrics associated to X_1 and X_2 in the sense of Theorem 3.7 are also combined to get a new one, namely the quasi-metric associated to X (again in the sense of Theorem 3.7). Conversely, if d and d' are two bounded weightable quasi-metrics on a universe U, if we combine them in a way that gives rise to a new bounded weightable quasi-metric D, we may also interprete that the associated fuzzy sets associated to d and d' are also aggregated into a new one, namely the fuzzy set associated to the quasi-metric D, in the sense of Proposition 3.2 and Remark 3.6. A typical combination of bounded weighted quasi-metrics is the following one: if d and d' are bounded weighted quasi-metrics on a universe U, and $\alpha > 0$ and $\beta > 0$ are strictly positive real numbers, it is clear that $D = \alpha d + \beta d'$ is a weightable quasi-metric, too.

Finally, another unexplored possible application follows from the relationship between fuzzy sets and representable total preorders (see [5]). Having in mind studies on *extensions* of representable total preorders from a set to a bigger set (see e.g. [21, 9, 4]), we may think about extensions of a universe, and consequently consider extensions of fuzzy sets on those universes. Furthermore, these new and bigger fuzzy sets would give rise to extensions of weightable quasi-metrics, in the sense of Theorem 3.10.

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