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On $*$-(σ,τ)-Lie ideals of $*$-prime rings with derivation

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Abstract

Let R be a $*-$prime ring with characteristic not 2, U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R and d be a nonzero derivation of R. Suppose σ,τ be two automorphisms of R such that $\sigma d = d\sigma, \tau d = d\tau$ and $*$ commutes with σ,τ,d. In the present paper it is shown that if $d^2(U) = (0)$, then $U \subseteq Z$.

This study is dedicated to our pioneer in this area, Professor Kazım Kaya.

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1. Introduction

Let R will be an associative ring with center Z. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $* : R \to R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*-$ring. A ring with an involution is said to $*-$prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*-$prime but the converse need not to hold general. As an example Oukhtite [8] justifies the above statement that is, R is a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R. Define involution $*$ on S as $(x,y)^* = (y,x)$. S

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is $*-$prime, but not prime. This example shows that $*-$prime rings constitute a more general class of prime rings. In all that follows the symbol $S_*(R)$, that was first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e. $S_*(R) = \{x \in R \mid x^* = \pm x\}$. An ideal M of R is said to be a $*-$ideal if $M^* = M$.

Let σ and τ two mappings from R into itself. For any $x, y \in R$, we write $[x, y]$ and $[x, y]_{\sigma,\tau}$, for $xy - yx$ and $x\sigma(y) - \tau(y)x$ respectively and make extensive use of basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$

$[xy, z] = [x, z]y + x[y, z]$

$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau} y$

$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$.

We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c \text{ for all } x \in R\}$ and call it $(\sigma, \tau)-$center of R. Note that $C_{1,1} = Z$, where $1 : R \longrightarrow R$ is the identity map. Recall that an additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. Kaya [3] first introduced the $(\sigma, \tau)-$Lie ideal as: Let U be an additive subgroup of R, $\sigma, \tau : R \longrightarrow R$ be two mappings. Then (i) U is a $(\sigma, \tau)-$right Lie ideal of R if $[U, R]_{\sigma,\tau} \subseteq U$. (ii) U is a $(\sigma, \tau)-$left Lie ideal of R if $[R, U]_{\sigma,\tau} \subseteq U$. (iii) U is a $(\sigma, \tau)-$Lie ideal of R if U is both a $(\sigma, \tau)-$right Lie ideal and $(\sigma, \tau)-$left Lie ideal of R. Every Lie ideal of R is a $(1, 1)-$left (and right) Lie ideal of R, where $1 : R \longrightarrow R$ is the identity map of R. But there exist $(\sigma, \tau)-$Lie ideals which are not Lie ideals (Such an example due to [3]). An $(\sigma, \tau)-$Lie ideal U of R is said to be a $* - (\sigma, \tau)-$Lie ideal if U is invariant under $*$, i.e. $U^* = U$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation determined by a. The commutativity of prime rings with derivation was initiated by Posner [9]. Over the last five decades, a great deal of work has been done on this subject. The following results have been proved for Lie ideals in [2]: Let R be a prime ring of characteristic different from 2, U is a nonzero Lie ideal of R and d a nonzero derivation. If any one of the following conditions is satisfied, then $U \subseteq Z$: (i) $d(U) = 0$ (ii) $d(U)a = 0$ or $ad(U) = 0$ with $a \neq 0$ (iii) $d^2(U) = 0$. In [4], Lee and Lee proved that if R is a prime ring of characteristic different from 2, U is a nonzero Lie ideal of R and d is a nonzero derivation such that $d^2(U) \subseteq Z$ then $U \subseteq Z$. Further, the above results were extended to $(\sigma, \tau)-$ Lie ideals of R in [1]. Oukhtite et al. showed that these results are valid for $*-$prime rings in [7]. In this work our main goal will be proving the above result for a nonzero $* - (\sigma, \tau)-$Lie ideal of a $*-$prime ring with characteristic not two.

2. Results

In the view of the definition of generalized derivation, one can easily notice that the following remark.

2.1. Remark. Let d be a derivation of R. If $d\sigma = \sigma d$, $d\tau = \tau d$, then

$$d([x, y]_{\sigma,\tau}) = [d(x), y]_{\sigma,\tau} + [x, d(y)]_{\sigma,\tau}, \text{ for all } x, y \in R.$$

2.2. Lemma. [5, Theorem 3.2] *Let R be a $*-$prime ring with characteristic not 2, I be a nonzero $*-$ideal of R and d be a nonzero derivation of R commutes with $*$. If $a \in S_*(R)$ and $[d(I), a] = 0$, then $a \in Z$. Furthermore, if $d(I) \subseteq Z$, then R is commutative.*

2.3. Lemma. [6, Theorem 2.2] *Let R be a $*-$prime ring and I be a nonzero $*-$ideal of R. If a, b in R are such that $aIb = aIb^* = (0)$, then $a = 0$ or $b = 0$.*

2.4. Lemma. [10, Lemma 2.8] *Let R be a $*-$prime ring and U be a nonzero $*-(\sigma, \tau)-$left Lie ideal of R such that τ commutes with $*$. If $U \subseteq C_{\sigma,\tau}$, then $U \subseteq Z$.*

2.5. Lemma. [10, Lemma 2.9] *Let R be a $*-$prime ring, U be a nonzero $*-(\sigma,\tau)-$left Lie ideal of R such that τ commutes with $*$ and $a \in R$. If $Ua = (0)$, then $a = 0$ or $U \subseteq Z$.*

2.6. Lemma. [10, Theorem 2.17] *Let R be a prime ring with characteristic not 2 and U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R such that τ commutes with $*$. If $U \nsubseteq Z$ and $U \nsubseteq C_{\sigma,\tau}$, then there exist a nonzero $*-$ideal M of R such that $[R,M]_{\sigma,\tau} \subseteq U$ and $[R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}$.*

2.7. Theorem. *Let R be a $*-$prime ring with characteristic not 2, U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d be a nonzero derivation of R and $*$ commutes with σ,τ and d. If $d(U) = (0)$, then $U \subseteq Z$.*

Proof. Suppose on the contrary that $U \nsubseteq Z$. By Lemma 2.4, we get $U \nsubseteq C_{\sigma,\tau}$. Hence, there exists a nonzero $*-$ideal M of R such that $[R,M]_{\sigma,\tau} \subseteq U$ but $[R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}$ by Lemma 2.6. For any $x \in R$ and $m \in M$,

$$[x,m]_{\sigma,\tau}\sigma(m) = [x\sigma(m),m]_{\sigma,\tau} \in U.$$

By the hypothesis, we have

$$0 = d([x,m]_{\sigma,\tau}\sigma(m)) = d([x,m]_{\sigma,\tau})\sigma(m) + [x,m]_{\sigma,\tau}d(\sigma(m))$$

and so

$$[x,m]_{\sigma,\tau}d(\sigma(m)) = 0, \text{ for all } x \in R, m \in M. \tag{2.1}$$

Replacing x by $xy, y \in R$ in (2.1) and using (2.1), we find that

$$[x,\tau(m)]Rd(\sigma(m)) = (0), \text{ for all } x \in R, m \in M.$$

Since τ is an automorphism of R, we can rewrite the above equation

$$\tau([y,m])Rd(\sigma(m)) = (0), \text{ for all } y \in R, m \in M. \tag{2.2}$$

Assume that $m \in M \cap S_*(R)$. In (2.2), replacing y by y^* and using $*\tau = \tau*$, we get

$$\tau^*([y,m])Rd(\sigma(m)) = (0), \text{ for all } y \in R, m \in M \cap S_*(R).$$

Thus

$$\tau([y,m])Rd(\sigma(m)) = \tau^*([y,m])Rd(\sigma(m)) = (0), \ \forall y \in R, m \in M \cap S_*(R)$$

is obtained. By the $*-$primeness of R, we have

$$[y,m] = 0 \text{ or } d(\sigma(m)) = 0, \text{ for all } y \in R, m \in M \cap S_*(R).$$

Since $m - m^* \in M \cap S_*(R)$ for all $m \in M$, we have

$$[y,m] = [y,m^*] \text{ or } d(\sigma(m)) = d(\sigma(m^*)), \text{ for all } y \in R, m \in M.$$

Now, let us define the sets $A = \{m \in M \mid [y,m] = [y,m^*], \ \forall y \in R\}$ and $B = \{m \in M \mid d(\sigma(m)) = d(\sigma(m^*))\}$. It is clear that, A and B are an additive subgroups of M such that $M = A \cup B$. But a group can not be an union of its proper subgroups. Therefore, it yields either $M = A$ or $M = B$. In $M = A$ case, $[y,m] = [y,m^*]$, for all $y \in R$. In (2.2) substituting y by y^*, we get

$$\tau^*([y,m])Rd(\sigma(m)) = (0), \ \forall y \in R, m \in M.$$

Since R is $*-$prime, we arrive

$$\tau([y,m]) = 0 \text{ or } d(\sigma(m)) = 0, \ \forall y \in R, m \in M.$$

In $M = B$ case, $d(\sigma(m)) = d(\sigma(m^*))$, for all $m \in M$. From (2.2), we have

$$\tau([y,m])Rd^*(\sigma(m)) = (0), \ \forall y \in R, m \in M.$$

Since R is $*-$prime, we get

$$\tau([y,m]) = 0 \text{ or } d(\sigma(m)) = 0,\ \forall y \in R, m \in M.$$

Expressing that, for both cases the results are the same and this means that

$$m \in Z \text{ or } d(\sigma(m)) = 0,\ \forall m \in M.$$

Define $K = \{m \in M \mid m \in Z\}$ and $L = \{m \in M \mid d(\sigma(m)) = 0\}$. Clearly each of K and L is additive subgroups of M. Moreover, M is the set-theoretic union of K and L. But a group can not be the set-theoretic union of its two proper subgroups, hence $K = M$ or $L = M$. In the former case, $M \subseteq Z$, which forces R to be commutative, and so, $U \subseteq Z$, a contradiction. In the latter case, $d(\sigma(M)) = (0)$. Since R is $*-$prime ring and $\sigma(M)$ is a nonzero $*-$ideal of R, we find that R is commutative by Lemma 2.2, a contradiction. This completes the proof. □

2.8. Theorem. *Let R be a $*-$prime ring with characteristic not 2, U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R, $0 \neq a \in R$, d be a nonzero derivation of R and $*$ be commute with σ, τ and d. If $ad(U) = (0)$ (or $d(U)a = (0)$), then $U \subseteq Z$.*

Proof. Assume that $U \nsubseteq Z$ and $ad(U) = (0)$. There exists a nonzero $*-$ideal M of R such that $[R,M]_{\sigma,\tau} \subseteq U$, but $[R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}$. For any $x \in R, m \in M$ and $[x,m]_{\sigma,\tau}\sigma(m) \in U$, we get

$$ad([x,m]_{\sigma,\tau}\sigma(m)) = 0$$

Expanding this equation and using the hypothesis, we have

$$a[x,m]_{\sigma,\tau}d(\sigma(m)) = 0, \text{ for all } x \in R, m \in M. \tag{2.3}$$

Substituting $d(u)x$ for x in (2.3) and using this equation, we arrive at

$$a[d(u),\tau(m)]Rd(\sigma(m)) = (0), \text{ for all } u \in U, m \in M. \tag{2.4}$$

Now, taking m^* instead of $m, m \in M \cap S_*(R)$ in the last equation, we obtain

$$a[d(u),\tau(m^*)]Rd(\sigma(m^*)) = (0).$$

Using $m^* = \pm m$ and $\sigma* = *\sigma, *d = d*$, we get

$$a[d(u),\tau(m)]Rd(\sigma(m))^* = (0), \text{ for all } u \in U, m \in M \cap S_*(R). \tag{2.5}$$

Combining (2.4) and (2.5) and using the $*-$primeness of R, we have

$$a[d(u),\tau(m)] = 0 \text{ or } d(\sigma(m)) = 0,\ \forall u \in U, m \in M \cap S_*(R).$$

Since $m - m^* \in M \cap S_*(R)$ for all $m \in M$, we have

$$a[d(u),\tau(m)] = a[d(u),\tau(m^*)] \text{ or } d(\sigma(m)) = d(\sigma(m^*)),\ \forall u \in U, m \in M.$$

Now, define $A = \{m \in M \mid a[d(u),\tau(m)] = a[d(u),\tau(m^*)], \text{ for all } u \in U\}$ and $B = \{m \in M \mid d(\sigma(m)) = d(\sigma(m^*))\}$. It is clear that, A and B are an additive subgroups of M such that $M = A \cup B$. But a group can not be an union of its proper subgroups. Therefore, it yields $M = A$ or $M = B$. In $M = A$ case, $a[d(u),\tau(m)] = a[d(u),\tau(m^*)]$, for all $u \in U, m \in M$. In (2.4) substituting m by m^*, we get

$$a[d(u),\tau(m)]Rd^*(\sigma(m)) = (0),\ \forall u \in U, m \in M.$$

Since R is $*-$prime, we arrive

$$a[d(u),\tau(m)] = 0 \text{ or } d(\sigma(m)) = 0,\ \forall u \in U, m \in M.$$

In $M = B$ case, $d(\sigma(m)) = d(\sigma(m^*))$, for all $m \in M$. From (2.4), we have

$$a[d(u),\tau(m)]Rd^*(\sigma(m)) = (0),\ \forall u \in U, m \in M.$$

Since R is $*-$prime, we get

$$a\left[d\left(u\right),\tau\left(m\right)\right]=0 \text{ or } d\left(\sigma\left(m\right)\right)=0,\ \forall u\in U,m\in M.$$

Note that, for the both cases the same results are obtained.

Let us define the sets $K=\{m\in M\mid a[d(u),\tau(m)]=0, \text{ for all } u\in U\}$ and $L=\{m\in M\mid d(\sigma(m))=0\}$. By a standard argument one of these must hold for all $m\in M$.

If $a[d(u),\tau(m)]=0$, for all $u\in U,m\in M$. Expanding this equation and using the hyphotesis, we get

$$a\tau(M)d(u)=(0),\ \text{for all } u\in U.$$

Substituting u^* for u in this equation and using $*d=d*$

$$a\tau(M)d(u)^*=(0)$$

and so

$$a\tau(M)d(u)=a\tau(M)d(u)^*=(0),\ \text{for all } u\in U.$$

Since $\sigma(M)$ a nonzero $*-$ideal of R and by Lemma 2.3, we have

$$a=0 \text{ or } d(U)=(0).$$

If $d(U)=(0)$, then $U\subseteq Z$ by Theorem 2.7, which is a contradiction.

If $d(\sigma(M))=0$, then R is commutative by Lemma 2.2, a contradiction. This completes the proof.

Now, we get $d(U)a=(0)$. Assume that $U\nsubseteq Z$ and using the same arguments in the beginning of the proof, we get $\tau(m)[x,m]_{\sigma,\tau}\in U$ for any $x\in R,m\in M$, and so

$$d(\tau(m)[x,m]_{\sigma,\tau})a=0.$$

Expanding this equation and using the hypothesis, we arrive at

$$d(\tau(m))[x,m]_{\sigma,\tau}a=0.$$

Replacing $xd(u)$ for x in this equation and applying the same lines above, we get the required result. □

Remark. Suppose that U a nonzero $*-(\sigma,\tau)-$right Lie ideal of R, d a derivation of R and $d*=*d$. For all $u,v\in U,x\in R$,

$$\begin{aligned}[d(u)+v,x]_{\sigma,\tau}&=[d(u),x]_{\sigma,\tau}+[v,x]_{\sigma,\tau}\\&=[d(u),x]_{\sigma,\tau}+[u,d(x)]_{\sigma,\tau}-[u,d(x)]_{\sigma,\tau}+[v,x]_{\sigma,\tau}\\&=d([u,x]_{\sigma,\tau})-[u,d(x)]_{\sigma,\tau}+[v,x]_{\sigma,\tau}\in d(U)+U.\end{aligned}$$

We conclude that $d(U)+U$ is a $(\sigma,\tau)-$right Lie ideal of R. Furthermore, $(d(U)+U)^*=d(U)^*+U^*=d(U^*)+U^*=d(U)+U$. Hence $d(U)+U$ is a $*-(\sigma,\tau)-$right Lie ideal of R. On the other hand, if $d^2(U)=(0)$, then $d(d(U)+U))\subset d(U)\subset d(U)+U$. Hence, without the loss of generalizing, we can assume that if U is a nonzero $*-(\sigma,\tau)-$right Lie ideal such that $d^2(U)=(0)$, then $d(U)\subset U$.

2.9. Theorem. *Let R be a $*-$prime ring with characteristic not 2, U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d be a nonzero derivation of R such that $d\tau=\tau d$, $d\sigma=\sigma d$ and $*$ be commute with σ,τ and d. If $d^2(U)=(0)$, then $d(U)\subseteq Z$.*

Proof. For any $x\in R$ and $u\in U$, $\tau\left(u\right)\left[x,u\right]_{\sigma,\tau}=\left[\tau\left(u\right)x,u\right]_{\sigma,\tau}\in U$. Taking $\tau\left(u\right)\left[x,u\right]_{\sigma,\tau}$ instead of u in the hypothesis, we get

$$d^2(\tau\left(u\right)\left[x,u\right]_{\sigma,\tau})=d^2(\tau\left(u\right))\left[x,u\right]_{\sigma,\tau}+2d(\tau\left(u\right))d(\left[x,u\right]_{\sigma,\tau})+\tau(u)d^2(\left[x,u\right]_{\sigma,\tau}).$$

Using $d\tau=\tau d$ and the hypothesis in the above relation, we arrive at

$$d(\tau\left(u\right))d(\left[x,u\right]_{\sigma,\tau})=0,\ \text{for all } u\in U,x\in R. \tag{2.6}$$

Replacing u by $u+d(v)$ in (2.6) and using (2.6), we have

$$d(\tau(u))d([x,d(v)]_{\sigma,\tau})=0,\ \text{for all } u,v\in U, x\in R.$$

Using that τ is an automorphism and $d\tau=\tau d$, we see that

$$d(u)\tau^{-1}(d([x,d(v)]_{\sigma,\tau}))=0,\ \text{for all } u,v\in U, x\in R.$$

By Theorem 2.8, we conclude that

$$U\subseteq Z\ \text{ or }\ d([x,d(v)]_{\sigma,\tau})=0,\ \text{for all } v\in U, x\in R.$$

If $U\subseteq Z$, then $d(U)\subseteq Z$, and so the proof is completed.

Now, we have $d([x,d(v)]_{\sigma,\tau})=0$, for all $u,v\in U, x\in R$. Applying the hypothesis, we get

$$[d(x),d(v)]_{\sigma,\tau}=0,\ \text{for all } v\in U, x\in R. \tag{2.7}$$

Taking $xd(u)$ instead of x in the above equation and using $d^2(u)=0$, we find that

$$[d(x)d(u),d(v)]_{\sigma,\tau}=0.$$

(2.7) yields that

$$[d(x),\tau(d(U))]d(u)=0,\ \text{for all } u\in U, x\in R.$$

By Theorem 2.8, we have

$$U\subseteq Z\ \text{ or }\ [d(x),\tau(d(U))]=0,\ \text{for all } x\in R.$$

If $U\subseteq Z$, then $d(U)\subseteq Z$. This implies that $[d(x),\tau(d(U))]=0$, for all $x\in R$. So, we must have $[d(x),\tau(d(U))]=0$, for all $x\in R$ for any cases. Since $U\cap S_*(R)\subseteq U$, we get

$$[d(R),\tau(d(U\cap S_*(R)))]=(0).$$

From Lemma 2.2, it implies that $\tau(d(U\cap S_*(R)))\subseteq Z$ and so $d(U\cap S_*(R))\subseteq Z$. Since $u-u^*, u+u^*\in U\cap S_*(R)$ for all $u\in U$, we have $d(u)-d(u^*), d(u)+d(u^*)\in Z$. Therefore $2d(u)\in Z$, for all $u\in U$. Since $char R\neq 2$, it is implies that $d(u)\in Z$, for all $u\in U$. Namely, $d(U)\subseteq Z$. This completes the proof. □

2.10. Theorem. *Let R be a $*-$prime ring with characteristic not 2, U be a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d be a nonzero derivation of R such that $d\tau=\tau d, \sigma d=d\sigma$ and $*$ be commute with σ,τ and d. If $d^2(U)=(0)$, then $U\subseteq Z$.*

Proof. Applying the same arguments that are used in the proof of Theorem 2.9, we get

$$d(\tau(u))d([x,u]_{\sigma,\tau})=0,\ \text{for all } u\in U, x\in R. \tag{2.8}$$

Replacing u by $u+v$ in (2.8) and using this, we have

$$d(\tau(u))d([x,v]_{\sigma,\tau})+d(\tau(v))d([x,u]_{\sigma,\tau})=0,\ \text{for all } u,v\in U, x\in R. \tag{2.9}$$

Multiplying (2.9) from the left by $d(\tau(u))$ and using $d(\tau(u))=\tau(d(u))\in Z$ by Theorem 2.9, we find that

$$\begin{aligned}0&=d(\tau(u))d(\tau(u))d([x,v]_{\sigma,\tau})+d(\tau(u))d(\tau(v))d([x,u]_{\sigma,\tau})\\&=d(\tau(u))^2d([x,v]_{\sigma,\tau})+d(\tau(v))d(\tau(u))d([x,u]_{\sigma,\tau}).\end{aligned}$$

By (2.8) it holds that

$$d(\tau(u))^2d([x,v]_{\sigma,\tau})=0,\ \text{for all } u,v\in U, x\in R. \tag{2.10}$$

For any $u\in U, x\in R$, $[x\sigma(u),u]_{\sigma,\tau}=[x,u]_{\sigma,\tau}\sigma(u)\in[R,U]_{\sigma,\tau}$. Taking $[x,u]_{\sigma,\tau}\sigma(u)$ instead of $[x,v]_{\sigma,\tau}$ in (2.10) and using this, we obtain

$$d(\tau(u))^2[x,v]_{\sigma,\tau}d(\sigma(v))=0,\ \text{for all } u,v\in U, x\in R. \tag{2.11}$$

Writing v by $v+w$ in (2.11) and using this, we have

$$d(\tau(u))^2 [x,v]_{\sigma,\tau} d(\sigma(w)) + d(\tau(u))^2 [x,w]_{\sigma,\tau} d(\sigma(v)) = 0, \ \forall u,v,w \in U, x \in R.$$

Multiplying the last equation from the right by $d(\sigma(v))$ and using $d(\sigma(v)) = \sigma(d(v)) \in Z$ by Theorem 2.9, we get

$$\begin{aligned} 0 &= d(\tau(u))^2 [x,v]_{\sigma,\tau} d(\sigma(w)) d(\sigma(v)) + d(\tau(u))^2 [x,w]_{\sigma,\tau} d(\sigma(v)) d(\sigma(v)) \\ &= d(\tau(u))^2 [x,v]_{\sigma,\tau} d(\sigma(v)) d(\sigma(w)) + d(\tau(u))^2 [x,w]_{\sigma,\tau} d(\sigma(v))^2. \end{aligned}$$

From (2.11), we conclude that

$$d(\tau(u))^2 [x,w]_{\sigma,\tau} d(\sigma(v))^2 = 0, \text{ for all } u,v,w \in U, x \in R.$$

Using $d(\sigma(v)) \in Z$, we obtain that

$$(2.12) \quad d(\tau(u))^2 [x,w]_{\sigma,\tau} R d(\sigma(v))^2 = (0), \text{ for all } u,v,w \in U, x \in R.$$

Replacing v by v^* in this equation, we get

$$d(\tau(u))^2 [x,w]_{\sigma,\tau} R d(\sigma(v^*))^2 = (0).$$

Since $*$ commutes with σ, τ and d, we get

$$(2.13) \quad d(\tau(u))^2 [x,w]_{\sigma,\tau} R (d(\sigma(v))^2)^* = (0), \text{ for all } u,v,w \in U, x \in R.$$

Equations (2.12) and (2.13) yields that

$$d(\tau(u))^2 [x,w]_{\sigma,\tau} = 0 \ \text{ or } \ d(\sigma(v))^2 = 0, \text{ for all } u,v,w \in U, x \in R.$$

If $d(\sigma(v))^2 = 0$ for all $v \in U$, then it implies that $d(u)^2 = 0$, and so, we get $d(\tau(u))^2 [x,w]_{\sigma,\tau} = 0$ for all $u,w \in U, x \in R$. Again using $d(\tau(u)) = \tau(d(u)) \in Z$, we have

$$(2.14) \quad d(\tau(u))^2 R [x,w]_{\sigma,\tau} = (0), \text{ for all } u,w \in U, x \in R.$$

Writing u by u^* in this equation, we get

$$(2.15) \quad d(\tau(u^*))^2 R [x,w]_{\sigma,\tau} = (0), \text{ for all } u,w \in U, x \in R.$$

Combining (2.14) and (2.15) equations and using the $*-$primeness of R, we arrive at

$$d(\tau(u)) = 0 \ \text{ or } \ [x,w]_{\sigma,\tau} = 0, \text{ for all } u,w \in U, x \in R.$$

If $d(\tau(u)) = 0$, for all $u \in U$, then $d(U) = (0)$, and so $U \subseteq Z$ by Theorem 2.7.

Now, we get $[x,w]_{\sigma,\tau} = 0$ for all $w \in U, x \in R$. Replacing x by vx and using this, we get

$$\begin{aligned} 0 &= [vx,w]_{\sigma,\tau} = v[x,\sigma(w)] + [v,w]_{\sigma,\tau} x \\ &= v[x,\sigma(w)] \end{aligned}$$

and so

$$U[x,\sigma(w)] = 0, \text{ for all } w \in U, x \in R.$$

According to Lemma 2.5, we obtain that $U \subseteq Z$. This completes the proof. □

2.11. Remark. Our assumption that $*d = d*$ implies that both the symmetric and skew-symmetric elements are stable under d. This assumption is commonly used in the literature and it would be interesting to see which of these results hold without this assumption.

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