Hyers–Ulam stability of first-order non-linear delay differential equations with fractional integrable impulses

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Abstract

This paper proves the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of first–order non–linear delay differential equations with fractional integrable impulses. Our approach uses abstract Grönwall lemma together with integral inequality of Grönwall type for piecewise continuous functions.

1. Introduction

Ulam, in 1940, queried [31] regarding the stability of functional equation for homomorphism in front of a Mathematical Colloquium: The question was " *When an approximate homomorphism from a group G_1 to a metric group G_2 can be approximated by an exact homomorphism?* ".

Within the next two years, Hyers [10] brilliantly gave a partial answer to this question for the case when G_1 and G_2 are assumed to be Banach spaces by using direct method. In 1978, Rassias [25] provided an extension of the Hyers–Ulam stability by introducing new function variables. As a result, another new stability concept, Hyers–Ulam–Rassias stability, was named by mathematicians. In fact, the most exciting result was of Rassias [25] that weakens the condition for the bound of the norm of Cauchy difference $f(x + y) - f(x) - f(y)$. For further details and discussions, we recommend the book by Jung [12].

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Many researchers paid attention to the stability properties of all kinds of equations since the work of Obłoza [21, 22]. We emphasize that Ulam's type stability problems have been taken up by a huge amount of mathematicians and the study of this region has grown–up to be one of the vital subjects in mathematical analysis. For more details on Hyers–Ulam stability, we recommend [3, 6, 7, 8, 9, 11, 13, 14, 15, 16, 19, 24, 26, 30, 34, 36, 37, 38, 39, 40, 41, 42, 29].

Many real world phenomenons are represented by smooth differential equations. However, the situation becomes quite different in the case when a physical phenomena has sudden changes in its state such as mechanical systems with impact, biological systems with heart beats, blood flows, population dynamics [1, 20], theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry [2], engineering [4], control theory, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses i.e impulsive differential equations.

An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active.

Fractional differential and integral equations plays a key role not only in mathematics but also in the modeling of various physical phenomena in physics, control systems and dynamical systems. In fact, fractional order derivatives and integrals are assumed to be more realistic and practical than derivatives and integrals of integral order. These are excellent tools to modeled memory and hereditary properties of several materials and processes.

To the best of our knowledge, the first mathematicians who investigated the Ulam's type stability of impulsive ordinary differential equations are Wang *et al.* [32]. Following their own work, in 2014, they proved the Hyers–Ulam–Rassias stability and generalized Hyers–Ulam–Rassias stability for impulsive evolution equations on a compact interval [33] which then they extended for infinite impulses in the same paper. Wang and Zhang [35] introduced a new class of differential equations; nonlinear differential equations with fractional integrable impulses which are more interesting. They presented four Bielecki–Ulam's type stabilities for this class of differential equations.

However, despite the situations where only impulsive factor is involved or delay effects happened we have a wide variety of evolutionary processes together delay and impulsive effects exists in their state. To modeled such phenomena which are subject to impulsive perturbations as the time delays an impulsive delay differential equation is used. In 2016, Zada *et al.* [39] using fixed point method discussed Hyers–Ulam stability and Hyers–Ulam–Rassias stability of first order impulsive delay differential equations. For more details on impulsive differential equations, we recommend [5, 17, 18, 23].

After studying the work done by Wang and Zhang [35] and Zada *et al.* [39], we are motivated to obtain the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of first–order non–linear delay differential equations with fractional integrable impulses of the form

$$(1.1) \quad \begin{cases} z^{'}(t) = F(t, z(t), z(h(t))), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ z(t) = I_{t_i, t}^{\alpha} g_i(t, z(t), z(h(t))), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \\ z(t) = \phi(t), \ t \in [s_0 - \lambda, s_0], \end{cases}$$

where $\lambda > 0$, $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \ldots t_m \le s_m \le t_m \le t_{m+1} = t_f$ are pre–fixed numbers, $F : (s_i, t_{i+1}] \times R^2 \to R$, $i = 0, 1, 2, \ldots, m$ is a continuous function, $g_i : (t_i, s_i] \times R^2 \to R$, $i = 1, 2, \ldots, m$ are continuous functions, $\phi : [s_0 - \lambda, s_0] \to R$ is history function and $I_{t_i,t}^\alpha g_i$ are so called Riemann–Liouville fractional integrals of order α and are given as:

$$I_{t_i,t}^\alpha g_i(t, z(t), z(h(t))) = \frac{1}{\Gamma(\alpha)} \int_{t_i}^t (t-s)^{\alpha-1} g_i(s, z(s), z(h(s))) ds.$$

Moreover, $h : [s_0 - \lambda, t_f] \to (s_i, t_{i+1}]$ is a continuous delay function such that $h(t) \le t$.

2. Preliminaries

In this section we list some important notations, definitions and lemmas that would be used in our main results. Through out this paper, the following spaces appear mostly.
a) $C(J, R)$ is the Banach space of all continuous real valued functions from J with norm $\|x\|_C = \max\{|x(t)| : t \in J\}$, where $J = [s_0 - \lambda, t_f]$ and R represents the set of real numbers.
b) $PC(J, R)$ denotes the Banach space of all functions $x : J \to R$ such that $x \in C((t_k, t_{k+1}], R]$, $k = 0, 1, 2, \ldots, m$ and there exists $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, \ldots, m$ such that $x(t_k^-) = x(t_k)$ with norm $\|x\|_{PC} = \max\{|x(t)| : t \in J\}$.
c) We set $PC^1(J, R) = \{x \in PC(J, R) : x' \in PC(J, R)\}$ is Banach space with norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.
Consider the following inequalities,

$$(2.1) \quad \begin{cases} \left| y'(t) - F(t, y(t), y(h(t))) \right| \le \epsilon, \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ \left| y(t) - I_{t_i,t}^\alpha g_i(t, y(t), y(h(t))) \right| \le \epsilon, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \end{cases}$$

$$(2.2) \quad \begin{cases} \left| y'(t) - F(t, y(t), y(h(t))) \right| \le \varphi(t), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ \left| y(t) - I_{t_i,t}^\alpha g_i(t, y(t), y(h(t))) \right| \le \kappa, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \end{cases}$$

where $\epsilon > 0$, $\kappa \ge 0$ and $\varphi \in PC(J, R^+)$ is an increasing function.

2.1. Definition. Equation (1.1) is Hyers–Ulam stable on J if for every $y \in PC^1(J, R)$ satisfying (2.1), there exists a solution $y_0 \in PC^1(J, R)$ of (1.1) with $|y_0(t) - y(t)| \le K\epsilon$, $K > 0$, for all $t \in J$.

2.2. Definition. Equation (1.1) is Hyers–Ulam–Rassias stable on J with respect to (φ, κ), if for every $y \in PC^1(J, R)$ satisfying (2.2), there exists a solution $y_0 \in PC^1(J, R)$ of (1.1) with $|y_0(t) - y(t)| \le M\varphi(t)$, $M > 0$, for all $t \in J$.

2.3. Definition. Let $(X; d)$ be a metric space. An operator $\Lambda : X \to X$ is a Picard operator if it has a unique fixed point $x^* \in X$ such that for every $x \in X$, the sequence $\{\Lambda^{(n)}(x)\}_{n \in N}$ converges to x^*.

2.4. Lemma. *(Grönwall Lemma[28]): If for $t \ge t_0 \ge 0$ we have,*

$$x(t) \le a(t) + \int_{t_0}^t b(s)x(s)ds + \sum_{t_0 < t_k < t} \xi_k x(t_k^-),$$

where $x, a, b \in PC[[t_0, \infty), R^+]$, a is non–decreasing and $b(t), \xi_k > 0$. Then for $t \ge t_0$ the following inequality works:

$$x(t) \le a(t) \prod_{t_0 < t_k < t} (1 + \xi_k) \exp \left(\int_{t_0}^t b(s)ds \right).$$

2.5. Lemma. *(Abstract Grönwall Lemma* [27]*): Let* $(X, d, \leq)$ *be an ordered metric space and* $\Lambda : X \to X$ *be an increasing Picard operator with fixed point* x^*. *Then for any* $x \in X$, $x \leq \Lambda(x)$ *implies* $x \leq x^*$ *and* $x \geq \Lambda(x)$ *implies* $x \geq x^*$, *where* x^* *is the fixed point of* Λ *in* X.

2.6. Remark. A function $y \in PC^1(J, R)$ satisfies (2.1) if and only if there is a function $f \in PC(J, R)$ and a sequence f_k (which depends on y) such that $|f(t)| \leq \epsilon$ for all $t \in J$, $|f_i| \leq \epsilon$ for all $i = 1, 2, \ldots, m$, and;

$$(2.3) \quad \begin{cases} y^{'}(t) = F(t, y(t), y(h(t))) + f(t), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ y(t) = I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) + f_i, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1). \end{cases}$$

We do similar remark for (2.2).

2.7. Lemma. *Every* $y \in PC^1(J, R)$ *that satisfies (2.1) also comes out perfect on the following inequality;*

$$\begin{cases} \left| y(t) - \phi(t_0) - \int_{s_i}^{t} F(s, y(s), y(h(s))) ds - I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) \right| \leq (t_f - s_i + m)\epsilon, \\ t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m, \\ \left| y(t) - I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) \right| \leq m\epsilon, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1). \end{cases}$$

Proof. If $y \in PC^1(J, R)$ satisfies (2.1), then by Remark 2.6 we have

$$(2.4) \quad \begin{cases} y^{'}(t) = F(t, y(t), y(h(t))) + f(t), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ y(t) = I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) + f_i, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \\ y(t) = \phi(t), \ t \in [s_0 - \lambda, s_0]. \end{cases}$$

Clearly the solution of (2.4) is given as

$$y(t) = \begin{cases} \phi(t_0) + \int_{s_i}^{t} \big(F(s, y(s), y(h(s))) + f(s) \big) ds + I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) + f_i, \\ t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, \\ I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) + f_i, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1). \end{cases}$$

For $t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1)$, we get

$$\begin{aligned} \left| y(t) - \phi(t_0) - \int_{s_i}^{t} F(s, y(s), y(h(s))) ds - I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) \right| \ &\leq \ \int_{s_i}^{t} |f(s)| ds + \sum_{i=1}^{m} |f_i| \\ &\leq \ (t - s_i + m)\epsilon \\ &\leq \ (t_f - s_i + m)\epsilon. \end{aligned}$$

Proceeding as above we derive

$$\left| y(t) - I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) \right| \ \leq \ m\epsilon, \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1).$$

We have similar remarks for (2.2). □

3. Main results

Now we are in the position to state our main results. First we are going to give our result on Hyers–Ulam stability.

3.1. Theorem. *If*
a) $F : (s_i, t_{i+1}] \times R^2 \to R$ *is continuous with the Lipschitz condition:*
$\left| F(t, x_1, x_2) - F(t, y_1, y_2) \right| \leq \sum_{k=1}^{2} L|x_k - y_k|$, $L > 0$, *for all* $t \in (s_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, m$
and $x_k, y_k \in R$, $k \in \{1, 2\}$,

b) $g_i : (t_i, s_i] \times R^2 \to R$ *satisfies the Lipschitz condition* $|g_i(t, u_1, u_2) - g_i(t, v_1, v_2)| \leq \sum_{k=1}^{2} L_{g_i}|u_k - v_k|$, $L_{g_i} > 0$, *for all* $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$ *and* $u_1, u_2, v_1, v_2 \in R$ *and*

c) $\left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t_f - s_i) \right) < 1$,

then equation (1.1) has
i) a unique solution in $PC^1(J, R)$;
ii) Hyers–Ulam stability on J.

Proof. i) Define an operator $\Lambda : PC(J, R) \to PC(J, R)$ by

$$
(3.1) \qquad (\Lambda z)(t) = \begin{cases} \phi(t), \ t \in [s_0 - \lambda, s_0], \\ I_{t_i, s_i}^{\alpha} g_i(s_i, z(s_i), z(h(s_i))), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \\ \phi(t_0) + I_{t_i, s_i}^{\alpha} g_i(s_i, z(s_i), z(h(s_i))) + \int_{s_i}^{t} F(s, z(s), z(h(s))) ds, \\ t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1). \end{cases}
$$

For any $z_1, z_2 \in PC(J, R)$, $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\left| (\Lambda z_1)(t) - (\Lambda z_2)(t) \right| &\leq \left| I_{t_i, s_i}^{\alpha} g_i(s_i, z_1(s_i), z_1(h(s_i))) - I_{t_i, s_i}^{\alpha} g_i(s_i, z_2(s_i), z_2(h(s_i))) \right| \\
&\quad + \int_{s_i}^{t} \left| F(s, z_1(s), z_1(h(s))) - F(s, z_2(s), z_2(h(s))) \right| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \left| g_i(s, z_1(s), z_1(h(s))) - g_i(s, z_2(s), z_2(h(s))) \right| ds \\
&\quad + L \int_{s_i}^{t} \left| z_1(s) - z_2(s) \right| ds + L \int_{s_i}^{t} \left| z_1(h(s)) - z_2(h(s)) \right| ds \\
&\leq \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \left| z_1(s) - z_2(s) \right| ds \\
&\quad + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \left| z_1(h(s)) - z_2(h(s)) \right| ds \\
&\quad + 2L \int_{s_i}^{t} \max_{s_i \leq s \leq t_{i+1}} |z_1(s) - z_2(s)| ds \\
&\leq \frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \max_{t_i \leq s \leq s_i} |z_1(s) - z_2(s)| ds + \\
&\quad 2L \int_{s_i}^{t} \max_{s_i \leq s \leq t_{i+1}} |z_1(s) - z_2(s)| ds \\
&\leq \left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t - s_i) \right) \|z_1 - z_2\| \\
&\leq \left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t_f - s_i) \right) \|z_1 - z_2\|.
\end{aligned}
$$

Following from (c), the operator is strictly contractive on $(s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, and hence a Picard operator on $PC(J, R)$. From (3.1), it follows that the unique fixed point of this operator is in fact the unique solution of (1.1) in $PC^1(J, R)$.

ii) Next, let $y \in PC^1(J, R)$ be a solution to (2.1). The unique solution $z \in PC^1(J, R)$ of the differential equation (1.1) is given by

$$
z(t) = \begin{cases}
\phi(t), \ t \in [s_0 - \lambda, s_0], \\
I_{t_i, s_i}^{\alpha} g_i(s_i, z(s_i), z(h(s_i))), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \ \alpha \in (0, 1), \\
\phi(t_0) + I_{t_i, s_i}^{\alpha} g_i(s_i, z(s_i), z(h(s_i))) + \displaystyle\int_{s_i}^{t} F(s, z(s), z(h(s)))ds, \ t \in (s_i, t_{i+1}], \\
i = 1, 2, \ldots, m, \ \alpha \in (0, 1).
\end{cases}
$$

We observe that for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, using 2.7 Lemma, we have

$$
\begin{aligned}
\left| y(t) - z(t) \right| &\leq \left| y(t) - \phi(t_0) - \int_{s_i}^{t} F(s, y(s), y(h(s)))ds - I_{t_i, t}^{\alpha} g_i(t, y(t), y(h(t))) \right| \\
&+ \left| I_{t_i, s_i}^{\alpha} g_i(s_i, z_1(s_i), z_1(h(s_i))) - I_{t_i, s_i}^{\alpha} g_i(s_i, z_2(s_i), z_2(h(s_i))) \right| \\
&+ \int_{s_i}^{t} \left| F(s, z_1(s), z_1(h(s))) - F(s, z_2(s), z_2(h(s))) \right| ds \\
&\leq (m + t_f - s_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha - 1} \left| z_1(s) - z_2(s) \right| ds \\
&+ \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha - 1} \left| z_1(h(s)) - z_2(h(s)) \right| ds + L \int_{s_i}^{t} \left| z_1(s) - z_2(s) \right| ds \\
&+ L \int_{s_i}^{t} \left| z_1(h(s)) - z_2(h(s)) \right| ds.
\end{aligned}
$$

Next, we show that the operator $T : PC(J, R^+) \to PC(J, R^+)$ given below is an increasing Picard operator:

$$
\begin{aligned}
(Tg)(t) &= (m + t_f - s_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha - 1} g(s)ds \\
&+ \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha - 1} g(h(s))ds + L \int_{s_i}^{t} g(s)ds + L \int_{s_i}^{t} g(h(s))ds.
\end{aligned}
$$

For any $g_1, g_2 \in PC(J, R^+)$, $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\left| (Tg_1)(t) - (Tg_2)(t) \right| &\leq \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \left| g_1(s) - g_2(s) \right| ds \\
&\quad + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \left| g_1(h(s)) - g_2(h(s)) \right| ds \\
&\quad + L \int_{s_i}^{t} \left| g_1(s) - g_2(s) \right| ds + L \int_{s_i}^{t} \left| g_1(h(s)) - g_2(h(s)) \right| ds \\
&\leq \frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \max_{t_i \leq s \leq s_i} |g_1(s) - g_2(s)| ds \\
&\quad + 2L \int_{s_i}^{t} \max_{s_i \leq s \leq t_{i+1}} |g_1(s) - g_2(s)| ds \\
&\leq \left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t - s_i) \right) \|g_1 - g_2\| \\
&\leq \left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t_f - s_i) \right) \|g_1 - g_2\|.
\end{aligned}
$$

Again from (c), the operator is strictly contractive on $(s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, and hence a Picard operator on $PC(J, R^+)$. Applying Banach contraction principle, T is Picard operator with unique fixed point $g^* \in PC(J, R^+)$ i.e.

$$
\begin{aligned}
g^*(t) &= (m + t_f - s_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} g^*(s) ds \\
&\quad + \frac{L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} g^*(h(s)) ds + L \int_{s_i}^{t} g^*(s) ds + L \int_{s_i}^{t} g^*(h(s)) ds.
\end{aligned}
$$

Since g^* is increasing, so $g^*(h(t)) \leq g^*(t)$ and hence we can write,

$$
\begin{aligned}
g^*(t) &\leq (m + t_f - s_i)\epsilon + \frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} g^*(s) ds + 2L \int_{s_i}^{t} g^*(s) ds \\
&\leq (m + t_f - s_i)\epsilon + \sum_{0 < s_i < t} \left(\frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} g^*(s) ds \right) + 2L \int_{s_i}^{t} g^*(s) ds.
\end{aligned}
$$

Using 2.4 Lemma, we get

$$
g^*(t) \leq (m + t_f - s_i)\epsilon \prod_{0 < s_i < t} \left(1 + \frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds \right) \exp\left(2L(t_f - s_i) \right).
$$

If we set $g = |y - z|$, then $g(t) \leq (Tg)(t)$ from which by using abstract Grönwall lemma, it follows that $g(t) \leq g^*$, thus

$$
|y(t) - z(t)| \leq (m + t_f - s_i)\epsilon \prod_{0 < s_i < t} \left(1 + \frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds \right) \exp\left(2L(t_f - s_i) \right).
$$

$\square$

In the following theorem, we state about the Hyers–Ulam–Rassias stability of (1.1) on J. The proof follows the same steps as that of above theorem. The remarked 2.7 Lemma for inequality (2.2) is consumed in the proof.

3.2. Theorem. *If*
a) $F : (s_i, t_{i+1}] \times R^2 \to R$ is continuous with the Lipschitz condition:
$\left| F(t, x_1, x_2) - F(t, y_1, y_2) \right| \leq \sum_{k=1}^{2} L|x_k - y_k|$, $L > 0$, for all $t \in (s_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, m$ and $x_k, y_k \in R$, $k \in \{1, 2\}$,

b) $g_i : (t_i, s_i] \times R^2 \to R$ satisfies the Lipschitz condition $|g_i(t, u_1, u_2) - g_i(t, v_1, v_2)| \leq \sum_{k=1}^{2} L_{g_i}|u_k - v_k|$, $L_{g_i} > 0$, for all $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$ and $u_1, u_2, v_1, v_2 \in R$ and

c) $\left(\frac{2L_{g_i}}{\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} ds + 2L(t_f - s_i) \right) < 1$ and

d) $\varphi \in PC(J, R^+)$ is increasing such that for some $\rho > 0$,

$$\int_{t_0}^{t} \varphi(r) dr \leq \rho \varphi(t),$$

then equation (1.1) has
i) a unique solution in $PC^1(J, R)$;
ii) Hyers–Ulam–Rassias stability on J.

4. Conclusion

In this paper, we have proved the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of equation (1.1) using Grönwall lemma and 2.7 Lemma. Also we have proved the unique solution of (1.1) in $PC^1(J, R)$. Our results guarantee that there is an exact solution $y(t)$ of (1.1) which is close to the approximate solution. In fact, our results are important when finding exact solution is quite difficult and hence are important in approximation theory etc.

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