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Exponential decay of thermo-elastic Bresse system with distributed delay term

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Abstract

The paper considered here is one-dimensional linear thermo-elastic Bresse system with a distributed delay term in the first equation. We prove the well-posedness and exponential stability result, this later will be shown without the usual assumption on the wave speeds. To achieve our goals, we make use of the semi-group method.

Keywords: Bresse system, Delay terms, Decay rate, Lyaponov method, Thermo-elastic.

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1. Introduction and previous results

In the present paper we are concerned with the Bresse system with a distributed delay term,

(1.1)

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2}\mu(s)\varphi_t\left(x, t-s\right)ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + kl\left(\varphi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases}$$

where $(x,t) \in (0,1) \times \mathbb{R}_+$ with the Dirichlet conditions:

(1.2) $$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = w(0,t) = w(1,t) = \theta(0,t) = 0, t > 0$$

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and the initial conditions

$$(1.3)\qquad \begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \psi(x,0)=\psi_0(x), \\ \psi_t(x,0)=\psi_1(x), w(x,0)=w_0(x), w_t(x,0)=w_1(x), x\in(0,1) \\ \theta(x,0)=\theta_0(x), \quad q(x,0)=q_0(x) \text{ in } (0,\infty) \text{ in } (0,\infty) \\ \varphi_t(x,-t)=f_0(x,t) \text{ in } (0,1)\times(0,\tau_2) \\ \varphi(0,t)=\psi_x(0,t)=w_x(0,t)=\theta(0,t)=0, \quad \forall t\geq 0 \\ \varphi_x(1,t)=\psi(1,t)=w(1,t)=q(1,t)=0, \quad \forall t\geq 0 \end{cases}$$

τ_1 and τ_2 are two real numbers with $0\leq\tau_1<\tau_2$, $\mu_0>0$ is a positive constant and $\mu:[\tau_1,\tau_2]\longrightarrow\mathbb{R}$ is an L^∞ function, $\mu\geq 0$ almost everywhere and the initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,f_0,\theta_0,q_0)$ belong to a suitable Sobolev space. We prove the well-posedness and establish both an exponential stability results depending on the following parameters

$$(1.4)\qquad \eta=\left(1-\frac{\alpha k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)-\frac{\gamma^2\alpha}{b} \text{ and } k=k_0$$

and under the assumption

$$(1.5)\qquad \mu_0\geq\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds.$$

Originally the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [5]):

$$(1.6)\qquad \begin{cases} \rho_1\varphi_{tt}=Q_x+lN+F_1 \\ \rho_2\psi_{tt}=M_x-Q+F_2 \\ \rho_1 w_{tt}=N_x-IQ+F_3 \end{cases}$$

where

$$(1.7)\qquad N=k_0\left(w_x-l\varphi\right), Q=k\left(\varphi_x+lw+\psi\right), M=b\psi_x$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. By w,φ and ψ we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1=\rho A=\rho I, k_0=EA, k=k'GA$ and $l=R^{-1}$. To material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denoting external forces.

System (1.6) is an undamped system and its associated energy remains constant when the time t evolves. To stabilize system (1.6), many damping terms have been considered by several authors. (see [1], [2], [3], [4],[8], [10], [19]).

By considering damping terms as infinite memories acting in the three equations, the system (1.6) have been recently studied in [8]

$$(1.8)\qquad \begin{cases} \rho_1\varphi_{tt}-Gh\left(\varphi_x+lw+\psi\right)_x-Ehl\left(w_x-l\varphi\right)+\int_0^\infty g_1(s)\varphi_{xx}(t-s)ds=0, \\ \rho_2\psi_{tt}-EI\psi_{xx}+Gh\left(\varphi_x+lw+\psi\right)+\int_0^\infty g_2(s)\psi_{xx}(t-s)ds=0 \\ \rho_1 w_{tt}-Eh\left(w_x-l\varphi\right)_x+lGh\left(\varphi_x+lw+\psi\right)+\int_0^\infty g_3(s)w_{xx}(t-s)ds=0 \end{cases}$$

where $(x,t)\in]0,L[\times\mathbb{R}_+, g_i:\mathbb{R}_+\rightarrow\mathbb{R}_+, i=1,2,3$ are given functions. The authors proved, under suitable conditions on the initial data and the memories g_i, that the system is well-posed and its energy converges to zero when time goes to infinity, and they provide a connection between the decay rate of energy and the growth of g_i at infinity. The proof is based on the semigroups theory for the well-posedness, and the energy method and the approach introduced in [7], for the stability.

In [2], the authors considered the Bresse system in bounded domain with delay terms in the internal feedbacks

$$(1.9)\qquad \begin{cases} \rho_1\varphi_{tt} - Gh\left(\varphi_x + lw + \psi\right)_x - Ehl\left(w_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh\left(\varphi_x + lw + \psi\right) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x,t-\tau_2) = 0 \\ \rho_1 w_{tt} - Eh\left(w_x - l\varphi\right)_x + lGh\left(\varphi_x + lw + \psi\right) + \widetilde{\widetilde{\mu_1}}w_t + \widetilde{\widetilde{\mu_2}}w_t(x,t-\tau_3) = 0 \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty), \tau_i > 0$ $(i = 1,2,3)$ are a time delays, $\mu_1, \mu_2, \widetilde{\mu_1}, \widetilde{\mu_2}, \widetilde{\widetilde{\mu_1}}, \widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subjected to the Dirichlet boundary conditions and to the initial conditions which belong to a suitable Sobolev space. First, the author proved the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, they studied the asymptotic behavior of solutions using multiplier method.
The Bresse system (1.6) is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l = 0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping, in this regard, we note the next references (see [9], [6] [12], [13], [14], [16], [18], [20], and [21]).

2. Well-posedness

We will prove that the system (1.1)-(1.3) is well posed using semi-group theory by introducing the following new variable as in [15]

$$(2.1)\qquad z(x,\rho,t,s) = \varphi_t\left(x, t - \rho s\right), x \in (0,1), \rho \in (0,1), s \in (\tau_1,\tau_2), t > 0.$$

Then, we have

$$(2.2)\qquad sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0 \text{ in } (0,1) \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2)$$

Therefore, problem (1.1) can be taken as

$$(2.3)\qquad \begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - lk_0\left(w_x - l\varphi\right) + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2}\mu(s)z\left(x,1,t,s\right)ds = 0, \\ sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0. \end{cases}$$

With the same boundary conditions and the initial additional conditions:

$$(2.4)\qquad \begin{cases} z\left(x,0,t,s\right) = \varphi_t\left(x,t\right) \text{ on } (0,1) \times (0,\infty) \times (\tau_1,\tau_2), \\ z\left(x,\rho,0,s\right) = f_0\left(x,\rho s\right) \text{ on } (0,1) \times (0,1) \times (\tau_1,\tau_2) \end{cases}$$

We set

$$U = \left(\varphi, \varphi_t, z, \psi, \psi_t, w, w_t, \theta, q\right)^T,$$

then

$$U' = \left(\varphi_t, \varphi_{tt}, z_t, \psi_t, \psi_{tt}, w_t, w_{tt}, \theta_t, q_t\right)^T.$$

Therefore, we can rewrite the problem (2.3)-(2.4) as

$$(2.5)\qquad \begin{cases} U'\left(t\right) + AU\left(t\right) = 0, \\ U\left(0\right) = \left(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, f_0, \theta, q\right), \end{cases}$$

We define the operator A as

$$A\begin{pmatrix}\varphi\\u\\z\\\psi\\v\\w\\\varpi\\\theta\\q\end{pmatrix}=\begin{pmatrix}-u\\-\frac{k}{\rho_1}\left(\varphi_x+lw+\psi\right)_x-\frac{k_0 l}{\rho_1}\left(w_x-l\varphi\right)+\frac{\mu_0}{\rho_1}\varphi_t+\frac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\mu(s)z\left(x,1,t,s\right)ds\\\left(\frac{1}{s}\right)z_\rho\\-v\\-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}\left(\varphi_x+lw+\psi\right)+\frac{\gamma}{\rho_2}\theta_x\\-\varpi\\-\frac{k_0}{\rho_1}\left(w_x-l\varphi\right)_x+\frac{kl}{\rho_1}\left(\varphi_x+lw+\psi\right)\\\frac{1}{\rho_3}q_x+\frac{\gamma}{\rho_3}\psi_{tx}\\\frac{\beta}{\alpha}q+\frac{1}{\alpha}\theta_x\end{pmatrix}$$

We now consider the following spaces

$$H_*^1(0,1)=\left\{h\in H^1(0,1):h(0)=0\right\},$$
$$\widetilde{H}_*^1(0,1)=\left\{h\in H^1(0,1):h(1)=0\right\},$$
$$H_*^2(0,1)=H^2(0,1)\cap H_*^1(0,1),$$
$$\widetilde{H}_*^2(0,1)=H^2(0,1)\cap \widetilde{H}_*^1(0,1),$$

and

$$\begin{aligned}\mathcal{H} \quad = \quad & H_*^1(0,1)\times L^2(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1),\\ &\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1),\\ &\times L^2\left((0,1),(\tau_1,\tau_2),H_0^1(0,1)\right).\end{aligned}$$

We will show that the operator A generates a C_0 semigroup on $\mathcal{H}$. Let us define on the Hilbert space $\mathcal{H}$ the inner product, for

$$U=(\varphi,u,z,\psi,v,w,\varpi,\theta,q)^T \ ,\overline{U}=\left(\overline{\varphi},\overline{u},\overline{z},\overline{\psi},\overline{v},\overline{w},\overline{\varpi},\overline{\theta},\overline{q}\right)^T$$

$$\begin{aligned}\left\langle U,\overline{U}\right\rangle_{\mathcal{H}} \quad = \quad & k\int_0^1\left(\varphi_x+\psi+lw\right)\left(\overline{\varphi}_x+\overline{\psi}+l\overline{w}\right)dx+k_0\int_0^1\left(w_x-l\varphi\right)\left(\overline{w}_x-l\overline{\varphi}\right)dx\\ &+\rho_1\int_0^1 u\overline{u}dx+\rho_2\int_0^1 v\overline{v}dx+\rho_1\int_0^1\varpi\overline{\varpi}dx+b\int_0^1\psi_x\overline{\psi_x}dx\\ &+\int_0^L\int_{\tau_1}^{\tau_2}s\mu(s)\int_0^1 z(x,\rho,s)\,\overline{z}(x,\rho,s)\,d\rho ds dx+\rho_3\int_0^1\theta\overline{\theta}dx+\alpha\int_0^1 q\overline{q}dx\end{aligned}\tag{2.6}$$

$\mathcal{H}$ is a Hilbert space for l small enough since, in this case, the above inner product is equivalent to the natural inner product defined on $\mathcal{H}$.
The domain of A is given by

$$D(A)=\left\{\begin{array}{c}U\in\mathcal{H}/\varphi\in H_*^2(0,1);\psi,w\in\widetilde{H}_*^2(0,1),u,\theta\in H_*^1(0,1);v,\varpi,q\in\widetilde{H}_*^1(0,1)\\ z\in L^2\left((0,1),(\tau_1,\tau_2),H_0^1(0,1)\right),u(x)=z(x,0,s)\text{ in }(0,1)\\ \varphi_x(1)=0,w_x(0)=\psi_x(0)=0.\end{array}\right\}.\tag{2.7}$$

We now prove that A is a maximal monotone operator. For this purpose we need the following two Lemmas.

2.1. Lemma. *The operator A is monotone and satisfies, for any $U\in D(A)$,*

$$\langle AU,U\rangle_{\mathcal{H}}=\beta\int_0^1 q^2dx+\left(\mu_0-\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^L u^2(x)\,dx.\tag{2.8}$$

Proof. For any $U\in D(A)$, using the inner product and integration by parts, estimate (2.8) can be easily shown. □

2.2. Lemma. *The operator $I+A$ is surjective.*

Proof. We need to show that for all $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7 f_8, f_9)^T \in \mathcal{H}$, there exists $U \in D(A)$ such that

(2.9) $\quad U + AU = \mathcal{F}.$

that is

$$(2.10)\quad \begin{cases} -u + \varphi = f_1 \in H_*^1(0,1), \\ -k(\varphi_x + lw + \psi)_x - k_0 l (w_x - l\varphi) + \rho_1 u + \mu_0 \varphi_t + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds = \rho_1 f_2 \in L^2(0,1), \\ z + s^{-1} z_\rho = f_3 \in L^2((0,1), H^1(0,1)), \\ -v + \psi = f_4 \in \widetilde{H}_*^1(0,1), \\ -b\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 v + \gamma\theta_x = \rho_2 f_5 \in L^2(0,1), \\ -\varpi + w = f_6 \in \widetilde{H}_*^1(0,1), \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 \varpi = \rho_1 f_7 \in L^2(0,1), \\ q_x + \gamma v_x + \rho_3 \theta = \rho_3 f_8 \in L^2(0,1), \\ (\beta + \alpha) q + \theta_x = \alpha f_9 \in L^2(0,1). \end{cases}$$

From $(2.10)_9$, we obtain

$$(2.11)\quad \theta = \alpha \int_0^x f_9(y)\, dy - (\beta + \alpha) \int_0^x q(y)\, dy,$$

then $\theta(0,t) = 0$. Inserting $u = \varphi - f_1, v = \psi - f_4, \varpi = w - f_6$and (2.11) into $(2.10)_2$, $(2.10)_5$, $(2.10)_7$,$(2.10)_8$ we get

$$(2.12)\quad \begin{cases} -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 \varphi + \mu_0 \varphi_t + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds = h_1 \in L^2(0.1), \\ -b\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 \psi - \gamma(\beta - \alpha) q = h_2 \in L^2(0.1), \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 w = h_3 \in L^2(0.1), \\ q_x + (\beta + \alpha) \int_0^x q(y)\, dy - \gamma\psi_x = h_4 \in L^2(0.1), \\ z + s^{-1} z_\rho = h_5 \in L^2(0.1), \end{cases}$$

where

$$(2.13)\quad \begin{cases} h_1 = \rho_1(f_1 + f_2), \\ h_2 = \rho_2(f_4 + f_5) - \alpha\gamma f_9, \\ h_3 = \rho_1(f_6 + f_7), \\ h_4 = -\gamma f_{4x} - \rho_3\left(f_8 - \alpha \int_0^x f_9(y)\, dy\right), \\ h_5 = z + s^{-1} z_\rho. \end{cases}$$

Furthermore, by (2.10) we can find as

$$(2.14)\quad z(x,0,s) = u(x) \text{ for } x \in (0,1), s \in (\tau_1, \tau_2),$$

and from (2.10), we have

$$(2.15)\quad z(x,\rho,s) + s^{-1} z_\rho(x,\rho,s) = f_3(x,\rho,s) \text{ on } (0,1) \times (0,1) \times (\tau_1, \tau_2)$$

From (2.14) and (2.15) we obtain,

$$(2.16)\quad z(x,\rho,s) = u(x) e^{-\rho s} + s e^{-\rho s} \int_0^\rho f_3(x,\sigma,s) e^{\sigma s} d\sigma.$$

So, from (2.10) on $(0,1) \times (0,1) \times (\tau_1, \tau_2)$,

$$(2.17)\quad z(x,\rho,s) = \varphi(x) e^{-\rho s} - f_1 e^{-\rho s} + s e^{-\rho s} \int_0^\rho f_3(x,\sigma,s) e^{\sigma s} d\sigma.$$

and, in particular,

$$z(x,1,s) = u(x) e^{-s} + z_0(x,s), x \in [0,1], s \in (\tau_1, \tau_2)$$

with

$$z_0 \in L^2((0,1) \times (\tau_1, \tau_2))$$

defined by

$$z(x,\rho,s) = -f_1 e^{-\rho s} + s e^{-\rho s} \int_0^\rho f_3(x,\sigma,s) e^{\sigma s} d\sigma, x \in [0,1], s \in (\tau_1, \tau_2).$$

To solve (2.13) we consider

$$a\left(\left(\varphi,\psi,w,q\right),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right)=L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right),\tag{2.18}$$

where

$$a:\left[H_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times L^2\left(0,1\right)\right]^2\longrightarrow\mathbb{R}$$

is the bilinear form given by

$$\begin{aligned}a\left(\left(\varphi,\psi,w,q\right),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right) &= k\int_0^1\left(\varphi_x+lw+\psi\right)\left(\widetilde{\varphi}_x+l\widetilde{w}+\widetilde{\psi}\right)dx+\left(\beta+\alpha\right)\int_0^1 q\widetilde{q}dx\\ &\quad+b\int_0^1\psi_x\widetilde{\psi}_xdx+\rho_2\int_0^1\psi\widetilde{\psi}dx-\gamma\left(\beta+\alpha\right)\int_0^1 q\widetilde{\psi}dx\\ &\quad+\rho_1\int_0^1\psi\widetilde{\psi}dx+\gamma\left(\beta+\alpha\right)\int_0^1\psi\widetilde{q}dx+\rho_1\int_0^1 w\widetilde{w}dx\\ &\quad+k_0\int_0^1\left(w_x-l\varphi\right)\left(\widetilde{w}_x-l\widetilde{\varphi}\right)dx+\int_0^1\mu_0\varphi\widetilde{\varphi}dx\\ &\quad+\int_0^1\varphi\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu\left(s\right)e^{-s}dsdx\\ &\quad+\rho_3\left(\beta+\alpha\right)\int_0^1\left(\int_0^x q\left(y\right)dy\int_0^x\widetilde{q}\left(y\right)dy\right)dx,\end{aligned}\tag{2.19}$$

and

$$L:\left[H_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times L^2\left(0,1\right)\right]\longrightarrow\mathbb{R},$$

is the linear form defined by

$$\begin{aligned}L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right) &= \int_0^1 h_1\widetilde{\varphi}dx+\int_0^1 h_2\widetilde{\psi}dx+\int_0^1 h_3\widetilde{w}dx\\ &\quad+\left(\alpha+\beta\right)\int_0^1 h_4\int_0^x\widetilde{q}\left(y\right)dydx\\ &\quad+\int_0^1\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu\left(s\right)z_0\left(x,s\right)dsdx.\end{aligned}\tag{2.20}$$

Now for $V=H_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times\widetilde{H}_*^1\left(0,1\right)\times L^2\left(0,1\right)$, equipped with the norm

$$\left\|\left(\varphi,\psi,w,q\right)\right\|_V^2=\left\|\left(\varphi_x+\psi+lw\right)\right\|_2^2+\left\|\left(w_x-l\varphi\right)\right\|_2^2+\left\|\psi_x\right\|_2^2+\left\|q\right\|_2^2$$

and using the fact that

$$\int_0^1\left(\varphi_x^2+\psi_x^2+w_x^2\right)dx\le c\int_0^1\left(\left(\varphi_x+\psi+lw\right)^2+\left(w_x-l\varphi\right)^2+\psi_x^2\right)dx\tag{2.21}$$

for l small enough, it follows that L and a are bounded. Furthermore, from the definition of a, we get

$$\begin{aligned}a\left(\left(\varphi,\psi,w,q\right),\left(\varphi,\psi,w,q\right)\right) &= k\int_0^1\left(\varphi_x+\psi+lw\right)^2dx+k_0\int_0^1\left(w_x-l\varphi\right)^2dx+b\int_0^1\psi_x^2dx\\ &\quad+\rho_2\int_0^1\psi^2dx+\left(\rho_1+\mu_0\right)\int_0^1\varphi^2dx+\rho_1\int_0^1 w^2dx\\ &\quad+\left(\beta+\alpha\right)\int_0^1 q^2dx+\rho_3\left(\beta+\alpha\right)^2\int_0^1\left(\int_0^x q\left(y\right)dy\right)^2\\ &\ge c\left\|\left(\varphi,\psi,w,q\right)\right\|_V^2.\end{aligned}$$

Thus a is coercive. Consequently, by Lax–Milgram Lemma, system (2.12) has a unique solution

$$\varphi \in H_*^1(0,1), \psi \in \widetilde{H}_*^1(0,1), w \in \widetilde{H}_*^1(0,1), q \in L^2(0,1)$$

Substituting φ, ψ, w and q into $(2.10)_1$, $(2.10)_3$, $(2.10)_5$, and $(2.10)_8$, respectively, we get

$$u \in H_*^1(0,1), v \in \widetilde{H}_*^1(0,1), \varpi \in \widetilde{H}_*^1(0,1), \theta \in H_*^1(0,1).$$

Now, if

$$\left(\widetilde{\psi}, \widetilde{w}, \widetilde{q}\right) \equiv (0,0,0) \in \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1),$$

then (2.19) reduces to

$$k\int_0^1 (\varphi_x + \psi + lw)\,\widetilde{\varphi}_x dx - k_0\int_0^1 (w_x - l\varphi)\,\widetilde{\varphi} dx + \rho_1\int_0^1 \varphi\widetilde{\varphi} dx = \int_0^1 h_1\widetilde{\varphi} dx, \tag{2.22}$$

for all $\widetilde{\varphi}$ in $H_*^1(0,1)$ which implies

$$-k\varphi_{xx} = k\psi_x + l(k+k_0)\,w_x - \left(k_0 l^2 + \rho_1\right)\varphi + h_1 \in L^2(0,1). \tag{2.23}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\varphi \in \widetilde{H}_*^2(0,1).$$

Moreover, (2.22) is also true for any $\phi \in C^1([0,1]), \phi(0) = 0$ which is in $H_*^1(0,1)$. Hence, for all $\phi \in C^1([0,1]), \phi(0) = 0$, we have

$$k\int_0^1 \varphi_x\phi_x dx - \int_0^1 \left(k\psi_x + l(k+k_0)\,w_x - \left(k_0 l^2 + \rho_1\right)\varphi + h_1\right)\phi dx = 0.$$

Thus, using integration by parts and bearing in mind (2.23), we get

$$\varphi_x(1)\,\phi(1) = 0, \quad \forall\phi \in C^1([0,1]), \phi(0) = 0.$$

Therefore,

$$\varphi_x(1) = 0.$$

Similarly, we get

$$\left\{\begin{array}{l} -b\psi_{xx} = -k\varphi_x - (k+\rho_2)\,\psi - lkw - \gamma(\alpha+\beta)\int_0^1 q\widetilde{\varphi} dx + h_2 \in L^2(0,1) \\ -kw_{xx} = -l(k+k_0)\,\varphi_x - lk\psi + \left(\rho_1 + l^2 k_0\right) w + h_3 \in L^2(0,1) \\ -q_x = \gamma\psi_x - (\alpha+\beta)\,\rho_3\int_0^x q(y)\,dy + h_4 \in L^2(0,1) \end{array}\right\} \tag{2.24}$$

thus, we have

$$\psi, w \in \widetilde{H}_*^2(0,1), q \in \widetilde{H}_*^1(0,1), w_x(0) = \psi_x(0) = 0.$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(A)$ such that (2.9) is satisfied.
Consequently, using Lemma 2.1 and Lemma 2.2, we conclude that A is a maximal monotone operator. Hence, by Lumer–Philips theorem (see [11] and [17]) we have the following well-posedness result: □

2.3. Theorem. *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution $U \in C\left(\mathbb{R}^+, \mathcal{H}\right)$ of problem (1.1)-(1.3). Moreover, if $U_0 \in D(A)$, then $U \in C\left(\mathbb{R}^+, D(A)\right) \cap C^1\left(\mathbb{R}^+, \mathcal{H}\right)$.*

3. Exponential stability

In this section, we state and, by using a multiplier technique, prove our stability result for the energy of the solution of system (1.1)-(1.3) given by

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+\rho_3\theta^2+\alpha q^2\right.\\
&\left.+k(\varphi_x+\psi+lw)^2+k_0(w_x-l\varphi)^2\right]dx \\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_s(s)\right| z^2(x,\rho,s,t)\,ds d\rho dx.
\end{aligned}
\tag{3.1}
$$

To achieve our goal, we need a several Lemmas.

3.1. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3). Then the energy functional, defined by (3.1) satisfies*

$$E'(t)\le -\beta\int_0^1 q^2dx - n_0\int_0^1\varphi_t^2dx\le 0, \forall t\ge 0, \tag{3.2}$$

Proof. Multiplying $(1.1)_1$, $(1.1)_2$, $(1.1)_3$, $(1.1)_4$, and $(1.1)_5$ by φ_t, ψ_t, w_t, θ, and q, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions we obtain the desired result (3.2). □

3.2. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3). Then the functional given by*

$$F_1(t):=\alpha\rho_3\int_0^1\theta\int_0^x q(y)\,dydx \tag{3.3}$$

satisfies, for any $\varepsilon_1>0$, the estimate

$$F_1'(t)\le -\frac{\rho_3}{2}\int_0^1\theta^2dx+\varepsilon_1\int_0^1\psi_t^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 q^2dx. \tag{3.4}$$

Proof. Taking the derivative of F_1, using the fourth and fifth equations in (1.1) and performing integration by parts, we get

$$F_1'(t)=-\rho_3\int_0^1\theta^2dx+\alpha\int_0^1 q^2dx+\alpha\gamma\int_0^1 q\psi_t dx-\beta\rho_3\int_0^1\theta\int_0^x q(y)\,dydx. \tag{3.5}$$

We then use Cauchy–Schwartz and Young's inequalities with $\varepsilon_1>0$ in (3.5) to obtain (3.4). □

3.3. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3). Then the functional*

$$F_2(t):=-\frac{\rho_2\rho_3}{\gamma}\int_0^1\theta\int_0^x\psi_t(y)\,dydx \tag{3.6}$$

satisfies, for any $\varepsilon_2,\varepsilon_3>0$, the estimate

$$
\begin{aligned}
F_2'(t) \le\ & -\frac{\rho_2}{\gamma}\int_0^1\psi_t^2dx+\varepsilon_2\int_0^1(\varphi_x+\psi+lw)^2dx \\
&+\varepsilon_3\int_0^1\psi_x^2dx+c\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^1\theta^2dx+c\int_0^1 q^2dx.
\end{aligned}
\tag{3.7}
$$

Proof. By differentiating F_2, then exploiting the second and fourth equations in (1.1), and integrating by parts, we get

$$\begin{aligned} F_2'(t) &= -\rho_2 \int_0^1 \psi_t^2 dx - \frac{\rho_2}{\gamma} \int_0^1 q\psi_t dx + \rho_3 \int_0^1 \theta^2 dx - \frac{b\rho_3}{\gamma} \int_0^1 \theta \psi_x dx \\ &\quad + \frac{k\rho_3}{\gamma} \int_0^1 (\varphi_x + \psi + lw) \int_0^x \theta(y)\, dy dx. \end{aligned} \tag{3.8}$$

Estimate (3.7) follows by using Cauchy–Schwartz and Young's inequalities. □

3.4. Lemma. *Let $(\varphi, \psi, w, \theta, q, z)$ be the solution of (1.1)-(1.3). Then the functional*

$$F_3(t) := \rho_1 \int_0^1 \varphi_t \left(\varphi + \int_0^x \psi(y)\, dy \right) dx \tag{3.9}$$

satisfies, for any $\varepsilon_4 > 0$, the estimate

$$\begin{aligned} F_3'(t) &\leq -\frac{k}{2} \int_0^1 (\varphi_x + \psi + lw)^2 dx - \frac{lk_0}{2} \int_0^1 (w_x - l\varphi)^2 dx + \varepsilon_4 \int_0^1 \psi_t^2 dx \\ &\quad + c\left(1 + \frac{1}{\varepsilon_4}\right) \int_0^1 \varphi_t^2 dx + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds dx \end{aligned} \tag{3.10}$$

Proof. Taking the derivative of $F_3(t)$, and using that,

$$z(x, \rho, s, 0) = f_0(x, \rho s) \ \text{ in } \ (0,1) \times (0,1) \times (0, \tau_2) \tag{3.11}$$

and integration by parts, we obtain

$$\begin{aligned} F_3'(t) &= \rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y)\, dy dx - \int_0^1 \left(\varphi + \int_0^x \psi(y)\, dy \right) \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t)\, ds dx \\ &\quad - k \int_0^1 (\varphi_x + \psi + lw)^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx - lk_0 \int_0^1 (w_x - l\varphi)^2 dx \\ &\quad - \mu_0 \int_0^1 \varphi_t \left(\varphi + \int_0^x \psi(y)\, dy \right) dx. \end{aligned} \tag{3.12}$$

Using Young's and Poincaré's, and Cauchy-Schwartz inequalities, for estimate the terms in the right hand side of (3.12)

$$\rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi(y)\, dy \right) dx \leq \varepsilon_4 \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_4} \int_0^1 \varphi_t^2 dx, \tag{3.13}$$

where $\varepsilon_4 > 0$. □

3.5. Lemma. *Let $(\varphi, \psi, w, \theta, q, z)$ be the solution of (1.1)-(1.3). Then the functional*

$$F_4(t) := \rho_2 \int_0^1 \psi \psi_t dx \tag{3.14}$$

satisfies the estimate

$$\begin{aligned} F_4'(t) &\leq -\frac{b}{2} \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ &\quad + \frac{k^2}{b} \int_0^1 (\varphi_x + \psi + lw)^2 dx + c \int_0^1 \theta^2 dx. \end{aligned} \tag{3.15}$$

Proof. Taking the derivative of F_4 and using the second equation in (1.1), it follows that

$$\begin{aligned} F_4'(t) &= -b\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \gamma\int_0^1 \psi_x\theta dx \\ &\quad -k\int_0^1 (\varphi_x+\psi+lw)\,dx. \end{aligned} \tag{3.16}$$

Using Young's and Poincaré's inequalities, we obtain estimate (3.15). □

3.6. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3). Then the functional*

$$F_5(t) := -\rho_1\int_0^1 \varphi_t\,(w_x - l\varphi)\,dx - \rho_1\int_0^1 w_t\,(\varphi_x+\psi+lw)\,dx \tag{3.17}$$

satisfies the estimate

$$\begin{aligned} F_5'(t) &\leq -lk_0\int_0^1 (w_x-l\varphi)^2\,dx - l\rho_1\int_0^1 w_t^2 dx + l\rho_1\int_0^1 \varphi_t^2 dx \\ &\quad +lk\int_0^1 (\varphi_x+\psi+lw)\,dx + c\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 (w_x-l\varphi)^2\,dx\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)dx \\ &\quad +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(s)|\,z^2(x,1,s,t)\,ds dx. \end{aligned} \tag{3.18}$$

Proof. By differentiating F_5 and using the first and the third equations in (1.1), we obtain

$$\begin{aligned} F_5'(t) &= -lk_0\int_0^1 (w_x-l\varphi)^2\,dx - l\rho_1\int_0^1 w_t^2 dx + l\rho_1\int_0^1 \varphi_t^2 dx \\ &\quad +lk\int_0^1 (\varphi_x+\psi+lw)\,dx + c\int_0^1 \psi_t^2 dx + \int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)(w_x-l\varphi)\,dx. \end{aligned}$$

Estimate (1.1) follows thanks to Cauchy-Schwartz inequality. □

3.7. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3). and let $k=k_0$. Then the functional*

$$F_6(t) := -\rho_1\int_0^1 (\varphi\varphi_t + ww_t)\,dx \tag{3.19}$$

satisfies the estimate

$$\begin{aligned} F_6'(t) &\leq -\rho_1\int_0^1 \varphi_t^2 dx - \rho_1\int_0^1 w_t^2 dx + c\int_0^1 \psi_x^2 dx + k_0\int_0^1 (w_x-l\varphi)^2\,dx \\ &\quad +c\int_0^1 (\varphi_x+\psi+lw)\,dx + \frac{1}{2}\int_0^1 \varphi^2 dx\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)dx \\ &\quad +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(s)|\,z^2(x,1,s,t)\,ds dx. \end{aligned} \tag{3.20}$$

Proof. A simple differentiation of F_6, using the first and third equations in (1.1), leads to

$$\begin{aligned} F_6'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - \rho_1\int_0^1 w_t^2 dx + c\int_0^1 \psi_x^2 dx + k_0\int_0^1 (w_x-l\varphi)^2\,dx \\ &\quad +c\int_0^1 (\varphi_x+\psi+lw)\,dx + \int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\varphi dx. \end{aligned}$$

Estimate (1.1) follows thanks to Cauchy-Schwartz inequality. □

3.8. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3) and let (1.4) holds. Then the functional*

$$
\begin{aligned}
F_7(t) \quad : \quad &= \rho_2\int_0^1 \psi_t(\varphi_x+\psi+lw)\,dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_x dx \\
&+\frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_t dx - \frac{b}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x+\psi+lw)\,dx \\
&-\frac{bl^2\rho_2}{k_0}\int_0^1 \psi\psi_t dx + \frac{bl\rho_1}{k_0}\int_0^1 w_t\psi dx \qquad (3.21)
\end{aligned}
$$

satisfies,

$$
\begin{aligned}
F_7'(t) \quad \le \quad & -\frac{k}{2}\int_0^1 (\varphi_x+\psi+lw)^2 dx + \varepsilon_7\int_0^1 w_t^2 dx + \frac{2b^2l^2}{k}\int_0^1 \psi_x^2 dx \\
&+\varepsilon_7'\int_0^1 (w_x-l\varphi)^2 dx + c\left(1+\frac{1}{\varepsilon_7}\right)\int_0^1 \psi_t^2 dx + c\left(1+\frac{1}{\varepsilon_7}\right)\int_0^1 q^2 dx \\
&+c\left(1+\frac{1}{\varepsilon_7'}\right)\int_0^1 \theta^2 dx + \frac{b}{\gamma\tau}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\
&+\frac{b\rho_1}{2k}\int_0^1 \psi_x^2 dx\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)dx + \frac{b\rho_1}{2k}\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\,z^2(x,1,s,t)\,dsdx \\
&+\frac{b\rho_3}{2\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\theta_t^2 dx \\
&+\frac{b\rho_3}{2\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z^2(x,1,s,t)\,dsdx \qquad (3.22)
\end{aligned}
$$

Proof. A simple differentiation of F_7 gives

$$
\begin{aligned}
F_7'(t) \quad = \quad & \rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi+lw)\,dx + \rho_2\int_0^1 \psi_t(\varphi_{xt}+\psi_t+lw_t)\,dx \\
&+\frac{b\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx - \frac{b\rho_1}{k}\int_0^1 \psi_t\varphi_{xt} dx + \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta_t\varphi_t dx \\
&+\frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta_t\varphi_{tt} dx - \frac{b}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q_t(\varphi_x+\psi+lw)\,dx \\
&-\frac{b}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q_t(\varphi_{xt}+\psi_t+lw_t)\,dx - \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_t^2 dx \\
&-\frac{bl^2\rho_2}{k_0}\int_0^1 \psi_{tt}\psi dx + \frac{bl\rho_1}{k_0}\int_0^1 w_{tt}\psi dx + \frac{bl\rho_1}{k_0}\int_0^1 w_t\psi_t dx \\
&+\frac{b\rho_1}{2k}\int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\psi_x dx \\
&+\frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\theta_t dx
\end{aligned}
$$

Estimate (1.1) follows thanks to Cauchy-Schwartz inequality. □

3.9. Lemma. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3) and (2.2). Then the functional*

$$
F_8(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,dsd\rho dx \qquad (3.23)
$$

satisfies, for some positive constant n_1, the following estimate

$$
\begin{aligned}
F_8'(t) \quad \leq \quad & -n_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx \\
& -n_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx + \mu_0 \int_0^1 \varphi_t^2 dx.
\end{aligned} \tag{3.24}
$$

Proof. Differentiating $F_8(t)$, and using the equation (2.2), we obtain,

$$
\begin{aligned}
F_8'(t) \quad = \quad & -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\, z(x,\rho,s,t)\, z_\rho(x,\rho,s,t)\, ds d\rho dx \\
= \quad & -\frac{d}{d\rho} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx \\
& - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx \\
= \quad & - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| \left[e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t)\right] ds d\rho dx \\
& - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned} \tag{3.25}
$$

Using the fact that $z(x,0,s,t) = \varphi_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$
\begin{aligned}
F_8'(t) \quad \leq \quad & - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s}\,|\mu(s)|\, z^2(x,1,s,t)\, ds d\rho dx + \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^1 \varphi_t^2 dx \\
& -n_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned} \tag{3.26}
$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Finally, setting $n_1 = e^{-\tau_2}$ and recalling (1.5), we obtain (3.24). □

3.10. Lemma. *Let $(\varphi, \psi, w, \theta, q, z)$ be the solution of (1.1)-(1.3). Then the functional*

$$
F_9(t) = -\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x w_t(y)\, dy dx - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_x + \psi + lw)(y)\, dy dx \tag{3.27}
$$

satisfies the estimate

$$
\begin{aligned}
F_9'(t) \quad \leq \quad & -\left(\frac{\rho_1}{2} - c\mu_0\right) \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 \int_0^1 w_t^2 dx \\
& + \left(1 + \varepsilon\mu_0 + \frac{1}{k}\right) \int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{\rho_1}{2} \int_0^1 \psi_t^2 dx \\
& + \frac{1}{k} \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds dx
\end{aligned} \tag{3.28}
$$

Proof. A simple differentiation of F_6, using the first equation in (1.1), leads to

$$
\begin{aligned}
F_9(t) \quad = \quad & -\rho_1 \int_0^1 (w_x - l\varphi)_t \int_0^x w_t(y)\, dy dx - \rho_1 \int_0^1 (w_x - l\varphi) \left(\int_0^x w_t(y)\, dy\right)_t dx \\
& -\rho_1 \int_0^1 \varphi_{tt} \int_0^x (\varphi_x + \psi + lw)(y)\, dy dx - \rho_1 \int_0^1 \varphi_t \left(\int_0^x (\varphi_x + \psi + lw)(y)\, dy\right)_t dx
\end{aligned}
$$

Now, we estimate the terms in the right hand side of (3.27) using Young's, Poincaré's, and Cauchy-Schwartz inequalities, with the fact that $k = k_0$, give (3.28) □

3.11. Theorem. *Let $(\varphi,\psi,w,\theta,q,z)$ be the solution of (1.1)-(1.3) and assume that $\eta=0$ and $k=k_0$. Then the energy functional (3.1) satisfies,*

$$E(t)\le c_0e^{-c_1t}, t\ge 0 \tag{3.29}$$

where c_0 and c_1 are positive constants.

For $N,N_i>0$, we set

$$\mathcal{L}(t):=NE(t)+\sum_{i=1}^{i=9}N_iF_i(t) \tag{3.30}$$

First, we must prove the equivalence between $E(t)$ and $\mathcal{L}(t)$.

3.12. Lemma. *For two positive constants c_1 and c_2, we have*

$$c_1E(t)\le \mathcal{L}(t)\le c_2E(t)\forall t\ge 0$$

Proof. Now, let

$$\mathfrak{L}(t)=\sum_{i=1}^{i=9}N_iF_i(t)$$

$$\begin{aligned}
|\mathfrak{L}(t)|\le\quad & N_1\alpha\rho_3\int_0^1|\theta\int_0^x q(y)|dydx+N_2\frac{\rho_2\rho_3}{\gamma}\int_0^1|\theta\int_0^x\psi_t(y)|dydx\\
&+N_3\rho_1\int_0^1|\varphi_t\left(\varphi+\int_0^x\psi(y)dy\right)|dx+N_4\rho_2\int_0^1|\psi\psi_t|dx\\
&+N_5\rho_1\int_0^1|\varphi_t(w_x-l\varphi)|dx+\rho_1\int_0^1|w_t(\varphi_x+\psi+lw)|dx\\
&+N_6\rho_1\int_0^1|(\varphi\varphi_t+ww_t)|dx+N_7\rho_2\int_0^1|\psi_t(\varphi_x+\psi+lw)|dx\\
&+N_7\frac{b\rho_1}{k}\int_0^1|\varphi_t\psi_x|dx+N_7\frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)\int_0^1|\theta\varphi_t|dx\\
&+N_7\frac{b}{\gamma}\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)\int_0^1|q(\varphi_x+\psi+lw)|dx+N_7\frac{bl^2\rho_2}{k_0}|\int_0^1\psi\psi_t|dx\\
&+N_7\frac{bl\rho_1}{k_0}\int_0^1|w_t\psi|dx+N_8\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|se^{-s\rho}|\mu(s)|z^2(x,\rho,s,t)|dsd\rho dx\\
&+N_9\rho_1\int_0^1|(w_x-l\varphi)\int_0^x w_t(y)|dydx+\rho_1\int_0^1|\varphi_t\int_0^x(\varphi_x+\psi+lw)(y)|dydx.
\end{aligned}$$

Exploiting Young's, Poincaré's, Cauchy-Schwartz inequalities, (3.1), and the fact that $e^{-s\rho}\le 1$ for all $\rho\in[0,1]$, we obtain

$$\begin{aligned}
\mathfrak{L}(t)|\quad\le\quad & c\int_0^1\left[\varphi_t^2+\psi_t^2+w_t^2+\psi_x^2+\theta^2+q^2+(\varphi_x+\psi+lw)^2+(w_x-l\varphi)^2\right]dx\\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu(s)|z^2(x,\rho,s,t)dsd\rho dx\\
\le\quad & cE(t)
\end{aligned}$$

Consequently, $|\mathcal{L}(t)-NE(t)|\le cE(t)$ which yields

$$(N-c)E(t)\le\mathfrak{L}(t)\le(n+c)E(t)$$

choosing N such that $(N-c)>0$ □

Proof of Theorem 3.11. By differentiating (3.30) and recalling (3.4), (3.7), (3.12), (3.15), (3.18), (3.20), (3.22), (3.24) and (3.28)

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & \left[N_3\left(c(1+\frac{1}{\varepsilon_4})\right) + N_5 l\rho_1 - N_6\rho_1 + N_8\mu_0 - N_9(\frac{\rho_1}{2} - c\mu_0) - N\mu_0\right]\int_0^1 \varphi_t^2 dx \\
& + \left[N_1\varepsilon_1 - N_2\frac{\rho_2}{\gamma} + N_3\varepsilon_4 + N_4\rho_2 + N_5 c + N_7 c(1+\frac{1}{\varepsilon_7}) + N_9\frac{\rho_1}{2}\right]\int_0^1 \psi_t^2 dx \\
& + \left[N_2\varepsilon_3 - N_4\frac{b}{2} + N_6 c + N_7(\frac{2b^2l^2}{k} + \frac{\mu_0 b\rho_1}{2k})\right]\int_0^1 \psi_x^2 dx \\
& + \left[-N_5 l\rho_1 - N_6\rho_1 + N_7\varepsilon_7 + N_9\rho_1\right]\int_0^1 w_t^2 dx \\
& + \left[-N_1\frac{\rho_3}{2} + N_2 c(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}) + N_4 c + N_7 c(1+\frac{1}{\varepsilon_7})\right]\int_0^1 \theta^2 dx \\
& + \left[N_1 c(1+\frac{1}{\varepsilon_1}) + N_2 c + N_7 c(1+\frac{1}{\varepsilon_7}) - N\beta\right]\int_0^1 q^2 dx \\
& + \left[N_2\varepsilon_2 - N_3\frac{k}{2} + N_4\frac{k^2}{b} + N_5 lk + N_6 c\right]\int_0^1 (\varphi_x + \psi + lw)^2 dx \\
& + \left[-N_3\frac{lk_0}{2} + N_5\frac{\mu_0}{2} + N_6 k_0 + N_7\varepsilon_7' - N_9 k_0\right]\int_0^1 (w_x - l\varphi)^2 dx \\
& + [N_8 n_1]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\
& + \left[-N_8 n_1 + \frac{N_6}{2} + \frac{N_5}{2} + cN_3\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|z^2(x,1,s,t)\,ds\,dx
\end{aligned}$$

At this point we choose N_8 large enough so that

$$-N_8 n_1 + \frac{N_6}{2} + \frac{N_5}{2} + cN_3 < 0$$

Once N_8 is fixed, we then choose N_2 large enough such that

$$N_1\varepsilon_1 - N_2\frac{\rho_2}{\gamma} + N_3\varepsilon_4 + N_4\rho_2 + N_5 c + N_7 c(1+\frac{1}{\varepsilon_7}) + N_9\frac{\rho_1}{2} < 0$$

we choose N_4,N_6,N_1,N_3 large enough such that

$$\begin{gathered}
N_2\varepsilon_3 - N_4\frac{b}{2} + N_6 c + N_7(\frac{2b^2l^2}{k} + \frac{\mu_0 b\rho_1}{2k}) < 0 \\
-N_5 l\rho_1 - N_6\rho_1 + N_7\varepsilon_7 + N_9\rho_1 < 0 \\
-N_1\frac{\rho_3}{2} + N_2 c(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}) + N_4 c + N_7 c(1+\frac{1}{\varepsilon_7}) < 0
\end{gathered}$$

$\max\{N_2\varepsilon_2 - N_3\frac{k}{2} + N_4\frac{k^2}{b} + N_5 lk + N_6 c, -N_3\frac{lk_0}{2} + N_5\frac{\mu_0}{2} + N_6 k_0 + N_7\varepsilon_7' - N_9 k_0\} < 0$
finally, we choose N large enough such that
$\max\{N_3\left(c(1+\frac{1}{\varepsilon_4})\right) + N_5 l\rho_1 - N_6\rho_1 + N_8\mu_0 - N_9(\frac{\rho_1}{2} - c\mu_0) - N\mu_0, N_1 c(1+\frac{1}{\varepsilon_1}) + N_2 c + N_7 c(1+\frac{1}{\varepsilon_7}) - N\beta < 0\}$
By (3.1), we obtain

$$\mathcal{L}'(t) \leq -\alpha_0 E(t) \qquad \forall t \geq 0$$

for some $\alpha_0 > 0$. A combination with Lemma 3.12 gives

$$\mathcal{L}'(t) \leq -k_1\mathcal{L}(t) \qquad \forall t \geq 0 \tag{3.31}$$

where $k_1 = \frac{\alpha_0}{c_2}$
Finally, a simple integration of (3.31) we obtain (3.29). $\square$

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