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The complementary geometric transmuted-G family of distributions: model, properties and application

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Abstract

We introduce a new family of continuous distributions called the complementary geometric transmuted-G family, which extends the transmuted family proposed by Shaw and Buckley (2007). Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, quantile and generating functions, entropies, order statistics and probability weighted moments are derived. Two special models of the introduced family are discussed in detail. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of two applications to real data sets. We provide some simulation results to assess the performance of the proposed model.

Keywords: Complementary geometric-G Family, Maximum likelihood, Order statistic, Probability weighted moment, Transmuted family.

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1. Introduction

The statistical literature is full of new families of distributions that extend classical distributions and at the same time become very important for statisticians due to their flexible properties. These new families have been extensively used in modelling data in several applied areas such as reliability, engineering and life testing. In recent years there has been an increased interest in developing more flexible generators for univariate continuous distributions by adding extra shape parameter(s) to the baseline distribution. Some well-known families are the beta-G by Eugene et al. (2002), the Kumaraswamy-G by Cordeiro and de Castro (2011), the McDonald-G by Alexander et al. (2012), the gamma-G by Zografos and Balakrishanan (2012), the Weibull-G by Bourguignon et al. (2014), the odd generalized exponential-G by Tahir et al. (2015), the transmuted exponentiated generalized-G by Yousof et al. (2015), the generalized transmuted-G by Nofal et al. (2017), the transmuted geometric-G by Afify et al. (2016a), the Kumaraswamy transmuted-G by Afify et al. (2016b), the exponentiated transmuted-G by Merovci et al. (2016), the Burr X-G by Yousof et al. (2016), the two-sided power-G by Korkmaz and Genc (2016) and the beta transmuted-H by Afify et al. (2016d).

Let $G(x;\varphi)$ be a baseline cumulative distribution function (cdf) and $g(x;\varphi)$ be the associated probability density function (pdf), where $\varphi = (\varphi_1, \varphi_2, \ldots)$ is a parameter vector. Then, the cdf and pdf of the *transmuted-G* (T-G) family of distributions are, respectively, given by

$$H(x;\lambda,\varphi) = G(x;\varphi)\left[1+\lambda-\lambda G(x;\varphi)\right] \tag{1.1}$$

and

$$h(x;\lambda,\varphi) = g(x;\varphi)\left[1+\lambda-2\lambda G(x;\varphi)\right], \tag{1.2}$$

where $|\lambda| \le 1$. It is noted that the T-G family is a mixture of the baseline and exponentiated-G (exp-G) distributions, the last one with power parameter equal to two. Further, we obtain the baseline distribution when $\lambda = 0$. For more details about the T-G family, see Shaw and Buckley (2007).

For a baseline random variable having pdf $h(x)$ and cdf $H(x)$, the *complementary geometric-H* (CGc-H) family is defined by the cdf (see Appendix A)

$$F(x;\theta,\varphi) = \frac{\theta H(x;\varphi)}{1-(1-\theta)H(x;\varphi)},$$

and the pdf given by

$$f(x;\theta,\varphi) = \frac{\theta h(x;\varphi)}{\left[1-(1-\theta)H(x;\varphi)\right]^2},$$

where $\theta \in (0,1)$. In this paper, we propose and study a new extension of the T-G family by adding one parameter in equation (1.1) to provide more flexibility to the generated family. We construct a new generator called the *complementary geometric transmuted-G* (CGcT-G) family by taking the T-G cdf in (1.1) as the baseline cdf H in the last two equations. Further, we give a comprehensive description of the mathematical properties of the new family. In fact, the CGcT-G family is motivated by its important flexibility in applications. By means of two applications, we show that the CGcT-G class provides better fits than at least seven other families each having the same number of parameters.

The rest of the paper is outlined as follows. Section 2 is devoted to some well-known distributions, which will be used in the empirical comparisons in Section 9. In Section 3, we define the CGcT-G family. A very useful linear representation for its pdf is derived in Section 4. In Section 5, we define two special models and provide plots of their pdfs and hazard rate functions (hrfs). In Section 6, we derive some mathematical properties including ordinary and incomplete moments, quantile and generating functions, entropies, probability weighted moments (PWMs) and order statistics. We provide some properties of the CGcT-Weibull (CGcTW) distribution in Section 7. Maximum likelihood estimation of the model parameters is addressed in Section 8. In Section 9, we give two applications to real data to illustrate the importance of the introduced family. Some simulation results assess the performance of the proposed model in Section 10. Finally, some concluding remarks are presented in Section 11.

2. Previous works

We shall refer to some competitive models to the introduced distribution, namely: the Kumaraswamy-transmuted exponentiated modified Weibull (Kw-TEMW) (Al-Babtain et al., 2015), transmuted exponentiated modified Weibull (TEMW) (Ashour and Eltehiwy, 2013), transmuted exponentiated Weibull geometric (TEWG) (Saboor et al., 2015), transmuted additive Weibull (TAW) (Elbatal and Aryal, 2013), Kumaraswamy modified Weibull (Kw-MW) (Cordeiro et al., 2014), beta Weibull (BW) (Lee et al., 2007), Kumaraswamy Weibull (Kw-W) (Cordeiro et al., 2010), additive Weibull (AW) (Xie and Lai, 1995), Weibull Lindley (WLi) (Bourguignon et al., 2014), Weibull gamma (WG) (Bourguignon et al., 2014), odd log-logistic Lindley (OLL-Li) (Ozel et al., 2016), generalized transmuted Lindley (GT-Li) (Nofal et al., 2017), Kumaraswamy Lindley (Kw-Li) (Cakmakyapan and Kadilar, 2014) and beta Lindley (BLi) (Merovci and Sharma, 2014) distributions. Their corresponding pdfs are given in Appendix B.

3. The CGcT-G family

In this section, we generalize the T-G family by incorporating one additional parameter to yield a more flexible generator. The CGcT-G family is given by the cdf (for $x > 0$)

$$F\left(x;\theta,\lambda,\varphi\right)=\frac{\theta G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]}{1-\left(1-\theta\right)G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]}. \tag{3.1}$$

The pdf corresponding to (3.1) is

$$f(x;\theta,\lambda,\varphi)=\frac{\theta g\left(x;\varphi\right)\left[1+\lambda-2\lambda G\left(x;\varphi\right)\right]}{\left\{1-\left(1-\theta\right)G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]\right\}^{2}}, \tag{3.2}$$

where $\theta\in(0,1)$ and $|\lambda|\leq 1$ are shape parameters. For $\lambda=0$, we obtain the complementary geometric-G (CGc-G) family. For $\theta\to 1$, we have the T-G family. Henceforth, we denote by $X\sim$CGcT-$G\left(\theta,\lambda,\varphi\right)$ a random variable having pdf (3.2). The reliability function (rf) and hrf of X are, respectively, given by

$$R\left(x;\theta,\lambda,\varphi\right)=\frac{1-G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]}{1-\left(1-\theta\right)G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]}$$

and

$$\tau\left(x;\theta,\lambda,\varphi\right)=\frac{\theta\ r\left(x;\lambda,\varphi\right)}{1-\left(1-\theta\right)G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]},$$

where $r\left(x;\lambda,\varphi\right)$ is the hrf of the T-G family.

4. Linear representation

In this section, we provide a very useful linear representation for the CGcT-G pdf. We omit the dependence of $F(x)$ and $f(x)$ on the model parameters. The cdf (3.1) can be expressed as

$$F(x)=\theta G\left(x\right)\left[1+\lambda-\lambda G\left(x\right)\right]\left\{1-\left(1-\theta\right)G\left(x\right)\left[1+\lambda-\lambda G\left(x\right)\right]\right\}^{-1}. \tag{4.1}$$

An expansion for equation (4.1) can be derived using the power series

$$(4.2)\qquad (1-z)^{-b}=\sum_{k=0}^{\infty}\frac{\Gamma(b+k)}{k!\;\Gamma(b)}\;z^{k},\;\;|z|\leq 1,\;\;b>0.$$

Applying (4.2) to the last term of (4.1) gives

$$F(x)=\sum_{k=0}^{\infty}\theta(1-\theta)^{k}\,(1+\lambda)^{k}\,G^{k+1}(x)\,[1+\lambda-\lambda G(x)]\,[1-p\;G\,(x)]^{k}\,,$$

where $p=\lambda/\,(1+\lambda)$.

Using the binomial expansion to $[1-pG\,(x)]^{k}$, we obtain

$$(4.3)\qquad [1-pG\,(x)]^{k}=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}p^{j}G^{j}(x).$$

Combining the last two equations gives

$$F(x)=\textstyle\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j}\,\theta(1-\theta)^{k}\,(1+\lambda)^{k}\,p^{j}\binom{k}{j}\left[(1+\lambda)\,G^{k+j+1}(x)-\lambda G^{k+j+2}(x)\right].$$

Then, we can write

$$(4.4)\qquad F(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\;[\upsilon_{k,j}\;\Pi_{k+j+1}\,(x)-\omega_{k,j}\;\Pi_{k+j+2}\,(x)]\,,$$

where $\Pi_{\alpha}\,(x)$ is the cdf of the exp-G family with power parameter α,

$$\upsilon_{k,j}=(-1)^{j}\,\theta(1-\theta)^{k}\,(1+\lambda)^{k+1}\,p^{j}\binom{k}{j}$$

and

$$\omega_{k,j}=(-1)^{j}\,\theta(1-\theta)^{k}\,(1+\lambda)^{k+1}\,p^{j+1}\binom{k}{j}.$$

By differentiating (4.4), the pdf (3.2) can be expressed as

$$\begin{aligned} f(x) \;&=\; \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j}\,\theta(1-\theta)^{k}\,(1+\lambda)^{k}\,p^{j}\binom{k}{j}\\ &\quad \Big[(1+\lambda)\,(k+j+1)\,g\,(x)\,G^{k+j}(x)-\lambda\,(k+j+2)\,g\,(x)\,G^{k+j+1}(x)\Big]. \end{aligned}$$

The last equation can be rewritten as

$$(4.5)\qquad f(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\;[\upsilon_{k,j}\;\pi_{k+j+1}\,(x)-\omega_{k,j}\;\pi_{k+j+2}\,(x)]\,,$$

where $\pi_{\alpha}\,(x)=\alpha g\,(x)\,G^{\alpha-1}\,(x)$ is the exp-G pdf with power parameter $\alpha>0$. Thus, some mathematical properties of the CGcT-G family can be determined from those properties of the exp-G family. Equations (4.4) and (4.5) are the main results of this section.

5. Special models

In this section, we provide two special models of the CGcT-G family. The pdf (3.2) will be most tractable when $g\,(x;\varphi)$ and $G\,(x;\varphi)$ have simple analytic expressions. These special models generalize some well-known distributions in the literature.

5.1. The CGcT-Weibull (CGcTW) distribution. Consider the pdf and cdf (for $x > 0$) $g(x) = \beta\alpha^{\beta}x^{\beta-1}\exp\left[-(\alpha x)^{\beta}\right]$ and $G(x) = 1 - \exp\left[-(\alpha x)^{\beta}\right]$, respectively, of the Weibull distribution with positive parameters α and β. Then, the cdf and pdf of the CGcTW model (for $x > 0$) are, respectively, given by

$$F(x) = \frac{\theta\left\{1 - \exp\left[-(\alpha x)^{\beta}\right]\right\}\left\{1 + \lambda\exp\left[-(\alpha x)^{\beta}\right]\right\}}{1 - (1-\theta)\left\{1 - \exp\left[-(\alpha x)^{\beta}\right]\right\}\left\{1 + \lambda\exp\left[-(\alpha x)^{\beta}\right]\right\}}$$

and

$$f(x) = \frac{\theta\beta\alpha^{\beta}x^{\beta-1}\exp\left[-(\alpha x)^{\beta}\right]\left\{1 - \lambda + 2\lambda\exp\left[-(\alpha x)^{\beta}\right]\right\}}{\left(1 - (1-\theta)\left\{1 - \exp\left[-(\alpha x)^{\beta}\right]\right\}\left\{1 + \lambda\exp\left[-(\alpha x)^{\beta}\right]\right\}\right)^{2}},$$

where $\theta \in (0,1)\,, |\lambda| \leq 1$ and $\beta > 0$ are shape parameters and $\alpha > 0$ is a scale parameter. The CGcTW distribution includes the complementary geometric Weibull (CGcW) distribution when $\lambda = 0$. If θ tends to 1, we have the transmuted Weibull (TW) distribution. For $\beta = 2$ and $\beta = 1$, we obtain the complementary geometric transmuted Rayleigh (CGcTR) and complementary geometric transmuted exponential (CGcTE) distributions, respectively. Figure 1 displays some possible shapes of the pdf and hrf of this distribution.

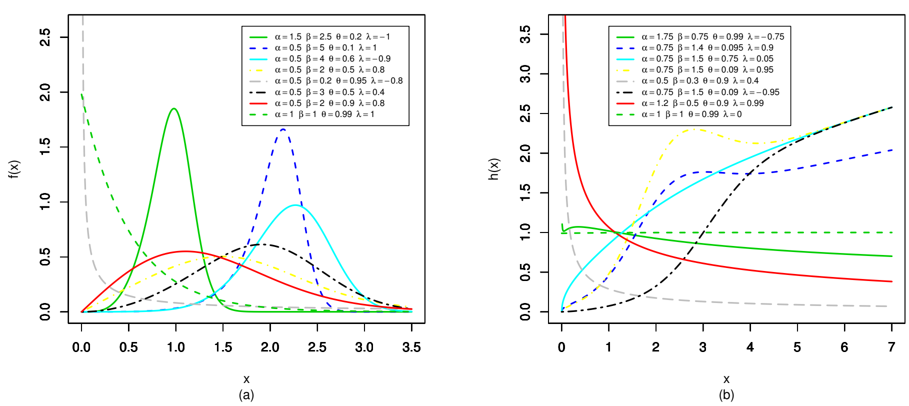

Figure 1. (a) The CGcTW pdf plots. (b) The CGcTW hrf plots.

5.2. The CGcT-Lindley (CGcTLi) distribution. The Lindley distribution with parameter $\alpha > 0$ has pdf and cdf (for $x > 0$) given by $g(x) = \frac{\alpha^2}{1+\alpha}(1+x)\exp(-\alpha x)$ and $G(x) = 1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)$, respectively. Then, the cdf and pdf of the CGcTLi distribution (for $x > 0$) are given by

$$F(x) = \frac{\theta\left[1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\left[1 + \lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]}{1 - (1-\theta)\left[1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\left[1 + \lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]}$$

and

$$f(x) = \frac{\frac{\theta\alpha^2}{1+\alpha}(1+x)\exp(-\alpha x)\left[1 - \lambda + 2\lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]}{\left\{1 - (1-\theta)\left[1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\left[1 + \lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\right\}^{2}},$$

respectively, where $\theta \in (0,1)$ and $|\lambda| \leq 1$ are shape parameters and $\alpha > 0$ is a scale parameter. The CGcTLi distribution reduces to the complementary geometric Lindley (CGcLi) distribution when $\lambda = 0$. If θ tends to 1, we obtain the transmuted Lindley (TLi) distribution. Plots of the pdf and hrf of the CGcTLi distribution are displayed in Figure 2 for some parameter values.

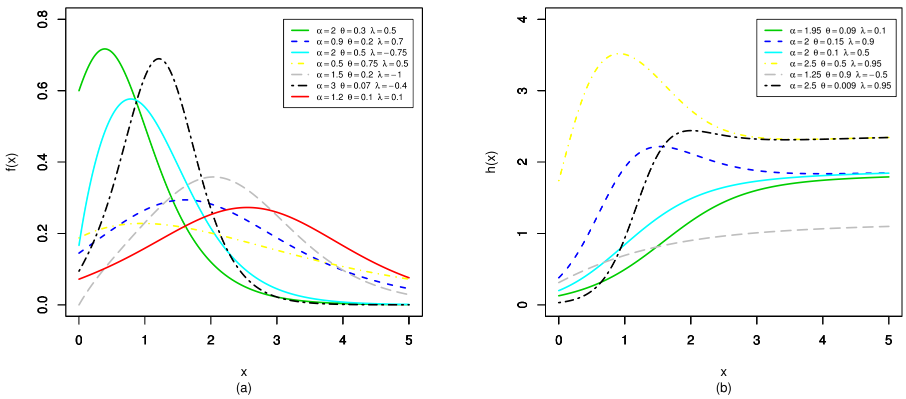

Figure 2. (a) Plots of the CGcTLi pdf. (b) Plots of the CGcTLi hrf.

6. Properties

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab because of their ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

6.1. Moments. Let Y_α be a random variable having exp-G pdf $\pi_\alpha(x)$. The rth ordinary moment of X, say μ'_r, follows from (4.5) as

$$\mu'_r = E\left(X^r\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \left[\upsilon_{k,j}\ E\left(Y_{k+j+1}^r\right) - \omega_{k,j}\ E\left(Y_{k+j+2}^r\right)\right]. \tag{6.1}$$

For $\alpha > 0$, we have

$$E\left(Y_\alpha^r\right) = \alpha \int_{-\infty}^{\infty} x^r\ g\left(x;\varphi\right)\ G\left(x;\varphi\right)^{\alpha-1}\ dx,$$

which can be computed numerically in terms of the baseline quantile function (qf) $Q_G\left(u;\varphi\right) = G^{-1}\left(u;\varphi\right)$ as

$$E\left(Y_\alpha^n\right) = \alpha \int_0^1 Q_G\left(u;\varphi\right)^n\ u^{\alpha-1} du.$$

Setting $r = 1$ in (6.1) gives the mean of X. The central moments (μ_n) and cumulants (κ_n) of X are determined from (6.1) as $\mu_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\ \mu_1'^k\ \mu'_{n-k}$ and $\kappa_n = \mu'_n - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\ \kappa_k\ \mu'_{n-k}$, respectively, where $\kappa_1 = \mu'_1$. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ are obtained from the third and fourth standardized cumulants.

The nth descending factorial moment of X (for $n = 1, 2, \ldots$) is

$$\mu'_{(n)} = E\left[X^{(n)}\right] = E\left[X\left(X-1\right) \times \ldots \times \left(X-n+1\right)\right] = \sum_{k=0}^{n} s\left(n,k\right) \mu'_k,$$

where $s\left(n,k\right) = (k!)^{-1}\left[d^k k^{(n)}/dx^k\right]_{x=0}$ is the Stirling number of the first kind.

6.2. Incomplete moments. The rth incomplete moment of X is defined by $m_r(y) = \int_{-\infty}^{y} x^r\ f(x)dx$. We can write from (4.5)

$$m_r(y) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\upsilon_{k,j}\ m_{r,k+j+1}(y) - \omega_{k,j}\ m_{r,k+j+2}(y)\right], \tag{6.2}$$

where

$$m_{r,\alpha}(y) = E\left(Y_\alpha^r\right) = \int_0^{G(y;\ \varphi)} Q_G^r\left(u;\varphi\right)\ u^{\alpha-1}\ du.$$

The integral $m_{r,\alpha}(y)$ can be determined analytically for special models with closed-form expressions for $Q_G\left(u;\varphi\right)$ or computed at least numerically for most baseline distributions.

An important application of the first incomplete moment refers to the mean deviations about the mean $[\delta_1 = E\left(|X-\mu'_1|\right)]$ and about the median $[\delta_2 = E(|X-M|)]$ of X given by

$$\delta_1 = 2\mu'_1\ F\left(\mu'_1\right) - 2m_1\left(\mu'_1\right) \qquad \text{and} \qquad \delta_2 = \mu'_1 - 2m_1(M),$$

respectively, where M is the median of X, $F\left(\mu'_1\right)$ is easily obtained from (3.1), $\mu'_1 = E(X)$ can follow from equation (6.1), and $m_1(z)$ can be determined from (6.2) with $r = 1$.

Another common application of the first incomplete moment refers to the Bonferroni and Lorenz curves, which are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, the Bonferroni and Lorenz curves are given by $B(\pi) = m_1(p)/\left(p\mu'_1\right)$ and $L(p) = m_1(p)/\mu'_1$, where $p = Q(\pi) = F^{-1}(\pi)$ can be determined numerically by inverting (3.1).

6.3. Quantile and generating functions. The qf of X ~CGcT-$G\left(\theta,\lambda,\varphi\right)$ follows by inverting (3.1), namely $x = Q(u) = F^{-1}(u)$.

$$\frac{\theta G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]}{1-\left(1-\theta\right)G\left(x;\varphi\right)\left[1+\lambda-\lambda G\left(x;\varphi\right)\right]} = u.$$

Rearranging terms gives the quadratic equation

$$\lambda G^2\left(x;\varphi\right) - (1+\lambda)G\left(x;\varphi\right) + \frac{u}{\theta+(1-\theta)u} = 0.$$

The two roots are

$$G\left(x;\varphi\right) = \frac{1+\lambda \pm \sqrt{\left(1+\lambda\right)^2 - 4\lambda u/\left[u+\theta\left(1-u\right)\right]}}{2\lambda}.$$

Of these, the minus sign gives the valid root. Hence, for $\lambda \neq 0$ and $u \in (0,1)$,

$$x = Q_G\left(\frac{1+\lambda - \sqrt{\left(1+\lambda\right)^2 - 4\lambda u/\left[u+\theta\left(1-u\right)\right]}}{2\lambda}\right),$$

where $Q_G(u) = G^{-1}(u)$ is the baseline qf. For $\lambda = 0$, we have

$$x = Q_G\left(u/\left[u+\theta\left(1-u\right)\right]\right).$$

Simulating the CGcT-G random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ follows (3.2).

The moment generating function (mgf) of X, say $M(t) = E\left[\exp\left(t\ X\right)\right]$, is determined from (4.5) as

$$M(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\upsilon_{k,j}\ M_{k+j+1}\left(t;\varphi\right) - \omega_{k,j}\ M_{k+j+2}\left(t;\varphi\right)\right],$$

where $M_{\alpha}\left(t;\varphi\right)$ is the generating function of Y_{α} given by

$$M_{\alpha}\left(t;\varphi\right) = \alpha\ \int_{-\infty}^{\infty}\exp\left(tx\right) g\left(x;\varphi\right) G^{\alpha-1}\left(x;\varphi\right) dx = \alpha\int_{0}^{1}\exp\left[t\ Q_G(u;\alpha)\right] u^{\alpha-1}du.$$

Both formulas can be computed numerically for most parent distributions.

6.4. Entropies. The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. It is defined by

$$I_{\gamma}\left(X\right) = \left(1-\gamma\right)^{-1}\ \log\left(\int_{-\infty}^{\infty} f^{\gamma}\left(x\right) dx\right),\ \gamma > 0 \text{ and } \gamma \neq 1.$$

Using the pdf (3.2), we can write

$$f^{\gamma}\left(x\right) = \frac{\theta^{\gamma} g^{\gamma}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]^{\gamma}}{\left\{1-\left(1-\theta\right) G\left(x\right)\left[1+\lambda-\lambda G\left(x\right)\right]\right\}^{2\gamma}}.$$

Applying (4.2) to the denominator gives

$$f^{\gamma}\left(x\right) = \sum_{k=0}^{\infty}\frac{\theta^{\gamma}\left(1+\lambda\right)^{k}\Gamma\left(2\gamma+k\right)}{k!\left(1-\theta\right)^{-k}\Gamma\left(2\gamma\right)} g^{\gamma}\left(x\right) G^{k}\left(x\right)\left[1-pG\left(x\right)\right]^{k}\left[1+\lambda-2\lambda G\left(x\right)\right]^{\gamma},$$

where $p = \lambda/\left(1+\lambda\right)$.

Based on (4.3) and after some algebra, we have

$$f^{\gamma}\left(x\right) = \textstyle\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(-1\right)^{j}\ \frac{\theta^{\gamma}(1+\lambda)^{k}p^{j}\Gamma(2\gamma+k)}{k!(1-\theta)^{-k}\Gamma(2\gamma)}\binom{k}{j} g^{\gamma}\left(x\right) G^{k+j}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]^{\gamma},$$

or, equivalently,

$$f^{\gamma}\left(x\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} m_{k,j}\ g^{\gamma}\left(x\right) G^{k+j}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]^{\gamma},$$

where

$$m_{k,j} = \left(-1\right)^{j}\ \frac{\theta^{\gamma}\left(1+\lambda\right)^{k} p^{j}\Gamma\left(2\gamma+k\right)}{k!\left(1-\theta\right)^{-k}\Gamma\left(2\gamma\right)}\binom{k}{j}.$$

Then, the Rényi entropy of the CGcT-G family reduces to

$$I_{\gamma}\left(X\right) = \left(1-\gamma\right)^{-1}\log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k} m_{k,j}\int_{-\infty}^{\infty} g^{\gamma}\left(x\right) G^{k+j}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]^{\gamma} dx\right].$$

The γ-entropy, say $H_{\gamma}\left(X\right)$, can be obtained as

$$H_{\gamma}\left(X\right) = \left(\gamma-1\right)^{-1}\log\left\{1-\left[\int_{-\infty}^{\infty} f^{\gamma}\left(x\right) dx\right]\right\},$$

which follows from the last equation, where $\gamma > 0$, $\gamma \neq 1$.

The Shannon entropy of a random variable X, say SE, is defined by

$$SE = E\left\{-\left[\log f\left(X\right)\right]\right\}.$$

It is a special case of the Rényi entropy when $\gamma \uparrow 1$. So, it follows by taking the limit of $I_{\gamma}\left(X\right)$ as γ tends to one.

6.5. PWMs. Generally, the PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist.

The (s,r)th PWM of X following the CGcT-G family, say $\rho_{s,r}$, is given by

$$\rho_{s,r} = E\left[X^s F^r(X)\right] = \int_{-\infty}^{\infty} x^s \ f\left(x\right) F^r(x) dx.$$

Using (3.1) and (3.2), we have

$$f\left(x\right) F^r\left(x\right) = \frac{\theta^{r+1} g\left(x\right) G^r\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]\left[1+\lambda-\lambda G\left(x\right)\right]^r}{\left\{1-\left(1-\theta\right) G\left(x\right)\left[1+\lambda-\lambda G\left(x\right)\right]\right\}^{r+2}}.$$

Applying the power series (4.2) gives

$$f\left(x\right) F^r\left(x\right) = \sum_{k=0}^{\infty} \frac{\theta^{r+1}\Gamma\left(k+r+2\right) g\left(x\right) G^{k+r}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]}{k!\left(1-\theta\right)^{-k}\left(1+\lambda\right)^{-k-r}\Gamma\left(r+2\right)\left[1-pG\left(x\right)\right]^{-(k+r)}},$$

where $p = \lambda/\left(1+\lambda\right)$. Using (4.3) and, after some simplifications, we obtain

$$\begin{aligned} f\left(x\right) F^r\left(x\right) &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\left(-1\right)^j \theta^{r+1} p^j \Gamma\left(k+r+2\right)}{k!\left(1-\theta\right)^{-k}\left(1+\lambda\right)^{-k-r}\Gamma\left(r+2\right)}\binom{k}{j} \\ &\quad \times g\left(x\right) G^{k+r+j}\left(x\right)\left[1+\lambda-2\lambda G\left(x\right)\right]. \end{aligned} \tag{6.3}$$

Then, we have

$$f\left(x\right) F^r\left(x\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[a_{k,j} \ \pi_{k+r+j+1}\left(x\right) - b_{k,j} \ \pi_{k+r+j+2}\left(x\right)\right], \tag{6.4}$$

where

$$a_{k,j} = \frac{\left(-1\right)^j \theta^{r+1}\left(1+\lambda\right)^{k+r+1} p^j \Gamma\left(k+r+2\right)}{k!\left(1-\theta\right)^{-k}\left(k+r+j+1\right)\Gamma\left(r+2\right)}\binom{k}{j}$$

and

$$b_{k,j} = \frac{\left(-1\right)^j 2\theta^{r+1}\left(1+\lambda\right)^{k+r+1} p^{j+1}\Gamma\left(r+k+2\right)}{k!\left(1-\theta\right)^{-k}\Gamma\left(r+2\right)}\binom{k}{j}.$$

Finally, the (s,r)th PWM of X can be expressed as an infinite linear combination of exp-G moments given by

$$\rho_{s,r} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[a_{k,j} \ E\left(Y_{k+r+j+1}^s\right) - b_{k,j} \ E\left(Y_{k+r+j+2}^s\right)\right].$$

6.6. Order statistics. Let $X_1, \ldots, X_n$ denote n independent and identically distributed CGcT-G random variables. Further, let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics from these n variables. The pdf of the ith order statistic $X_{(i)}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}\left(x\right) = \frac{f\left(x\right)}{\mathrm{B}\left(i, n-i+1\right)} \sum_{s=0}^{n-i}\left(-1\right)^s \binom{n-i}{s} F^{s+i-1}(x). \tag{6.5}$$

Using (6.3), we obtain

$$f\left(x\right) F^{s+i-1}\left(x\right) = \sum_{k=0}^{\infty} \frac{\theta^{s+i}\Gamma(k+s+i+1) g(x) G^{k+s+i-1}(x)[1+\lambda-2\lambda G(x)]}{k!(1-\theta)^{-k}(1+\lambda)^{-k-s-i+1}\Gamma(r+2)[1-pG(x)]^{-(k+s+i-1)}}.$$

Using (6.4), we can write

$$f(x)\,F^{s+i-1}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^j\,\theta^{s+i}p^j\Gamma(k+s+i+1)}{k!\,(1-\theta)^{-k}(1+\lambda)^{-k-s-i+1}\Gamma(s+i+1)}\binom{k}{j}$$

$$\times g(x)\,G^{k+s+i+j-1}(x)\left[1+\lambda-2\lambda G(x)\right]. \qquad (6.6)$$

Substituting (6.6) in equation (6.5), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{s=0}^{n-i}\left[a^{**}_{k,s}\ \pi_{k+s+i+j}(x) - b^{**}_{k,s}\ \pi_{k+s+i+j+1}(x)\right],$$

where $\pi_\alpha(x)$ is the exp-G pdf with power parameter α,

$$a^{**}_{k,s} = \frac{(-1)^{j+s}\,\theta^{s+i}(1-\theta)^k(1+\lambda)^{k+s+i}\,p^j\Gamma(k+s+i+1)}{k!\,(k+s+i+j)\,\mathrm{B}(i,n-i+1)\,\Gamma(s+i+1)}\binom{k}{j}\binom{n-i}{s}$$

and

$$b^{**}_{k,s} = \frac{(-1)^{j+s}2\theta^{s+i}(1-\theta)^k(1+\lambda)^{k+s+i}p^{j+1}\Gamma(k+s+i+1)}{k!(k+s+i+j+1)\mathrm{B}(i,n-i+1)\Gamma(s+i+1)}\binom{k}{j}\binom{n-i}{s}.$$

We note that the pdf of the CGcT-G order statistics is a linear combination of exp-G pdfs. Based on the last equation, the properties of $X_{i:n}$ can follow from those properties of Y_α. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X^q_{i:n}\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{s=0}^{n-i}\left[a^{**}_{k,s}\ E\left(Y^q_{k+s+i+j}\right) - b^{**}_{k,s}\ E\left(Y^q_{k+s+i+j+1}\right)\right]. \qquad (6.7)$$

The L-moments of X can be written as infinite weighted linear combinations of suitable means of the CGcT order statistics determined from equation (6.7) with $q=1$. Then, we can write

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E\left(X_{r-d:r}\right),\ r\geq 1.$$

7. The CGcTW properties

In this section, we derive some properties of the CGcTW distribution using the general properties discussed in Sections 4 and 6. According to equation (4.5), the CGcTW pdf can be expressed as

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\upsilon_{k,j}\ \pi_{k+j+1}(x) - \omega_{k,j}\ \pi_{k+j+2}(x)\right],$$

where $\pi_\eta(x)$ is the exponentiated Weibull (EW) pdf with power parameter η. Thus, several mathematical properties of the CGcTW distribution can be obtained simply from those properties of the EW model.

Let T be a random variable having the EW distribution with positive parameters α, β and δ. Then, the pdf and cdf of T are given by

$$g(t) = \delta\beta\alpha^\beta t^{\beta-1}\exp\left[-(\alpha t)^\beta\right]\left\{1-\exp\left[-(\alpha t)^\beta\right]\right\}^{\delta-1}$$

and

$$G(t) = \left\{1-\exp\left[-(\alpha t)^\beta\right]\right\}^{\delta}.$$

For any $n > -b$, Al-Hussaini and Ahsanullah (2015) derived the nth ordinary and incomplete moments of T as

$$\mu_n' = \frac{\delta\Gamma\left(1+n/b\right)}{a^n}\sum_{l=0}^{\infty}\frac{c_l\left(\delta\right)}{\left(l+1\right)^{n/b}} \text{ and } \varphi_n(t) = \frac{\delta\gamma\left(1+n/b,\left(a/t\right)^b\right)}{a^n}\sum_{l=0}^{\infty}\frac{c_l\left(\delta\right)}{\left(l+1\right)^{n/b}},$$

where $c_l\left(\delta\right) = \left(-1\right)^l\delta\left(\delta-1\right)\ldots\left(\delta-l\right)/\left(l+1\right)!$.

- **Moments:** From Section 6.1, the rth ordinary moment of the CGcTW distribution can be expressed (for $r > -b$) as

$$\mu_r' = \frac{\Gamma\left(1+r/b\right)}{a^r}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\left[\left(k+j+1\right)\upsilon_{k,j}\frac{c_l\left(k+j+1\right)}{\left(l+1\right)^{r/b}} - \left(k+j+2\right)\omega_{k,j}\frac{c_l\left(k+j+2\right)}{\left(l+1\right)^{r/b}}\right].$$

- **Incomplete moments:** From Section 6.2, the rth incomplete moment of the CGcTW model is given (for $r > -b$) by

$$m_r(y) = \frac{\gamma\left(1+\frac{r}{b},\left(\frac{a}{y}\right)^b\right)}{a^r}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\left[\left(k+j+1\right)\upsilon_{k,j}\frac{c_l\left(k+j+1\right)}{\left(l+1\right)^{r/b}} - \left(k+j+2\right)\omega_{k,j}\frac{c_l\left(k+j+2\right)}{\left(l+1\right)^{r/b}}\right].$$

- **mgf:** The mgf of the CGcTW model is given by

$$M(t) = \sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\left(-1\right)^l_1\Psi_0\left[\begin{array}{c}\left(1,-\beta^{-1}\right)\\ -\end{array};\left(l+1\right)^{\frac{-1}{\beta}}\frac{t}{\alpha}\right]\left[\upsilon_{k,j}\binom{k+j+1}{l+1} - \omega_{k,j}\binom{k+j+2}{l+1}\right],$$

  where ${}_p\Psi_q(\cdots)$ is the complex parameter Wright generalized hypergeometric function with p numerator and q denominator parameters (Kilbas et al., 2006, Equation (1.9)) defined by the power series

$${}_p\Psi_q\left[\begin{array}{c}\left(\alpha_1,A_1\right),\ldots,\left(\alpha_p,A_p\right)\\ \left(\beta_1,B_1\right),\ldots,\left(\beta_q,B_q\right)\end{array};z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_jn\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_jn\right)}\frac{z^n}{n!}$$

  for $z \in \mathbb{C}$, where $\alpha_j,\ \beta_k \in \mathbb{C}$, $A_j,\ B_k \neq 0$, $j = \overline{1,p}$, $k = \overline{1,q}$ and the series converges for $1+\sum_{j=1}^{q}B_j - \sum_{j=1}^{p}A_j > 0$.
- **PWMs:** From Section 6.5, we have

$$\rho_{s,r} = \frac{\Gamma\left(1+s/b\right)}{a^s}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\left[\left(k+r+j+1\right)a_{k,j}\frac{c_l\left(k+r+j+1\right)}{\left(l+1\right)^{s/b}} - \left(k+r+j+2\right)b_{k,j}\frac{c_l\left(k+r+j+2\right)}{\left(l+1\right)^{s/b}}\right].$$

- **Order statistics:** From Section 6.6, the qth moments of $X_{i:n}$ for the CGcTW distribution can be written as

$$E\left(X_{i:n}^{q}\right) = \frac{\Gamma\left(1+q/b\right)}{a^{q}} \sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\sum_{s=0}^{n-i}\left[\left(k+s+i+j\right)a_{k,s}^{**}\frac{c_{l}\left(k+s+i+j\right)}{\left(l+1\right)^{q/b}} - \left(k+s+i+j+1\right)b_{k,s}^{**}\frac{c_{l}\left(k+s+i+j+1\right)}{\left(l+1\right)^{q/b}}\right].$$

8. Maximum likelihood estimation

In this section, we consider estimation of the unknown parameters of the CGcT-G family from complete samples by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from this family with parameters θ, λ and φ. Let $\xi = (\lambda, \theta, \varphi^{\intercal})^{\intercal}$ be the $p \times 1$ parameter vector. Then, the log-likelihood function for ξ, say $\ell = \ell\left(\xi\right)$, is given by

$$\ell = n\log\theta + \sum_{i=0}^{n}\log g\left(x;\varphi\right) + \sum_{i=0}^{n}\log s_i - 2\sum_{i=0}^{n}\log p_i, \tag{8.1}$$

where $s_i = 1+\lambda-2\lambda G\left(x_i;\varphi\right)$, $p_i = 1+\left(\theta-1\right)G\left(x_i;\varphi\right)\left[1+\lambda\overline{G}\left(x_i;\varphi\right)\right]$ and $\overline{G}\left(x_i;\varphi\right) = 1-G\left(x_i;\varphi\right)$.

The score vector components, $\mathbf{U}\left(\xi\right) = \frac{\partial\ell}{\partial\xi} = \left(\frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\varphi_k}\right)^{\intercal} = \left(U_\theta, U_\lambda, U_{\varphi_k}\right)^{\intercal}$, are given by

$$U_\theta = \frac{n}{\theta} - \sum_{i=0}^{n}\frac{2}{p_i}G\left(x_i;\varphi\right)\left[1+\lambda\overline{G}\left(x_i;\varphi\right)\right],$$

$$U_\lambda = \sum_{i=0}^{n}\frac{1-2G\left(x_i;\varphi\right)}{s_i} - \left(\theta-1\right)\sum_{i=0}^{n}\frac{2}{p_i}G\left(x_i;\varphi\right)\overline{G}\left(x_i;\varphi\right)$$

and

$$U_{\varphi_k} = \sum_{i=0}^{n}\frac{g_k'\left(x_i;\varphi\right)}{g\left(x_i;\varphi\right)} - 2\lambda\sum_{i=0}^{n}\frac{G_k'\left(x_i;\varphi\right)}{s_i} - \left(\theta-1\right)\sum_{i=0}^{n}\frac{2}{p_i}\left\{\lambda G\left(x_i;\varphi\right)\overline{G}_k'\left(x_i;\varphi\right) + G_k'\left(x_i;\varphi\right)\left[1+\lambda\overline{G}\left(x_i;\varphi\right)\right]\right\},$$

where $g_k'\left(x_i;\varphi\right) = \partial g\left(x_i;\varphi\right)/\partial\varphi_k$ and $G_k'\left(x_i;\varphi\right) = \partial G\left(x_i;\varphi\right)/\partial\varphi_k$ for $k = 1, ..., p-2$.

Setting the nonlinear system of equations $U_\theta = U_\lambda = U_{\varphi_k} = \mathbf{0}$ and solving them simultaneously yields the maximum likelihood estimate (MLE) $\widehat{\xi} = (\widehat{\theta}, \widehat{\lambda}, \widehat{\varphi}^{\intercal})^{\intercal}$ of $\xi = (\theta, \lambda, \varphi^{\intercal})^{\intercal}$. These equations cannot be solved analytically and a statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

The MLEs can also be obtained by maximizing (8.1) directly by using `R` (`optim` function), `SAS` (`PROC NLMIXED`), `Ox` program (`MaxBFGS` sub-routine) or a `MATHCAD` program. In Sections 9 and 10, we used the `optim` function in `R`. We maximized (8.1) using a wide range of starting values. The starting values were taken in a fine scale. For the CGcTW distribution, for example, they were taken to correspond to all combinations of $\alpha = 1, 2, \ldots, 10$, $\beta = 1, 2, \ldots, 10$, $\theta = 0.1, 0.2, \ldots, 0.9$ and $\lambda = -0.9, -0.7, \ldots, 0.9$. For the CGcTLi distribution, for example, the starting values were taken to correspond to all combinations of $\alpha = 1, 2, \ldots, 10$, $\theta = 0.1, 0.2, \ldots, 0.9$ and $\lambda = -0.9, -0.7, \ldots, 0.9$. The call to `optim` converged about 98 percent of the time. When the calls to `optim` did converge, the maximum likelihood solution was unique. The unique solution was verified by using the PROC NLMIXED function in SAS. None of the unique solutions corresponded to boundaries of the parameter spaces.

We experimented maximization of (8.1) for a wide range of choices for G that are smooth (smooth in the sense of continuity and differentiability) and for a wide range of starting values. The reported observations held for each choice. That is, optim converged about 98 percent of the time, the maximum likelihood solution was unique when optim did converge and none of the unique solutions corresponded to boundaries of the parameter spaces. Generally, the likelihood surface was smooth whenever G was smooth.

For asymptotic interval estimation of the model parameters, we require the observed information matrix, whose elements are given by

$$U_{\theta\theta} = \tfrac{-n}{\theta^2} + \textstyle\sum_{i=0}^{n} \tfrac{2}{p_i^2} G^2\left(x_i;\varphi\right)\left[1+\lambda\overline{G}\left(x_i;\varphi\right)\right]^2, \quad U_{\theta\lambda} = \textstyle\sum_{i=0}^{n} \tfrac{-2}{p_i^2} G\left(x_i;\varphi\right)\overline{G}\left(x_i;\varphi\right),$$

$$U_{\theta\varphi_k} = \sum_{i=0}^{n} \frac{-2}{p_i^2}\left\{\lambda G\left(x_i;\varphi\right)\overline{G}_k^{\prime}\left(x_i;\varphi\right) + G_k^{\prime}\left(x_i;\varphi\right)\left[1+\lambda\overline{G}\left(x_i;\varphi\right)\right]\right\},$$

$$U_{\lambda\lambda} = -\sum_{i=0}^{n} \frac{\left[1-2G\left(x_i;\varphi\right)\right]^2}{s_i^2} + (\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G^2\left(x_i;\varphi\right)\overline{G}^2\left(x_i;\varphi\right),$$

$$\begin{aligned} U_{\lambda\varphi_k} &= -(\theta-1)\sum_{i=0}^{n}\frac{2}{p_i^2}\left\{G\left(x_i;\varphi\right)\overline{G}_k^{\prime}\left(x_i;\varphi\right)\left[1+(\theta-1)G\left(x_i;\varphi\right)\right]\right. \\ &\quad \left. + G_k^{\prime}\left(x_i;\varphi\right)\overline{G}\left(x_i;\varphi\right)\right\} - \sum_{i=0}^{n}\frac{2}{s_i^2}G_k^{\prime}\left(x_i;\varphi\right) \end{aligned}$$

and

$$\begin{aligned}
U_{\varphi_k\varphi_j} &= \sum_{i=0}^{n}\frac{1}{g^2(x_i;\varphi)}\left[g(x_i;\varphi)\,g''_{kj}(x_i;\varphi)-g'_k(x_i;\varphi)\,g'_j(x_i;\varphi)\right]\\
&-\lambda\sum_{i=0}^{n}\frac{2}{s_i^2}\left[s_iG''_{kj}(x_i;\varphi)+2\lambda G'_k(x_i;\varphi)\,G'_j(x_i;\varphi)\right]\\
&-\lambda(\theta-1)\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,\overline{G}''_{kj}(x_i;\varphi)+\lambda G'_j(x_i;\varphi)\,\overline{G}'_k(x_i;\varphi)\\
&-\lambda(\theta-1)\sum_{i=0}^{n}\frac{2}{p_i^2}G'_k(x_i;\varphi)\,\overline{G}'_j(x_i;\varphi)+G''_{kj}(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&-(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G^2(x_i;\varphi)\,\overline{G}''_{kj}(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&-\lambda(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,G'_j(x_i;\varphi)\,\overline{G}'_k(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&-\lambda(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,G'_k(x_i;\varphi)\,\overline{G}'_j(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&-\lambda(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,G''_{kj}(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]^2\\
&+\lambda^2(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G^2(x_i;\varphi)\,\overline{G}'_k(x_i;\varphi)\,\overline{G}'_j(x_i;\varphi)\\
&+\lambda(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,G'_j(x_i;\varphi)\,\overline{G}'_k(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&+\lambda(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G(x_i;\varphi)\,G'_k(x_i;\varphi)\,\overline{G}'_j(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]\\
&+(\theta-1)^2\sum_{i=0}^{n}\frac{2}{p_i^2}G'_k(x_i;\varphi)\,G'_j(x_i;\varphi)\left[1+\lambda\overline{G}(x_i;\varphi)\right]^2,
\end{aligned}$$

where $k,j=1,...,p-2$, $g'_j(x_i;\varphi)=\partial g(x_i;\varphi)/\partial\varphi_j$, $g''_{kj}(x_i;\varphi)=\partial^2 g(x_i;\varphi)/\partial\varphi_k\partial\varphi_j$, $G'_j(x_i;\varphi)=\partial G(x_i;\varphi)/\partial\varphi_j$ and $G''_{kj}(x_i;\varphi)=\partial^2 G(x_i;\varphi)/\partial\varphi_k\partial\varphi_j$.

Under standard regularity conditions when $n\to\infty$, the distribution of $\widehat{\xi}$ can be approximated by a multivariate normal $N_p(0,J(\widehat{\xi})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\xi})$ is the total observed information matrix evaluated at $\widehat{\xi}$. The method of re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Interval estimates may also be obtained using the bootstrap percentile method. Likelihood ratio tests can be performed for the proposed family of distributions in the usual way.

9. Applications

We provide two applications to real data to illustrate the flexibility of the CGcTW and CGcTLi models introduced in Section 5. We determine the MLEs of the model parameters and their standard errors. The goodness-of-fit statistics for these models are compared with other competitive models. In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion

(AIC), consistent Akaike information criterion ($CAIC$), Hannan-Quinn information criterion ($HQIC$), Bayesian information criterion (BIC), $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood, Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics. These statistics are widely used to determine how closely a specific cdf fits the empirical distribution of a given data set. Generally, the smaller these statistics are, the better the fit.

9.1. The data sets. The first data set refers to nicotine measurements made from several brands of cigarettes in 1998, collected by the Federal Trade Commission, which is an independent agency of the US government, whose main mission is the promotion of consumer protection. The report entitled tar, nicotine, and carbon monoxide of the smoke of 1,206 varieties of domestic cigarettes for the year of 1998 consists of the data sets and some information about the source of the data, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. The free form data set can be found at *http://pw1.netcom.com/rda vis2/smoke.html*. These data have been used by Afify et al. (2016c) to fit the Marshall-Olkin additive Weibull distribution. The second data set corresponds to uncensored observations on the breaking stress of carbon fibres (in Gba) as reported in Cordeiro et al. (2014).

9.2. The fitted models. We shall compare the fits of the CGcTW and CGcTLi distributions with those of other competitive models to both data sets.

Tables 1 and 3 provide the values of $-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^* and A^* for models fitted to both data sets. The MLEs and their corresponding standard errors (in parentheses) for the fitted models are reported in Tables 2 and 4. These numerical results are obtained using the `Mathcad` program.

In Table 1, we compare the fits of the CGcTW model with the Kw-TEMW, TEMW, TEWG, TAW, Kw-MW, BW, Kw-W and AW models to the nicotine data. We note that the CGcTW distribution has the lowest values for all goodness-of-fit statistics among all fitted models. So, the CGcTW model could be chosen as the best model to explain the nicotine data.

In Table 3, we compare the fits of the CGcTLi model with the Kw-Li, WLi, WG, BLi, OLL-Li, BW and GT-Li models to the carbon fibres data. The figures in this table reveal that the CGcTLi model has the lowest values for $-2\widehat{\ell}$ AIC, $CAIC$, $HQIC$, BIC, W^* and A^* statistics among all fitted models to these data. Then, the CGcTLi model can be chosen as the best model.

It is quite clear from the figures in Tables 1 and 3 that the CGcTW and CGcTLi distributions provide the best fits to both data sets. So, these new distributions can be better models than other competitive distributions. The plots of the fitted CGcTW and CGcTLi pdfs and other fitted pdfs discussed before are displayed in Figures 3 and 4. These plots also reveal that the CGcTW and CGcTLi distributions provide the best fits to both data sets. Figures 5 and 6 display the fitted cdfs and the QQ plots for both the CGcTW and CGcTLi models. It is evident from these plots that the two models provide closer fits to the two data sets.

Table 1. The statistics$-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^{*} and A^{*} for the nicotine data.

Model	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^{*}	A^{*}
CGcTW	212.777	220.777	220.894	226.903	236.162	0.34713	1.92288
Kw-TEMW	215.674	229.674	230.005	240.396	256.599	0.37863	2.08814
TEMW	215.967	225.967	226.143	233.625	245.199	0.38319	2.14169
TEWG	216.832	226.832	227.009	234.491	246.064	0.4391	2.38503
TAW	217.393	227.393	227.569	235.051	246.625	0.37208	2.08766
Kw-MW	221.938	231.938	232.114	239.596	251.17	0.43426	2.52687
BW	225.173	233.173	233.29	239.3	248.559	0.49664	2.89774
Kw-W	226.184	234.184	234.302	240.311	249.57	0.5325	3.08454
AW	226.581	234.581	234.698	240.707	249.966	0.55222	3.17512

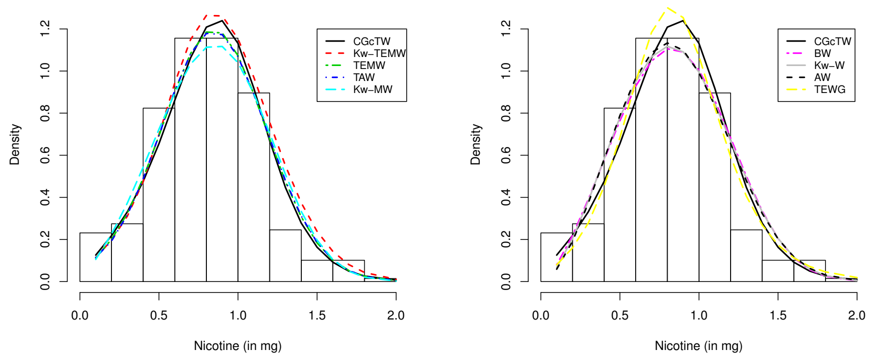

Figure 3. Fitted pdfs of the CGcTW distribution and other models for the nicotine data.

Table 2. MLEs and their standard errors (in parentheses) for the nicotine data.

Model	Estimates			
CGcTW	$\widehat{\alpha}$= 2.0778 (0.635)	$\widehat{\beta}$= 1.5694 (0.309)	$\widehat{\theta}$= 0.1218 (0.124)	$\widehat{\lambda}$= −0.2907 (0.529)
BW	$\widehat{\alpha}$= 0.6686 (0.578)	$\widehat{\beta}$= 3.1645 (0.426)	$\widehat{a}$= 0.7784 (0.163)	$\widehat{b}$= 3.0922 (8.174)
Kw-W	$\widehat{\alpha}$= 0.6157 (0.392)	$\widehat{\beta}$= 3.1187 (0.698)	$\widehat{a}$= 0.8395 (0.233)	$\widehat{b}$= 3.7931 (6.921)
AW	$\widehat{\alpha}$= 1.135 (0.062)	$\widehat{\beta}$= 0.3084 (0.1)	$\widehat{\gamma}$= 0.0002 (0.001369)	$\widehat{\theta}$= 2.7219 (0.114)
Kw-TEMW	$\widehat{\alpha}$= 0.113 (0.22)	$\widehat{\beta}$= 2.316 (0.62)	$\widehat{\gamma}$= 1.436 (1.71)	$\widehat{\alpha}$= 2.033 (1.145)
	$\widehat{\lambda}$= −0.902 (0.197)	$\widehat{a}$= 0.47 (0.213)	$\widehat{b}$= 1.079 (1.828)	
TEMW	$\widehat{\alpha}$= 0.6977 (0.492)	$\widehat{\beta}$= 2.5908 (0.265)	$\widehat{\gamma}$= 1.1925 (0.259)	$\widehat{\alpha}$= 1.5007 (0.487)
	$\widehat{\lambda}$= −0.6328 (0.228)			
TEWG	$\widehat{\alpha}$= 9.5829 (6.182)	$\widehat{\beta}$= 0.8057 (0.155)	$\widehat{\theta}$= 3.1388 (1.464)	$\widehat{p}$= 0.9841 (0.011)
	$\widehat{\lambda}$= −0.0876 (0.449)			
TAW	$\widehat{\alpha}$= 1.2252 (0.239)	$\widehat{\beta}$= 0.8994 (0.091)	$\widehat{\gamma}$= 0.433 (0.229)	$\widehat{\theta}$= 2.6404 (0.267)
	$\widehat{\lambda}$= −0.8831 (0.147)			
Kw-MW	$\widehat{\alpha}$= 0.6145 (0.09)	$\widehat{\beta}$= 0.4466 (0.364)	$\widehat{\gamma}$= 0.5622 (0.353)	$\widehat{a}$= 4.3285 (3.595)
	$\widehat{b}$= 6.7039 (6.728)			

Table 3. The statistics$-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^* and $A^*$for the carbon fibres data.

Model	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
CGcTLi	169.976	175.976	176.364	178.572	182.545	0.051	0.298
Kw-Li	171.419	177.419	177.807	180.015	183.988	0.079	0.469
WLi	171.570	177.570	177.957	180.166	184.139	0.079	0.483
WG	171.957	179.957	180.613	183.418	188.716	0.083	0.517
BLi	175.019	181.019	181.406	183.615	187.588	0.144	0.768
OLL-Li	175.993	179.993	180.183	181.723	184.372	0.160	0.845
BW	184.150	192.150	192.806	195.611	200.909	0.275	1.505
GT-Li	187.590	195.590	196.249	199.054	204.352	0.308	1.684

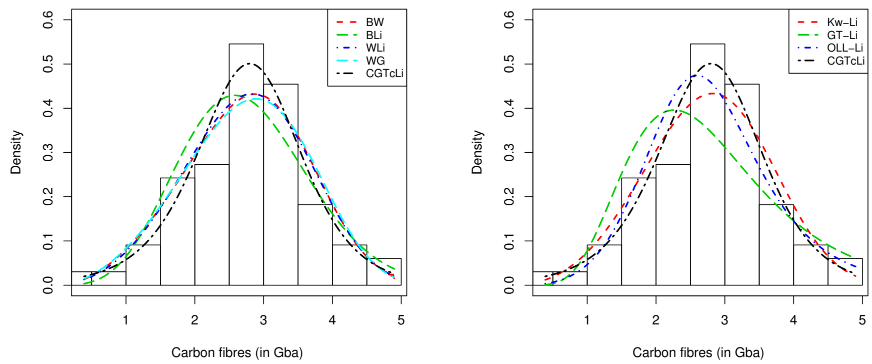

Figure 4. Fitted pdfs of the CGcTLi distribution and other models for the carbon fibres data.

Table 4. MLEs and their standard errors (in parentheses) for the carbon fibres data.

Model	Estimates			
OLL-Li	$\widehat{a}$= 2.963 (0.313)	$\widehat{\alpha}$= 0.488 (0.017)		
CGcTLi	$\widehat{\theta}$= 0.0075 (0.002)	$\widehat{\lambda}$= 0.999 ($8.840 \cdot 10^{-10}$)	$\widehat{\alpha}$= 1.211 (0.052)	
Kw-Li	$\widehat{a}$= 2.320 (0.283)	$\widehat{b}$= 1.031 (3.010)	$\widehat{\alpha}$= 0.052 (0.043)	
WLi	$\widehat{a}$= 20.266 (44.463)	$\widehat{b}$= 2.278 (0.224)	$\widehat{\alpha}$= 0.219 (0.109)	
BLi	$\widehat{a}$= 3.648 (0.631)	$\widehat{b}$= $2.687 \cdot 10^4$ (8.673)	$\widehat{\alpha}$= 0.004 (0.0006)	
WG	$\widehat{a}$= 1.265 (3.376)	$\widehat{b}$= 2.149 (3.461)	$\widehat{\alpha}$= 1.315 (3.472)	$\widehat{\beta}$= 0.3021 (0.994)
BW	$\widehat{a}$= 3.124 (1.103)	$\widehat{b}$= 0.102 (0.0131)	$\widehat{\alpha}$= 2.334 (0.002)	$\widehat{\beta}$= 1.051 (0.002)
GT-Li	$\widehat{a}$= 7.046 (1.796)	$\widehat{\lambda}$= 0.0002 (0.118)	$\widehat{b}$= 3.339 (0.207)	$\widehat{\alpha}$= 1.246 (0.109)

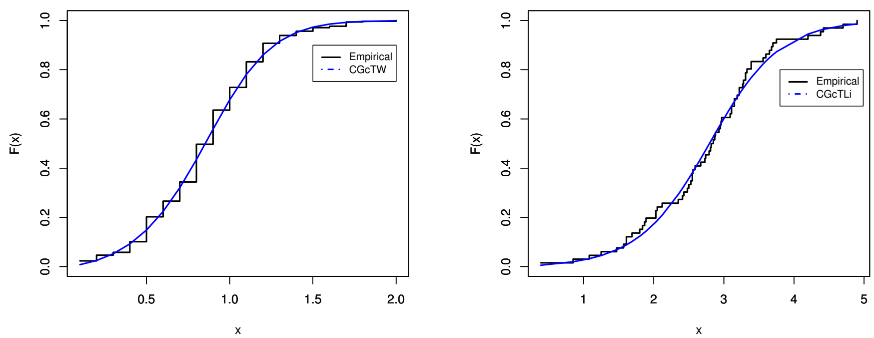

Figure 5. Fitted cdfs of the CGcTW and CGcTLi distributions.

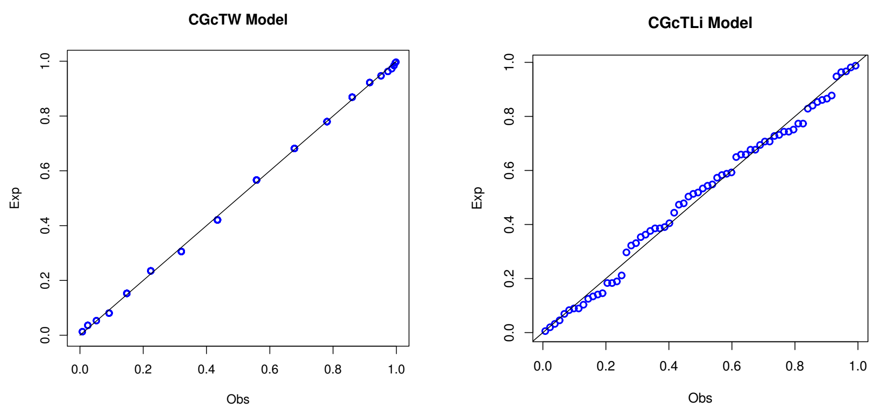

Figure 6. QQ plots of the CGcTW and CGcTLi distributions.

10. Simulation Study

In this section, we evaluate the performance of the maximum likelihood method for estimating the CGcTW and CGcTLi parameters using a Monte Carlo simulation study with $1,000$ replications. We calculate the mean square errors (MSEs) of the parameter estimates, estimated average lengths (ALs) and coverage probabilities (CPs) using the R software.

We generate $N = 1,000$ samples of sizes $n = 50,\ 55, \ldots, 1000$ from the CGcTW distribution with $\theta = \lambda = \alpha = \beta = 0.5$. The numerical results for the above measures are shown in the plots of Figures 7. It is noted, from these plots, that the estimated biases decrease when the sample size n increases. Further, the estimated MSEs decay toward zero as n increases. This fact reveals the consistency property of the MLEs. The CPs are near to 0.95 and approach to the nominal value when the sample size increases. Moreover, if the sample size increases, the ALs decrease in each case.

For the CGcTLi distribution, we consider the following combinations: I: $\theta = 0.3, \alpha = 0.5, \lambda = 0.5$; II: $\theta = 0.2, \alpha = 1, \lambda = 1$; III: $\theta = 0.7, \alpha = 2.5, \lambda = 1.5$; IV: $\theta = 0.9, \alpha = 3, \lambda = 2$.

Let $(\widehat{\theta}, \widehat{\alpha}, \widehat{\lambda})$ be the MLEs of the CGcTLi parameters and $(s_{\widehat{\theta}}, s_{\widehat{\alpha}}, s_{\widehat{\lambda}})$ be the standard errors of the MLEs. The MSEs, ALs and CPs can be estimated by the following equations:

$$MSE_{\varepsilon}(n) = \frac{1}{N}\sum_{i=0}^{N}\left(\widehat{\varepsilon}_i - \varepsilon\right), \qquad AL_{\varepsilon}(n) = \frac{3.919928}{N}\sum_{i=0}^{N} s_{\widehat{\varepsilon}_i}$$

and

$$CP_{\varepsilon}(n) = \frac{1}{N}\sum_{i=0}^{N} I\left(\widehat{\varepsilon}_i - 1.9599 s_{\widehat{\varepsilon}_i}, \widehat{\varepsilon}_i + 1.9599 s_{\widehat{\varepsilon}_i}\right)$$

The empirical results are given in Table 5. The figures in this table indicate that the estimates are quite stable and, more importantly, are close to the true parameter values for these sample sizes.

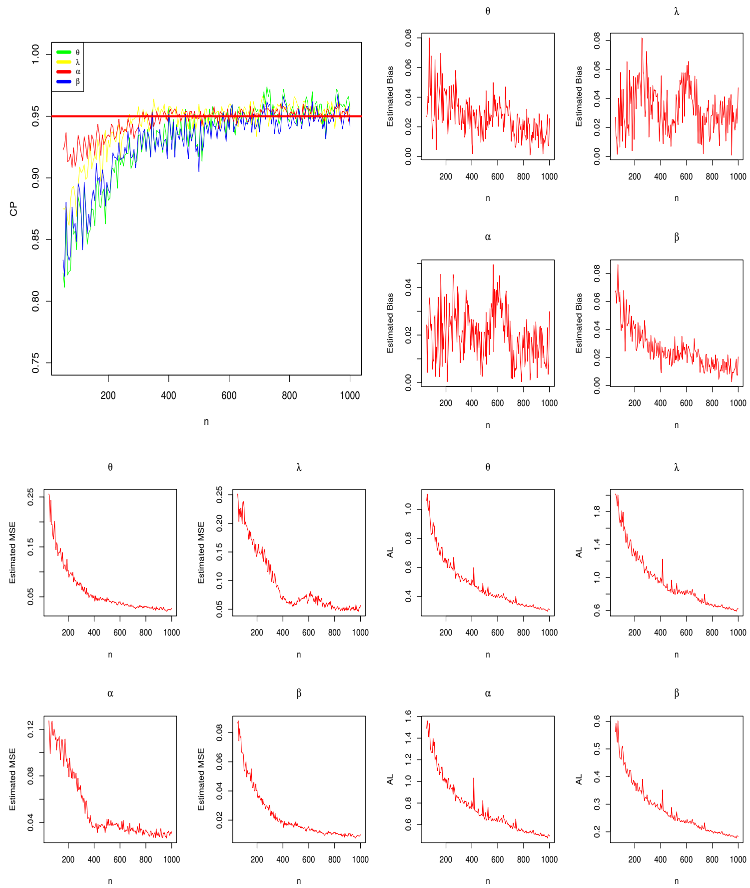

Figure 7. Estimated CPs, biases, MSEs and ALs of the selected parameter vector.

Table 5: Means, MSEs, ALs and
CPs of the estimates under the maximum likelihood method for the CGcTLi distribution.

	n	Mean			MSE			AL			CP		
		θ	α	λ	θ	α	λ	θ	α	λ	θ	α	λ
I	50	0.403	0.514	0.699	0.282	0.013	0.537	1.257	0.346	1.479	0.753	0.963	0.857
	100	0.402	0.497	0.648	0.153	0.006	0.127	0.975	0.257	1.305	0.864	0.887	0.863
	250	0.341	0.502	0.594	0.038	0.005	0.064	0.645	0.193	0.747	0.988	0.950	0.922
	500	0.328	0.500	0.514	0.015	0.002	0.010	0.438	0.109	0.683	0.940	0.934	0.938
	1000	0.298	0.500	0.512	0.005	0.0008	0.005	0.316	0.084	0.346	0.922	0.931	0.951
II	50	0.726	0.694	1.764	0.563	0.003	1.253	1.874	0.246	1.463	0.938	0.948	0.849
	100	0.574	0.804	1.821	0.113	0.001	0.984	1.146	0.143	0.843	0.940	0.953	0.876
	250	0.540	0.872	1.582	0.044	0.0007	0.126	0.734	0.112	0.479	0.941	0.951	0.907
	500	0.518	0.986	1.233	0.019	0.0003	0.039	0.498	0.089	0.246	0.943	0.951	0.931
	1000	0.504	0.999	1.041	0.009	0.0001	0.108	0.345	0.041	0.069	0.943	0.952	0.948
III	50	0.642	2.709	1.996	0.031	0.395	0.943	0.646	2.186	1.250	0.877	0.934	0.862
	100	0.658	2.553	1.865	0.015	0.135	0.741	0.378	1.452	1.114	0.941	0.964	0.905
	250	0.689	2.537	1.684	0.003	0.054	0.493	0.213	0.873	0.951	0.934	0.969	0.934
	500	0.692	2.516	1.602	0.001	0.028	0.342	0.145	0.642	0.776	0.936	0.955	0.945
	1000	0.699	2.504	1.523	0.0006	0.012	0.204	0.065	0.463	0.542	0.947	0.948	0.946
IV	50	0.837	3.184	2.909	0.025	0.199	0.952	0.675	1.507	1.466	0.946	0.963	0.895
	100	0.874	3.036	2.883	0.012	0.115	0.760	0.383	1.301	1.377	0.936	0.957	0.919
	250	0.889	3.022	2.512	0.004	0.0461	0.355	0.224	0.756	0.950	0.932	0.941	0.942
	500	0.895	2.997	2.305	0.001	0.018	0.196	0.137	0.662	0.755	0.953	0.936	0.945
	1000	0.898	3.011	2.085	0.0007	0.011	0.097	0.076	0.358	0.362	0.969	0.922	0.952

The figures in Table 5 indicate that the MSEs decrease when n increases. The simulation study also reveals that the maximum likelihood method is appropriate for estimating the CGcTLi parameters. In fact, the MSEs of the parameters tend to be closer to zero when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can often be improved by using bias adjustments to these estimators. The CPs are near to 0.95. When the sample size increases, the CPs approach to the nominal value. The ALs decrease for all cases.

11. Conclusions

The idea of generating new extended models from classic ones has been of great interest among researchers in the past decade. We have proposed a new *complementary geometric transmuted-G* (CGcT-G) family of distributions, which extends the transmuted class (Shaw and Buckley, 2007) by adding two extra shape parameters. Many well-known distributions emerge as special cases of the proposed family by using special parameter values. We have provided some mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, quantile and generating functions, Rényi and Shannon entropies, order statistics and probability weighted moments. The maximum likelihood estimation of the model parameters has been investigated and the observed information matrix has been determined. By means of two real data sets, we have verified that special cases of the CGcT-G family can provide better fits than other models generated from well-known families.

Appendix A: Important background

Let $r(t)$ be the pdf of a random variable $T \in [a,b]$ for $-\infty < a < b < \infty$ and let $W[H(x)]$ be a function of the cdf of a random variable X such that $W[H(x)]$ satisfies the following conditions:

$$\begin{cases} (i) & W[H(x)] \in [a,b], \\ (ii) & W[H(x)] \text{ is differentiable and monotonically non-decreasing, and} \\ (iii) & W[H(x)] \to a \quad \text{as} \quad x \to -\infty \text{ and } W[H(x)] \to b \quad \text{as} \quad x \to \infty. \end{cases}$$

Recently, Alzaatreh et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_a^{W[H(x)]} r(t)\ dt, \tag{11.1}$$

where $W[H(x)]$ satisfies the above conditions. The pdf corresponding to (11.1) is given by

$$f(x) = \left\{ \frac{d}{dx}\ W[H(x)] \right\}\ r\{\ W[H(x)]\}. \tag{11.2}$$

For the complementary exponential-geometric (CEGc) distribution introduced by Louzada-Neto et al. (2011), the pdf and cdf are, respectively, given by

$$f(x) = \frac{\alpha\theta\ \exp(-\alpha x)}{[\theta + (1-\theta)\exp(-\alpha x)]^2}$$

and

$$F(x) = \frac{\theta[1 - \exp(-\alpha x)]}{\theta + (1-\theta)\exp(-\alpha x)},$$

where $\alpha > 0$ is the scale parameter and $0 < \theta < 1$ is the shape parameter.

For $W[H(x)] = -\log[1 - H(x;\boldsymbol{\varphi})]$ and $r(t)$ the pdf of the CEGc distribution with $\alpha = 1$, we define the cdf of the new *complementary geometric-H* (CGc-H) family of distributions by

$$\begin{aligned} F(x;\theta,\boldsymbol{\varphi}) &= \int_0^{-\log[1-H(x;\boldsymbol{\varphi})]} \frac{\theta\ \exp(-t)}{[\theta + (1-\theta)\exp(-t)]^2}\ dt \\ &= \frac{\theta H(x;\boldsymbol{\varphi})}{1 - (1-\theta) H(x;\boldsymbol{\varphi})},\ \theta \in (0,1), \end{aligned} \tag{11.3}$$

where $H(x;\boldsymbol{\varphi})$ is the baseline cdf depending on a parameter vector $\boldsymbol{\varphi}$ and $\theta \in (0,1)$ is an additional shape parameter. The pdf corresponding to (11.3) becomes

$$f(x;\theta,\boldsymbol{\varphi}) = \frac{\theta h(x;\boldsymbol{\varphi})}{[1 - (1-\theta) H(x;\boldsymbol{\varphi})]^2}.$$

Appendix B: Existing literature

The pdfs of the competitive distributions used in the application section are given below:

- The Kw-TEMW pdf given by

$$\begin{aligned} f(x) &= ab\alpha \mathrm{e}^{-\alpha x - \gamma x^\beta} \left(\alpha + \gamma\beta x^{\beta-1}\right) \left[1 + \lambda - 2\lambda \left(1 - \mathrm{e}^{-\alpha x - \gamma x^\beta}\right)^\alpha\right] \\ &\times \left(1 - \mathrm{e}^{-\alpha x - \gamma x^\beta}\right)^{a\alpha - 1} \left[1 + \lambda - \lambda\left(1 - \mathrm{e}^{-\alpha x - \gamma x^\beta}\right)^\alpha\right]^{a-1} \\ &\times \left\{1 - \frac{\left[1+\lambda-\lambda\left(1-\mathrm{e}^{-\alpha x-\gamma x^\beta}\right)^\alpha\right]^a}{\left(1-\mathrm{e}^{-\alpha x - \gamma x^\beta}\right)^{-a\alpha}}\right\}^{b-1}. \end{aligned}$$

- The TEMW pdf given by

$$f\left(x\right)=\alpha\left(\alpha+\gamma\beta x^{\beta-1}\right)\mathrm{e}^{-\alpha x-\gamma x^{\beta}}\left(1-\mathrm{e}^{-\alpha x-\gamma x^{\beta}}\right)^{\alpha-1}$$
$$\times\left[1+\lambda-2\lambda\left(1-\mathrm{e}^{-\alpha x-\gamma x^{\beta}}\right)^{\alpha}\right].$$

- The TEWG pdf given by

$$f\left(x\right)=\frac{\theta(1-p)\beta\alpha^{\beta}x^{\beta-1}\ \exp\mathrm{e}^{-(\alpha x)^{\beta}}}{\left\{1-p\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{\theta}\right\}^{2}}\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{\theta-1}\left\{1+\lambda-2\lambda\frac{(1-p)\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{\theta}}{1-p\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{\theta}}\right\}.$$

- he TAW pdf given by

$$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\mathrm{e}^{-\alpha x^{\theta}-\gamma x^{\beta}}\left[1-\lambda+2\lambda\mathrm{e}^{-\alpha x^{\theta}-\gamma x^{\beta}}\right].$$

- The Kw-MW pdf given by

$$f\left(x\right)=ab\gamma\left(\beta+\alpha x\right)x^{\beta-1}\mathrm{e}^{\alpha x-\gamma x^{\beta}\mathrm{e}^{\alpha x}}\left(1-\mathrm{e}^{-\gamma x^{\beta}\mathrm{e}^{\alpha x}}\right)^{a-1}$$
$$\times\left[1-\left(1-\mathrm{e}^{-\gamma x^{\beta}\mathrm{e}^{\alpha x}}\right)^{a}\right]^{b-1}.$$

- The BW pdf given by

$$f\left(x\right)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}\mathrm{e}^{-b(\alpha x)^{\beta}}\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}.$$

- The Kw-W pdf given by

$$f\left(x\right)=ab\beta\alpha^{\beta}x^{\beta-1}\ \mathrm{e}^{-(\alpha x)^{\beta}}\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a}\right\}^{b-1}.$$

- The AW pdf given by

$$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\mathrm{e}^{-\alpha x^{\theta}-\gamma x^{\beta}}.$$

- The WLi pdf given by

$$f(x)=\frac{ab\alpha^{2}}{1+\alpha}(1+x)\mathrm{e}^{-\alpha x}\frac{\left(1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{b-1}}{\left(\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{b+1}}\mathrm{e}^{-a\left[\frac{1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}}{\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}}\right]^{b}}.$$

- The WG pdf given by

$$f(x)=\frac{ab\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-x/\beta}\frac{[\gamma(\alpha,x/\beta)/\Gamma(\alpha)]^{b-1}}{[1-\gamma(\alpha,x/\beta)/\Gamma(\alpha)]^{b+1}}\mathrm{e}^{-a\left[\frac{\gamma(\alpha,x/\beta)/\Gamma(\alpha)}{1-\gamma(\alpha,x/\beta)/\Gamma(\alpha)}\right]^{b}}.$$

- The OLL-Li pdf given by

$$f(x)=\frac{a\alpha^{2}}{1+\alpha}(1+x)\mathrm{e}^{-\alpha x}\frac{\left(1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{a-1}\left(\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{a-1}}{\left[\left(1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{a}+\left(\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right)^{a}\right]^{2}}.$$

- The GT-Li pdf given by

$$f(x)=\frac{\alpha^{2}\mathrm{e}^{-\alpha x}}{1+\alpha}(1+x)\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right]^{a-1}$$
$$\times\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right]^{b}\right\}.$$

- The Kw-Li pdf given by

$$f\left(x\right)=\frac{ab\alpha^{2}(1+x)}{(1+\alpha)}\mathrm{e}^{-\alpha x}\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right]^{a-1}\left\{1-\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right]^{a}\right\}^{b-1}.$$

- The BLi pdf given by

$$f\left(x\right)=\frac{\alpha^{2}(1+x)}{B(a,b)(1+\alpha)}\mathrm{e}^{-\alpha x}\left\{\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right\}^{b-1}\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\right]^{a-1}.$$

The parameters of the pdfs above are all positive real numbers except for the parameters λ and p, where $|\lambda|\leq 1$ and $p\in[0,1)$.

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