Kamenev-type oscillation criteria for second order matrix differential systems with damping

Yasemin Başçı*

Abstract

By using the positive linear functional, including the generalized averaging technique, some new Kamenev-type oscillation criteria are established for the second order matrix differential system

$$(r(t)P(t)\psi(X(t))K(X'(t)))' + p(t)R(t)\psi(X(t))K(X'(t))$$
$$+Q(t)F(X'(t))G(X(t)) = 0.$$

The results improve and generalize those given in some previous papers.

This paper is dedicated to the memory of Professor Aydın Tiryaki (June 1956-May 2016).

1. Introduction

Consider the second order matrix differential system of the form

(1.1) $(r(t)P(t)\psi(X(t))K(X'(t)))' + p(t)R(t)\psi(X(t))K(X'(t))$

$$+Q(t)F(X'(t))G(X(t)) = 0, t \geq t_0,$$

where $t_0 \geq 0$ and r, p, P, ψ, K, R, Q and G satisfy the following conditions:

(1) $r \in C^1([t_0, \infty), (0, \infty))$, $p \in C([t_0, \infty), (-\infty, \infty))$;

_____

*Department of Mathematics, Faculty of Arts and Sciences, Abant Izzet Baysal University, 14280 Gölköy, Bolu, Turkey, Email: `basci_y@ibu.edu.tr`

(2) $P(t) = P^T(t) > 0$, $Q(t) \geq 0$, $R(t) = R^T(t) > 0$ for $t \geq t_0$, P, Q and R are $n \times n$ matrices real valued continuous functions on the interval $[t_0, \infty)$, and $P(t)$ and $R(t)$ are commutative. By A^T we mean the transpose of the matrix A;

(3) ψ, K, G, $F \in C^1(\mathbb{R}^{n^2}, \mathbb{R}^{n^2})$, and $\psi^{-1}(X(t))$, $K^{-1}(X'(t))$ and $G^{-1}(X(t))$ exist and $F(X') \geq 0$ for all real matrix $X \neq 0$.

We now denote by M the linear space of $n \times n$ real matrices, $I_n \in M$ the identity matrix and S the subspace of all symmetric matrices in M. For any A, B, $C \in S$, we write $A \geq B$ to mean that $A - B \geq 0$, that is, $A - B$ is positive semi-definite, and $A > B$ to mean that $A - B > 0$, that is, $A - B$ is positive definite. Note that $A \pm B$, and A' are also symmetric matrices, where $'$ denotes the first derivative. We will use some properties of this ordering, that is, $A \geq B$ implies that $C^T A C \geq C^T B C$.

We call a matrix function solution $X(t) \in C^2([t_0, \infty), \mathbb{R}^{n^2})$ of (1.1) is prepared nontrivial if $det X(t) \neq 0$ for at least one $t \in [t_0, \infty)$ and $X(t)$ satisfies the equation

$$(1.2) \qquad G^T(X(t))P(t)\psi(X(t))K(X'(t)) - (K(X'(t)))^T \psi^T(X(t))P(t)G(X(t)) \equiv 0,$$

$$(1.3) \qquad G^T(X(t))R(t)\psi(X(t))K(X'(t)) - (K(X'(t)))^T \psi^T(X(t))R(t)G(X(t)) \equiv 0$$

and

$$(1.4) \qquad \psi^T(X(t))G'(X(t))X'(t)K^{-1}(X'(t))$$
$$-(K^T(X'(t)))^{-1}(X'(t))^T(G'(X(t)))^T\psi(X(t)) \equiv 0, \quad t \geq t_0.$$

A prepared solution $X(t)$ of (1.1) is called oscillatory if $det X(t)$ has arbitrarily large zeros; otherwise, it is called nonoscillatory.

For $n = 1$, oscillatory and nonoscillatory behavior of solutions for various classes of second-order differential equations have been widely discussed in the literature (see, for example, [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 25, 23, 26, 27, 28, 33, 35, 36, 41] and references quoted there in).

The oscillatory properties for (1.1) with its special case

$$(1.5) \qquad (X'(t))' + Q(t)X(t) = 0, t \geq t_0 > 0,$$

and

$$(1.6) \qquad (P(t)X'(t))' + Q(t)X(t) = 0, t \geq t_0 > 0,$$

are important in the mechanical systems associated with (1.1). Therefore, such properties have been studied quite extensively. Some criteria for oscillation of systems (1.5) and (1.6) have established (see [3, 4, 17, 18, 31, 32, 42, 43])

Oscillation results based on Kamenev-type criterion for Eq.(1.6) can also be found in earlier papers of Etgen and Pawlowski [5], Erbe et al. [4], Meng et al. [17]. On the other hand, Wang et al. [32] and Wang [30] also studied for Eq.(1.6). Motivated by ideas of Philos [19], Kong [12] and Wang [29], Wang [30] used the functions of the form $H(t, s)k(s)$ instead of $H(t, s)$. By using the generalized Riccati technique and the averaging technique, he established several new interval criteria for oscillation of system (1.6). In 2003, Wang et al. [32] obtained new Kamenev-type oscillation criteria by using the generalized Riccati technique and the averaging technique for Eq.(1.6).

In 2005, Yang and Cheng [38] studied for the linear matrix differential system with damped

$$(1.7) \qquad (P(t)X'(t))' + r(t)P(t)X'(t) + Q(t)X(t) = 0.$$

They obtained Kamenev-type oscillation criteria for Eq.(1.7).

In 2006, Sun and Meng [24] using a positive linear functional established some oscillation criteria of Kamenev-type

$$(1.8) \qquad (P(t)X'(t))' + R(t)X'(t) + Q(t)X(t) = 0.$$

In 2002, Yang [40] extended some results of Li and Agarwal [15] to the nonlinear matrix differential system

$$(1.9) \qquad (r(t)X'(t))' + p(t)X'(t) + Q(t)F(X'(t))G(X(t)) = 0, t \ge t_0 \ge 0.$$

Also, in 2006, motivated by the work of Wong [34], Yancong and Fanwei [37] studied for (1.9). They obtained some results different from those of Yang [40] for (1.9).

In 2003, Yang and Tang [39] obtained new oscillation criteria for the nonlinear matrix differential system

$$(1.10) \qquad (r(t)P(t)X'(t))' + p(t)P(t)X'(t) + Q(t)F(X'(t))G(X(t)) = 0, t \ge t_0 \ge 0.$$

In this paper, the authors improved the theorems of Yang [40] and generalized the results of Li and Agarwal [15] and Rogovchenko [20].

Motivated by the idea of Li and Agarwal [15], Yang and Tang [39] and Yang [40], in this paper we establish the oscillation theorems of Kamenev-type by using the generalized averaging technique and positive linear functionals. Our results make use of the oscillatory properties of the damping term and some of the them extend and generalized the main results given in [15, 20, 38, 40].

The rest of paper is arranged as follows. In Section 2, several definitions and a lemma are introduced. Motivated by [15, 39, 40], several Kamenev-type oscillation criteria for (1.1) are established in Section 3. Finally, in Section 4 several examples that dwell upon the sharpness of our results are presented.

2. Definitions and lemma

2.1. Definition. Denote by M the linear space of $n \times n$ real matrices, by $I_n \in M$ the identity matrix and S the subspace of all symmetric matrices in M. A linear functional L on M is said to be "positive" if $L(A) > 0$ for any $A \in S$ and $A > 0$.

2.2. Definition. Let $D = \{(t,s) : t \ge s \ge t_0\}$, $D_0 = \{(t,s) : t > s > t_0\}$, $\rho \in C^1(D, \mathbb{R})$ and $k \in C^1([t_0, \infty), (0, \infty))$. We say that a pair of real-valued functions (ρ, k) belongs to a function class $\mathfrak{H}$, if there exist functions $h_1, h_2 \in C^1(D_0, \mathbb{R})$ satisfying the following conditions:

(H1) $\rho(t,t) = 0$ for $t \ge t_0$; $\rho(t,s) > 0$ on D_0;
(H2) $\frac{\partial}{\partial t}(\rho(t,s)k(t)) = h_1(t,s)\sqrt{\rho(t,s)k(t)}$, $\forall (t,s) \in D_0$;
(H3) $\frac{\partial}{\partial s}(\rho(t,s)k(s)) = -h_2(t,s)\sqrt{\rho(t,s)k(s)}$, $\forall (t,s) \in D_0$.

2.3. Lemma. *Let $X(t)$ be a nontrivial prepared solution of (1.1) and $\det X(t) \ne 0$ for $t_0 \ge 0$. Then for all $a \in C^1((t_0, \infty), (0, \infty))$ the matrix function*

$$(2.1) \qquad W(t) = a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t))$$

satisfies the equation

$$(2.2) \qquad W'(t) = \frac{a'(t)}{a(t)}W(t) - \frac{p(t)}{r(t)}R(t)P^{-1}(t)W(t) - a(t)Q(t)F(X'(t))$$
$$- \frac{W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t)}{a(t)r(t)}.$$

Proof. From (1.1), we obtain

$$W'(t) = a'(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t))$$
$$+a(t)(r(t)P(t)\psi(X(t))K(X'(t)))'G^{-1}(X(t))$$
$$+a(t)r(t)P(t)\psi(X(t))K(X'(t))(G^{-1}(X(t)))'$$
$$= \frac{a'(t)}{a(t)}W(t) - a(t)p(t)R(t)\psi(X(t))K(X'(t))G^{-1}(X(t)) - a(t)Q(t)F(X'(t))$$
$$-a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t))G'(X(t))X'(t)G^{-1}(X(t))$$
$$= \frac{a'(t)}{a(t)}W(t) - \frac{p(t)}{r(t)}R(t)P^{-1}(t)W(t) - a(t)Q(t)F(X'(t))$$
$$-\frac{W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t)}{a(t)r(t)}.$$

$\square$

3. Main results

In this section, by using the generalized averaging technique, we establish the Kamenev-type oscillation criterion for the system of (1.1). Before we state our next theorems, we need two lemmas.

3.1. Lemma. *Let $X(t)$ be a prepared solution of (1.1) such that $\det X(t) \neq 0$ on $[c,b) \subset [t_0, \infty)$. Assume that all conditions stated in Section 1 are satisfied and suppose also that for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0$$

and $P(t)$ and $R(t)$ are commutative with

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$$

for $t \geq t_0$. Moreover, for $a \in C^1([t_0, \infty), (0, \infty))$ let

$$W(t) = a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t)), t \in [c,b).$$

Then, for any $(\rho, k) \in \mathcal{H}$, we get

$$(3.1) \qquad \int_c^b \rho(b,s)k(s)Q(s)F(X'(s))ds \leq \rho(b,c)k(c)W(c)$$

$$+\int_c^b \frac{1}{4a(s)r(s)}\bigg(h_2(b,s)a(s)r(s)I_n$$

$$-\sqrt{\rho(b,s)k(s)}\Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big]\bigg)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Proof. By Lemma 2.3 and (1.2), $W(t)$ is symmetric and satisfies the Riccati equation (2.2). Also, from (1.3), it can be seen that

$$R(t)P^{-1}(t)W(t)$$

is symmetric. Multiplying both sides of (2.2) by $\rho(t,s)k(s)$, integrating it with respect

to s from c to t for $t \in [c, b)$ and using integration by parts and (H1),(H3), we obtain

$$\int_c^t \rho(t,s)k(s)a(s)Q(s)F(X'(s))ds = -\int_c^t \rho(t,s)k(s)W'(s)ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{a'(s)}{a(s)}W(s)ds - \int_c^t \rho(t,s)k(s)\frac{p(s)}{r(s)}R(s)P^{-1}(s)W(s)ds$$

$$-\int_c^t \rho(t,s)k(s)\frac{W(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)W(s)}{a(s)r(s)}ds$$

$$(3.2) \quad = \rho(t,c)k(c)W(c) - \int_c^t h_2(t,s)\sqrt{\rho(t,s)k(s)}W(s)ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{a'(s)}{a(s)}W(s)ds - \int_c^t \rho(t,s)k(s)\frac{p(s)}{r(s)}R(s)P^{-1}(s)W(s)ds$$

$$-\int_c^t \rho(t,s)k(s)\frac{W(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)W(s)}{a(s)r(s)}ds.$$

Denote

$$(3.3) \quad Z(t) = W(t) - \frac{1}{2}\left(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\right)$$

$$\times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}.$$

Since $R(t)P^{-1}(t)W(t)$ is symmetric, and $P(t)$ and $R(t)$ are commutative with $G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$, then we obtain

$$Z^T(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)Z(t)$$

$$= \left(W(t) - \frac{1}{2}\left[a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\right]\right.$$

$$\left.\times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}\right)$$

$$\times G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)$$

$$\times \left(W(t) - \frac{1}{2}\left[a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\right]\right.$$

$$\left.\times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}\right)$$

$$(3.4) \quad = W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t) - a'(t)r(t)W(t)$$

$$+a(t)p(t)R(t)P^{-1}(t)W(t) + \frac{1}{4}\left(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\right)^2$$

$$\times P(t)\psi(X(t)))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}.$$

Then from (3.4), (3.2) can be written as

$$\int_c^t \rho(t,s)k(s)a(s)Q(s)F(X'(s))ds = \rho(t,c)k(c)W(c) - \int_c^t h_2(t,s)\sqrt{\rho(t,s)k(s)}Z(s)ds$$

$$-\int_c^t \frac{1}{2}h_2(t,s)\sqrt{\rho(t,s)k(s)}\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{a'(s)}{a(s)}W(s)ds - \int_c^t \rho(t,s)k(s)\frac{p(s)}{r(s)}R(s)P^{-1}(s)W(s)ds$$

$$-\int_c^t \rho(t,s)k(s)\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds$$

$$-\int_c^t \rho(t,s)k(s)\frac{a'(s)}{a(s)}W(s)ds + \int_c^t \rho(t,s)k(s)\frac{p(s)}{r(s)}R(s)P^{-1}(s)W(s)ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)^2}{4a(s)r(s)}$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$(3.5) \qquad = \rho(t,c)k(c)W(c) - \int_c^t h_2(t,s)\sqrt{\rho(t,s)k(s)}Z(s)ds$$

$$-\int_c^t \frac{1}{2}h_2(t,s)\sqrt{\rho(t,s)k(s)}\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$-\int_c^t \rho(t,s)k(s)\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)^2}{4a(s)r(s)}$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Let

$$V(s) = \sqrt{\rho(t,s)k(s)}(UZU)(s) + \frac{1}{2}h_2(t,s)I_n,$$

where

$$U(s) = \left[\frac{1}{a(s)r(s)}G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)\right]^{1/2}.$$

Then (3.5) can be written as

$$\int_c^t \rho(t,s)k(s)a(s)Q(s)F(X'(s))ds = \rho(t,c)k(c)W(c)$$

$$-\int_c^t h_2(t,s)U^{-1}(s)V(s)U^{-1}(s)ds + \int_c^t \frac{1}{2}h_2^2(t,s)(U^{-1}(s))^2ds$$

$$-\int_c^t \frac{1}{2}h_2(t,s)\sqrt{\rho(t,s)k(s)}\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$-\int_c^t U^{-1}(s)V^2(s)U^{-1}(s)ds + \int_c^t h_2(t,s)U^{-1}(s)V(s)U^{-1}(s)ds$$

$$-\int_c^t \frac{1}{4}h_2^2(t,s)(U^{-1}(s))^2ds$$

$$+\int_c^t \rho(t,s)k(s)\frac{\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)^2}{4a(s)r(s)}$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$= \rho(t,c)k(c)W(c) - \int_c^t U^{-1}(s)V^2(s)U^{-1}(s)ds + \int_c^t \frac{1}{4}h_2^2(t,s)(U^{-1}(s))^2ds$$

1254

$$-\int_c^t \frac{1}{2}h_2(t,s)\sqrt{\rho(t,s)k(s)}\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$
$$+\int_c^t \rho(t,s)k(s)\frac{\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{4a(s)r(s)}$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$
$$\le \rho(t,c)k(c)W(c) + \int_c^t \frac{1}{4a(s)r(s)}\Big(h_2(t,s)a(s)r(s)I_n$$
$$-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big]\Big)^2$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Letting $t \to b^-$ in the above inequality, we obtain (3.1). $\square$

3.2. Lemma. *Let $X(t)$ be prepared solution of (1.1) such that $detX(t) \neq 0$ on $(\overline{a},c] \subset [t_0,\infty)$. Assume that all conditions stated in Section 1 are satisfied and suppose also that for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0$$

and $P(t)$ and $R(t)$ are commutative with
$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$$
for $t \ge t_0$. Moreover, for $a \in C^1([t_0,\infty),(0,\infty))$ let

$$W(t) = a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t)), t \in (\overline{a},c].$$

Then, for any $(\rho,k) \in \mathcal{H}$, we get

(3.6) $$\int_{\overline{a}}^c \rho(s,\overline{a})k(s)a(s)Q(s)F(X'(s))ds \le -\rho(c,\overline{a})k(c)W(c)$$
$$+\int_{\overline{a}}^c \frac{1}{4a(s)r(s)}\Big(h_1(s,\overline{a})a(s)r(s)I_n$$
$$+\sqrt{\rho(s,\overline{a})k(s)}\Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big]\Big)^2$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Proof. Similar to the proof of Lemma 3.1, multiplying both sides of (2.2) (with t replaced by s) by $\rho(s,t)k(s)$, integrating it with respect to s from t to c for $t \in (\overline{a},c]$, using (H1),(H2) and (3.3), and rearranging the terms, we find

$$\int_t^c \rho(s,t)k(s)a(s)Q(s)F(X'(s))ds = -\rho(c,t)k(c)W(c)+\int_t^c h_1(s,t)\sqrt{\rho(s,t)k(s)}Z(s)ds$$
$$+\int_t^c \frac{1}{2}h_1(s,t)\sqrt{\rho(s,t)k(s)}\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$-\int_t^c \rho(s,t)k(s)\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds$$

$$+\int_t^c \rho(s,t)k(s)\frac{\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)^2}{4a(s)r(s)}$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Let
$$V_1(s) = \sqrt{\rho(s,t)k(s)}(UZU)(s) - \frac{1}{2}h_1(s,t)I_n,$$
where
$$U(s) = \left[\frac{1}{a(s)r(s)}G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)\right]^{1/2}.$$

It follows that
$$\int_t^c \rho(s,t)k(s)a(s)Q(s)F(X'(s))ds \leq -\rho(c,t)k(c)W(c)$$

$$-\int_t^c U^{-1}(s)V_1^2(s)U^{-1}(s)ds + \int_t^c \frac{1}{4}h_1^2(s,t)(U^{-1}(s))^2 ds$$

$$+\int_t^c \frac{1}{2}h_1(s,t)\sqrt{\rho(s,t)k(s)}\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$+\int_t^c \rho(s,t)k(s)\frac{\left(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right)^2}{4a(s)r(s)}$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$

$$\leq -\rho(c,t)k(c)W(c) + \int_t^c \frac{1}{4a(s)r(s)}\left(h_1(s,t)a(s)r(s)I_n\right.$$

$$\left.+\sqrt{\rho(s,t)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Letting $t \to \overline{a}$ in the above inequality, we obtain (3.6). □

Now we state the following theorems of Kamenev-type.

3.3. Theorem. *Assume that all conditions stated in Section 1 are satisfied and suppose also that for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0$$

for $t \geq t_0$, and $P(t)$ and $R(t)$ are commutative with
$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$$
for $t \geq t_0$. Let $(\rho(t,s), k(s)) \in \mathcal{H}$ and $\partial(\rho(t,s)k(s))/\partial s \leq 0$ be continuous for $t \geq s \geq t_0$. Suppose further that there exist $a \in C^1([t_0,\infty), (0,\infty))$ and a positive linear functional L on M such that

$$(3.7) \qquad \limsup_{t\to\infty} \frac{1}{\rho(t,t_0)}L\left[\int_{t_0}^t \left\{\rho(t,s)k(s)a(s)Q(s)F(X'(s))\right.\right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n-a(s)p(s)R(s)P^{-1}(s)\Big]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big\}ds\Big]=\infty.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory on $[t_0,\infty)$.

Proof. Suppose to the contrary that there exists a prepared solution $X(t)$ of the system (1.1) which is any nontrivial prepared solution of (1.1) in $[t_1,\infty)$ is not oscillatory. Without loss of generality, assume that $det X(t)\neq 0$, $t\geq t_1\geq t_0$. Define a matrix function $W(t)$ on $[t_1,\infty)$ by (2.1). Then by Lemma 2.3, $W(t)$ satisfies (2.2). On multiplying (2.2) (with t replaced by s)by $\rho(t,s)k(s)$, integrating with respect to s from t_1 to t for $t\geq t_1\geq t_0$, and following the procedure of the proof of Lemma 3.1, we obtain for $t\geq t_1\geq t_0$

$$\int_{t_1}^t\Big\{\rho(t,s)k(s)a(s)Q(s)F(X'(s))$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n-a(s)p(s)R(s)P^{-1}(s)\Big]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big\}ds$$

$$\leq\rho(t,t_1)k(t_1)W(t_1).$$

From $\frac{\partial(\rho(t,s)k(s))}{\partial s}\leq 0$, we have

$$\int_{t_0}^t\Big\{\rho(t,s)k(s)a(s)Q(s)F(X'(s))$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n-a(s)p(s)R(s)P^{-1}(s)\Big]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big\}ds$$

$$=\left(\int_{t_0}^{t_1}+\int_{t_1}^t\right)\Big\{\rho(t,s)k(s)a(s)Q(s)F(X'(s))$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n-a(s)p(s)R(s)P^{-1}(s)\Big]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big\}ds$$

$$\leq\rho(t,t_0)\left[\int_{t_0}^{t_1}k(s)a(s)Q(s)F(X'(s))ds+W(t_1)k(t_1)\right],$$

which implies for $t\geq t_0$

$$\limsup_{t\to\infty}\frac{1}{\rho(t,t_0)}L\Big[\int_{t_0}^t\Big\{\rho(t,s)k(s)a(s)Q(s)F(X'(s))$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n-\sqrt{\rho(t,s)k(s)}\Big[a'(s)r(s)I_n-a(s)p(s)R(s)P^{-1}(s)\Big]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big\}ds\Big]$$

$$\leq L\left[\int_{t_0}^{t_1} k(s)Q(s)F(X'(s))ds + W(t_1)k(t_1)\right] < \infty,$$

which contradicts (3.7). So,this completes the proof Theorem 3.3. $\qquad\square$

3.4. Corollary. *Assume that all conditions stated in Section 1 are satisfied and suppose also that for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) \geq A$$

and

$$F(X'(t)) \geq B, \qquad t \in [t_0, \infty),$$

where $A, B \in S$ are constant positive definite matrices, and A is commutative with $P(t)$ and $R(t)$. Let $(\rho(t,s), k(s)) \in \mathcal{H}$ and $\partial(\rho(t,s)k(s))/\partial s \leq 0$ be continuous for $t \geq s \geq t_0$. Suppose further that there exist $a \in C^1([t_0, \infty), (0, \infty))$ and a positive linear functional L on S such that

$$(3.8) \qquad \limsup_{t\to\infty} \frac{1}{\rho(t,t_0)} L\left[\int_{t_0}^t \left\{\rho(t,s)k(s)a(s)Q(s)B - \frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n\right.\right.\right.$$

$$\left.\left.\left. -\sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2 P(s)A^{-1}\right\}ds\right] = \infty.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory.

Under the modification of the hypotheses of Theorem 3.3 and Corollary 3.4, we can obtain the following results, respectively.

3.5. Theorem. *Let the condition (3.7) in Theorem 3.3 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{\rho(t,t_0)} L\left[\int_{t_0}^t \rho(t,s)k(s)a(s)Q(s)F(X'(s))ds\right] = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{\rho(t,t_0)} L\left[\int_{t_0}^t \frac{1}{a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n\right.\right.$$

$$\left.\left. -\sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2\right.$$

$$\left.\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds\right] < \infty$$

with the other conditions unchanged. Then any prepared solution $X(t)$ of (1.1) is oscillatory on $[t_0, \infty)$.

3.6. Corollary. *Let the condition (3.8) in Corollary 3.4 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{\rho(t,t_0)} L\left[\int_{t_0}^t \rho(t,s)k(s)a(s)Q(s)Bds\right]ds = \infty$$

1258

and

$$\limsup_{t\to\infty} \frac{1}{\rho(t,t_0)} L\left[\int_{t_0}^{t} \frac{1}{a(s)r(s)} \left(h_2(t,s)a(s)r(s)I_n \right.\right.$$
$$\left.\left. -\sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \right] \right)^2 P(s)A^{-1}ds \right] < \infty$$

with the other conditions unchanged. Then any prepared solution $X(t)$ of (1.1) is oscillatory on $[t_0,\infty)$.

The following theorem is an immediate consequence of Lemma 3.1 and Lemma 3.2.

3.7. Theorem. *Assume that all conditions stated in Section 1 are satisfied and suppose also that for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0,$$

and $P(t)$ and $R(t)$ are commutative with
$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$$
for $t \in [t_0,\infty)$. Assume that for some $c \in (\overline{a},b)$, $(\rho,k) \in \mathcal{H}$, $a \in C^1([t_0,\infty),(0,\infty))$, and for any prepared solution $X(t)$ of (1.1),

$$\frac{1}{\rho(c,\overline{a})} \int_{\overline{a}}^{c} \rho(s,\overline{a})k(s)a(s)Q(s)F(X'(s))ds$$
$$+\frac{1}{\rho(b,c)} \int_{c}^{b} \rho(b,s)k(s)a(s)Q(s)F(X'(s))ds$$
$$> \frac{1}{\rho(c,\overline{a})} \int_{\overline{a}}^{c} \frac{1}{4a(s)r(s)} \left(h_1(s,\overline{a})a(s)r(s)I_n \right.$$
$$\left. +\sqrt{\rho(s,\overline{a})k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \right] \right)^2$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds$$
$$+\frac{1}{\rho(b,c)} \int_{c}^{b} \frac{1}{4a(s)r(s)} \left(h_2(b,s)a(s)r(s)I_n \right.$$
$$\left. -\sqrt{\rho(b,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \right] \right)^2$$
$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}ds.$$

Then for any every prepared solution $X(t)$ of (1.1), $detX(t)$ has at least one zero in $(\overline{a},b)$.

Now, we state some corollaries of Theorem 3.7.

3.8. Corollary. *Assume that all conditions stated in Section 1 hold and suppose also that for each $\tau \geq t_0$, there exists $(\rho,k) \in \mathcal{H}$, $a \in C^1([t_0,\infty),(0,\infty))$,*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0$$

and $P(t)$ and $R(t)$ are commutative with
$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)),$$

such that for any prepared solution $X(t)$ of (1.1),

$$(3.9) \quad \limsup_{t \to \infty} \int_\tau^t \left\{ \rho(s,\tau)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_1(s,\tau)a(s)r(s)I_n + \sqrt{\rho(s,\tau)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\left. \times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\right\}ds > 0$$

and

$$(3.10) \quad \limsup_{t \to \infty} \int_\tau^t \left\{ \rho(t,s)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n - \sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\left. \times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\right\}ds > 0.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory.

3.9. Corollary. *Let the conditions (3.9) and (3.10) in Corollary 3.8 be respectively replaced by*

$$(3.11) \quad \limsup_{t \to \infty} \frac{1}{\rho(t,\tau)} \int_\tau^t \left\{ \rho(s,\tau)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_1(s,\tau)a(s)r(s)I_n + \sqrt{\rho(s,\tau)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\left. \times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\right\}ds > 0$$

and

$$(3.12) \quad \limsup_{t \to \infty} \frac{1}{\rho(t,\tau)} \int_\tau^t \left\{ \rho(t,s)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n - \sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\left. \times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\right\}ds > 0.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory.

From Corollary 3.9, we obtain the following results:

3.10. Corollary. *Assume all conditions stated in Section 1 hold. Suppose further that there exists $A > 0$ such that for each $X \in M$,*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) \geq A$$

and

$$F(X'(t)) \geq B, \qquad t \in [t_0, \infty),$$

where $A, B \in S$ are constant positive definite matrices, and A is commutative with $P(t)$ and $R(t)$. Suppose that for each $\tau \geq t_0$, there exists $(\rho, k) \in \mathcal{H}$ and $a \in C^1([t_0, \infty), (0, \infty))$

such that

$$(3.13) \quad \limsup_{t\to\infty} \frac{1}{\rho(t,\tau)} \int_\tau^t \left\{ \rho(s,\tau)k(s)a(s)Q(s)B - \frac{1}{4a(s)r(s)} \left(h_1(s,\tau)a(s)r(s)I_n \right.\right.$$
$$\left.\left. + \sqrt{\rho(s,\tau)k(s)} \Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \Big] \right)^2 P(s)A^{-1} \right\} ds > 0$$

and

$$(3.14) \quad \limsup_{t\to\infty} \frac{1}{\rho(t,\tau)} \int_\tau^t \left\{ \rho(t,s)k(s)a(s)Q(s)B - \frac{1}{4a(s)r(s)} \left(h_2(t,s)a(s)r(s)I_n \right.\right.$$
$$\left.\left. - \sqrt{\rho(t,s)k(s)} \Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \Big] \right)^2 P(s)A^{-1} \right\} ds > 0.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory.

3.11. Remark. In the special case of Eq.(1.1) with $\psi(X(t)) = I_n$, $K(X'(t)) = X'(t)$ and $p(t) = 0$, for $a(t) = 1$ Theorem 3.3, Theorem 3.5, Corollary 3.4, Corollary 3.6, Theorem 3.7, Corollaries 3.8-3.10 give Theorem 4.1, Theorem 4.2, Corollary 4.1 and Corollary 4.2, Theorem 4.3, Corollaries 5.1-5.3 of Yang and Tang [39], with $a(t) = 1$ and $p(t) = 0$, respectively.

3.12. Remark. When $P(t) = I_n$, $R(t) = I_n$, $\psi(X(t)) = I_n$ and $K(X'(t)) = X'(t)$ in (1.1), for $k(t) \equiv 1$ Theorem 3.7, Corollaries 3.9 and 3.10 give Theorem 3.5, Corollaries 3.6 and 3.7 in Yang [40], respectively.

Now, let $k(t) \equiv 1$ and $\rho(t,s) = (t-s)^\lambda$, where $\lambda > 1$ is a constant. Then, it follows from (ρ, k) in Definition 2.2 that

$$h_1(t,s) = \lambda(t-s)^{\frac{\lambda}{2}-1}$$

and

$$h_2(t,s) = \lambda(t-s)^{\frac{\lambda}{2}-1}.$$

Based on Corollary 3.10 we obtain the following corollary.

3.13. Corollary. *Assume the conditions of Corollary 3.10 hold. Suppose further that for each $\tau \geq t_0$, there exists $a \in C^1([t_0,\infty),(0,\infty))$ such that*

$$(3.15) \quad \limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{ (s-\tau)^\lambda a(s)Q(s)B - \frac{1}{4a(s)r(s)} \left(\lambda(s-\tau)^{\frac{\lambda}{2}-1} a(s)r(s) \right.\right.$$
$$\left.\left. + (s-\tau)^{\frac{\lambda}{2}} \Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \Big] \right)^2 P(s)A^{-1} \right\} ds > 0$$

and

$$(3.16) \quad \limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{ (t-s)^\lambda a(s)Q(s)B - \frac{1}{4a(s)r(s)} \left(\lambda(t-s)^{\frac{\lambda}{2}-1} a(s)r(s) \right.\right.$$
$$\left.\left. - (t-s)^{\frac{\lambda}{2}} \Big[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s) \Big] \right)^2 P(s)A^{-1} \right\} ds > 0.$$

Then any prepared solution $X(t)$ of (1.1) is oscillatory.

3.14. Remark. When $P(t) = I_n$, $R(t) = I_n$, $\psi(X(t)) = I_n$ and $K(X'(t)) = X'(t)$ in Eq.(1.1) Corollary 3.13 gives Corollary 3.8 in Yang [40]. Also, when $\psi(X(t)) = I_n$, $K(X'(t)) = X'(t)$ and $p(t) = 0$ in (1.1) with $a(t) = 1$, Corollary 3.13 is the same Corollary 5.4 in Yang and Tang [39], with $a(t) = 1$ and $p(t) = 0$.

Let $r(t) = 1$, $p(t) = 0$, $P(t) = I_n$, $\psi(X) = I_n$, $K(X') = X'$, $F(X') = I_n$ and $G(X) = I_n$. Then (1.1) reduces to the linear matrix equation

$$(3.17) \quad X''(t) + Q(t)X(t) = 0,$$

and for $a(t) = 1$ and $k(t) \equiv 1$, Corollary 3.13 reduces to the following corollary.

3.15. Corollary. *If $Q(t)$ be a continuous positive definite for all $[t_0, \infty)$, assume, for each $\tau \geq t_0$ and some $\lambda > 1$, that*

$$(3.18) \quad \limsup_{t \to \infty} \frac{1}{t^{\lambda-1}} \int_\tau^t (s-\tau)^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)} I_n$$

and

$$(3.19) \quad \limsup_{t \to \infty} \frac{1}{t^{\lambda-1}} \int_\tau^t (t-s)^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)} I_n.$$

Then every prepared solution $X(t)$ of (3.17) is oscillatory.

Let $p(t) = 0$. Then (1.1) reduces to the nonlinear matrix equation

$$(3.20) \quad (r(t)P(t)\psi(X(t))K(X'(t)))' + Q(t)F(X'(t))G(X(t)) = 0.$$

Define $B(t) = \int_\tau^t \frac{1}{r(s)}ds$, $t \geq \tau \geq t_0$, and let

$$\rho(t,s) = [B(t) - B(s)]^\lambda, t \geq t_0,$$

where $\lambda > 1$ is a constant. For $k(t) \equiv 1$ and $a(t) = 1$, Corollary 3.8 reduces to the following corollary.

3.16. Corollary. *Assume for each $X \in M$,*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) \geq I_n.$$

Let $P^{-1}(t) \geq I_n$, $F(X(t)) \geq I_n$, $\lim_{t\to\infty} B(t) = \infty$ holds and $Q(t)$ be a continuous positive definite for all $[t_0, \infty)$. Suppose further that there exists $\lambda > 1$ such that for each $\tau \geq t_0$ the following inequalities are satisfied:

$$(3.21) \quad \limsup_{t \to \infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t [B(s) - B(\tau)]^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)} I_n$$

and

$$(3.22) \quad \limsup_{t \to \infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t [B(t) - B(s)]^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)} I_n.$$

Then every prepared solution $X(t)$ of (3.20) is oscillatory.

Proof. It is easy to see that

$$h_1(t,s) = \lambda[B(t) - B(s)]^{\frac{\lambda}{2}-1}\frac{1}{r(t)}$$

and

$$h_2(t,s) = \lambda[B(t) - B(s)]^{\frac{\lambda}{2}-1}\frac{1}{r(s)}$$

in Corollary 3.8. Note that

$$\int_\tau^t \frac{1}{4}r(s)h_1^2(s,\tau)I_n ds = \frac{\lambda^2}{4(\lambda-1)}[B(t) - B(\tau)]^{\lambda-1}I_n$$

and

$$\int_\tau^t \frac{1}{4}r(s)h_2^2(t,s)I_n ds = \frac{\lambda^2}{4(\lambda-1)}[B(t) - B(\tau)]^{\lambda-1}I_n.$$

From $\lim_{t\to\infty} B(t) = \infty$, we obtain

(3.23) $\quad \lim_{t\to\infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t \frac{1}{4}r(s)h_1^2(s,\tau)I_n ds = \frac{\lambda^2}{4(\lambda-1)}I_n$

and

(3.24) $\quad \lim_{t\to\infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t \frac{1}{4}r(s)h_2^2(t,s)I_n ds = \frac{\lambda^2}{4(\lambda-1)}I_n.$

From (3.21) and (3.23), we get

$$\limsup_{t\to\infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t \left\{ \rho(s,\tau)Q(s)F(X'(s)) \right.$$

$$\left. -\frac{1}{4}h_1^2(s,\tau)r(s)P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1} \right\}ds$$

$$\geq \limsup_{t\to\infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t \left(\rho(s,\tau)Q(s) - \frac{1}{4}h_1^2(s,\tau)r(s)I_n \right)ds$$

$$= \limsup_{t\to\infty} \frac{1}{B^{\lambda-1}(t)} \int_\tau^t [B(s) - B(\tau)]^\lambda Q(s)ds - \frac{\lambda^2}{4(\lambda-1)}I_n > 0.$$

So, (3.9) holds. Similarly, (3.22) and (3.24) imply that (3.10) holds. From Corollary 3.8, it follows that every prepared solution of Eq.(3.20) is oscillatory. $\qquad\square$

3.17. Remark. Let $\psi(X(t)) = I_n$ and $K(X'(t)) = X'(t)$ in Corollary 3.16. Then Corollary 3.16 gives Corollary 5.5 of Yang and Tang [39], with $p(t) = 0$ and $v(t) = 1$.

3.18. Remark. $G'(X(t)) > 0$ in [39] and [40]. But in our paper, $G'(X(t))$ does not have to be positive definite matrix.

4. Examples

In this section, we will show the application of our oscillation criteria with three examples. We will see that the equations in the examples are oscillatory based on the results in Section 3, though the oscillations cannot be demonstrated by results of Yang and Tang [39] and Yang [40] since $K(X'(t)) \neq X'(t)$ and $\psi(X(t)) \neq I_n$.

4.1. Example. Let $t \geq 1$. Consider the following matrix differential system:

$$(4.1) \quad \left(t^{3/2}X(t)X'(t)\right)' - \sqrt{t}X(t)X'(t) + \frac{\sqrt{t}}{4}\left[2X^2(t) - I_n\right] = 0.$$

Then $r(t) = t^{3/2}$, $p(t) = -\sqrt{t}$, $P(t) = I_n$, $R(t) = I_n$, $Q(t) = \frac{\sqrt{t}}{4}I_n$, $\psi(X) = X$, $K(X') = X'$, $F(X') = I_n$, $G(X) = 2X^2 - I_n$, $G'(X) = 4X$ and

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) = 4I_n > 0.$$

Also, let $a(t) = \frac{1}{t^{3/2}}$, $k(t) \equiv 1$ and $\rho(t,s) = [\ln t - \ln s]^2$. Then, for $\tau \geq 1$, we obtain from Corollary 3.9 that if

$$\limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \rho(s,\tau)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_1(s,\tau)a(s)r(s)I_n + \sqrt{\rho(s,\tau)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s))^{-1}\bigg\}ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln s - \ln \tau]^2}{4s} - \frac{1}{16}\left(\frac{2}{s} - \frac{[\ln s - \ln \tau]}{2s}\right)^2 \right\}I_n ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln s - \ln \tau]^2}{4s} - \frac{1}{4s^2} + \frac{[\ln s - \ln \tau]}{8s^2} - \frac{[\ln s - \ln \tau]^2}{64s^2} \right\}I_n ds$$

$$> \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln s - \ln \tau]^2}{4s} - \frac{1}{4s^2} + \frac{[\ln s - \ln \tau]}{8s^2} - \frac{[\ln s - \ln \tau]^2}{64s} \right\}I_n ds$$

$$> \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left(-\frac{1}{4s^2} \right)I_n ds = 0$$

and similarly,

$$\limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \rho(t,s)k(s)a(s)Q(s)F(X'(s)) \right.$$

$$-\frac{1}{4a(s)r(s)}\left(h_2(t,s)a(s)r(s)I_n - \sqrt{\rho(t,s)k(s)}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2$$

$$\times P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s))^{-1}\bigg\}ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln t - \ln s]^2}{4s} - \frac{1}{16}\left(\frac{2}{s} + \frac{[\ln t - \ln s]}{2s}\right)^2 \right\}I_n ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln t - \ln s]^2}{4s} - \frac{1}{4s^2} - \frac{[\ln t - \ln s]}{8s^2} - \frac{[\ln t - \ln s]^2}{64s^2} \right\}I_n ds$$

$$> \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left\{ \frac{[\ln t - \ln s]^2}{4s} - \frac{1}{4s^2} - \frac{[\ln t - \ln s]^2}{8s} - \frac{[\ln t - \ln s]^2}{64s} \right\}I_n ds$$

$$> \limsup_{t\to\infty} \frac{1}{[\ln t - \ln \tau]^2} \int_\tau^t \left(-\frac{1}{4s^2}\right) I_n ds = 0.$$

So, all the assumptions of Corollary 3.9 are satisfied and every prepared solutions of (4.1) is oscillatory. In fact, $X(t) = sin\sqrt{t}I_2$ is an oscillatory solution of (4.1).

4.2. Example. Let $t \geq 1$. Consider the following matrix differential system:

$$(4.2) \qquad \left(X'(t) + (X'(t))^{-3}\right)' + \frac{\mu}{t^2} X(t) = 0.$$

Then $r(t) = 1$, $p(t) = 0$, $P(t) = I_n$, $R(t) = I_n$, $Q(t) = \frac{\mu}{t^2} I_n$, $\psi(X) = I_n$, $K(X') = X' + (X')^{-3}$, $F(X') = I_n = B > 0$, $G(X) = X$, $G'(X) = I_n$ and

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) = I_n + (X'(t))^2 \geq I_n = A > 0.$$

Also, let $a(t) = 1$. Then, for $\lambda > 1$ and $\tau \geq 1$, we obtain from Corollary 3.13 that if

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{(s-\tau)^\lambda a(s)Q(s)B - \frac{1}{4a(s)r(s)}\left(\lambda(s-\tau)^{\frac{\lambda}{2}-1}a(s)r(s)I_n\right.\right.$$

$$\left.\left. +(s-\tau)^{\frac{\lambda}{2}}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2 P(s)A^{-1}\right\}ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{(s-\tau)^\lambda \frac{\mu}{s^2}I_n - \frac{1}{4}\left(\lambda(s-\tau)^{\frac{\lambda}{2}-1}\right)^2 I_n\right\}ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{(s-\tau)^\lambda \frac{\mu}{s^2} - \frac{1}{4}\lambda^2 (s-\tau)^{\lambda-2}\right\}I_n ds$$

$$= \mu\frac{1}{\lambda-1}I_n > \frac{\lambda^2}{4(\lambda-1)}I_n$$

or

$$\mu > \frac{\lambda^2}{4} > \frac{1}{4}$$

and similarly,

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{(t-s)^\lambda a(s)Q(s)B - \frac{1}{4a(s)r(s)}\left(\lambda(t-s)^{\frac{\lambda}{2}-1}a(s)r(s)I_n\right.\right.$$

$$\left.\left. -(t-s)^{\frac{\lambda}{2}}\left[a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\right]\right)^2 P(s)A^{-1}\right\}ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_\tau^t \left\{(t-s)^\lambda \frac{\mu}{s^2} - \frac{1}{4}\left(\lambda(t-s)^{\frac{\lambda}{2}-1}\right)^2 I_n\right\}ds > 0,$$

then any prepared solution $X(t)$ of (4.2) is oscillatory.

4.3. Example. Let $t \geq t_0 > 1$. Consider the following matrix differential system:

$$(4.3) \qquad \left(tX^2(t)X'(t)\right)' + \frac{\mu}{t(\ln t)^2}\left[X^5(t) + \frac{1}{3}X^3(t)\right] = 0.$$

Then $r(t) = t$, $P(t) = I_n$, $Q(t) = \frac{\mu}{t(\ln t)^2}$, $\psi(X(t)) = X^2(t)$, $K(X') = X'$, $F(X') = I_n$, $G(X(t)) = X^5(t) + \frac{1}{3}X^3(t)$, $G'(X(t)) = 5X^4(t) + X^2(t)$ and

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) = 5X^2(t) + I_n \geq I_n.$$

So, we get

$$B(t) = \int_\tau^t \frac{1}{r(s)}ds = \int_\tau^t \frac{1}{s}ds = \ln t - \ln \tau.$$

For $\lambda > 1$, we obtain from Corollary 3.16 that if

(4.4) $\qquad \limsup_{t \to \infty} \frac{1}{(\ln t - \ln \tau)^{\lambda-1}} \int_\tau^t [\ln s - \ln \tau]^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)}I_n$

and

(4.5) $\qquad \limsup_{t \to \infty} \frac{1}{(\ln t - \ln \tau)^{\lambda-1}} \int_\tau^t [\ln t - \ln s]^\lambda Q(s)ds > \frac{\lambda^2}{4(\lambda-1)}I_n,$

then every prepared solutions of (4.3) is oscillatory. That is, (4.4) and (4.5) are equivalent to

$$\limsup_{t \to \infty} \frac{1}{(\ln t - \ln \tau)^{\lambda-1}} \int_\tau^t [\ln s - \ln \tau]^\lambda \frac{\mu}{s(\ln s)^2}I_n ds$$

$$= \frac{\mu}{\lambda-1}I_n > \frac{\lambda^2}{4(\lambda-1)}I_n$$

or

$$\mu > \frac{\lambda^2}{4} > \frac{1}{4}.$$

References

[1] Baker, J.W. *Oscillation theorems for a second-order damped nonlinear differential equation*, SIAM J. Appl. Math. **25**, 37-40, 1973.

[2] Butler, G.J. *The oscillatory behavior of a second-order nonlinear differential equation with damping*, J. Math. Anal. Appl. **57**, 273-289, 1977.

[3] Butler, G.J., Erbe, L.H. *Oscillation results for self-adjoint differential systems*, J. Math. Anal. Appl. **115**, 470-481, 1986.

[4] Erbe, L.E., Kong, Q., Ruan, S. *Kamenev-type theorems for second-order matrix differential systems*, Proc. Amer. Math. Soc. **117**, 957-962, 1993.

[5] Etgen, G.J., Pawlowski, J.F. *Oscillation criteria for second-order self-adjoint matrix differential systems*, Pacific J. Math. **66**, 99-110, 1976.

[6] Grace, S.R. *Oscillation theorems for second-order nonlinear differential equations with damping*, Math. Nachr. **141**, 117-127,1989.

[7] Grace, S.R. *Oscillation criteria for second-order nonlinear differential equations with damping*, J. Austral. Math. Soc. Ser. A, **49**, 43-54, 1990.

[8] Grace, S.R., Lalli, B.S. *Integral averaging technique for the oscillation of second-order nonlinear differential equations*, J. Math. Anal. Appl. **149**, 277-311, 1990.

[9] Grace, S.R., Lalli, B.S., Yeh, C.C. *Oscillation theorems for nonlinear second-order differential equations with a nonlinear damping term*, SIAM J. Math. Anal. **15**, 1082-1093, 1984.

[10] Hartman, P. *On nonoscillatory linear differential equations of second-order*, Amer. J. Math. **74**, 389-400, 1952.

[11] Kamenev, I.V. *An integral criterion for oscillation of linear differential equations*, Mat. Zametki, **23**, 249-251, 1978.

[12] Kong, Q. *Interval criteria for oscillation of second-order linear ordinary differential equations*, J. Math. Anal. Appl. **229**, 258-270, 1999.

[13] Li, H.J. *Oscillation criteria for second-order linear differential equations*, J. Math. Anal. Appl. **194**, 217-234, 1995.

[14] Li, W.T. *Oscillation of certain second-order nonlinear differential equations*, J. Math. Anal. Appl. **217**, 1-14, 1998.

[15] Li, W.T., Agarwal,R.P. *Interval oscillation criteria related to integral averaging technique for certain nonlinear differential equations*, J. Math. Anal. Appl. **245**, 171-188, 2000.

[16] Li, W.T., Agarwal, R.P. *Interval oscillation criteria for second-order nonlinear equations with damping*, Computers Math. Appl. **40**, 217-230, 2000.

[17] Meng, F., Wang, J., Zheng, Z. *A note on Kamenev-type theorem for second-order matrix differential systems*, Proc. Amer. Math. Soc. **126**, 391-395, 1998.

[18] Parhi, N., Praharaj, P. *Oscillation of linear second order matrix differential equations*, J. Math. Anal. Appl. **221**, 287-305, 1998.

[19] Philos, Ch.G. *Oscillation theorems for linear differential equations of second-order*, Arch. Math.(Basel), **53**, 482-492, 1989.

[20] Rogovchenko, Y.V. *Oscillation criteria for certain nonlinear differential equations*, J. Math. Anal. Appl. **229**, 399-416, 1999.

[21] Rogovchenko, Y.V., Tuncay, F. *Interval oscillation of second-order nonlinear differential equation with a damping term*, Discrete and Continuous Dynamical Systems Supplement, 883-891, 2007.

[22] Rogovchenko, Y.V., Tuncay, F. *Oscillation criteria for second-order nonlinear differential equations with damping*, Nonlinear Analysis, **69**, 208-221, 2008.

[23] Sun, Y.G. *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping*, J. Math. Anal. Appl. **291**, 341-351, 2004.

[24] Sun, Y.G., Meng, F.W. *Oscillation results for matrix differential systems with damping*, Appl. Mathl. Comput. **175**, 212-220, 2006.

[25] Swanson, C.A. *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

[26] Tiryaki, A., Ayanlar, B. *Oscillation theorems for nonlinear second-order differential equations with damping*, Acta Math. Hungar. **89** (1-2), 1-13, 2000.

[27] Tiryaki, A., Çakmak, D. *Integral averages and oscillation criteria of second-order nonlinear differential equations*, Comput. Math. Appl. **47**, 1495-1506, 2004.

[28] Tiryaki, A., Zafer, A. *Oscillation criteria for second-order nonlinear differential equations with damping*, Turk. J. Math. **24**, 185-196, 2000.

[29] Wang, Q.R. *Oscillation criteria for second-order matrix differential systems*, Arch. Math.(Basel), **76**, 385-390, 2001.

[30] Wang, Q.R. *Interval criteria for oscillation of certain matrix differential systems*, J. Math. Anal. Appl. **276**, 373-395, 2002.

[31] Wang, Q.R. *Oscillation of self-adjoint matrix differential systems*, Appl. Math. Letters, **17**, 1299-1305, 2004.

[32] Wang, Q.R., Wu, X.M., Zhu, S.M. *Kamenev-type oscillation criteria for second-order matrix differential systems*, Appl. Math. Letters, **16**, 821-826, 2003.

[33] Wintner, A. *A criterion of oscillatory stability*, Quart. Appl. Math. **7**, 115-117, 1949.

[34] Wong, J.S.W. *Oscillation criteria for a forced second-order linear differential equations*, J Math. Anal. Appl. **231**, 235-240, 1999.

[35] Wong, J.S.W. *On Kamenev-type oscillation theorems for second-order differential equations with damping*, J. Math. Anal. Appl. **258**, 244-257, 2001.

[36] Yan, J. *Oscillation theorems for second-order linear differential equations with damping*, Proc. Amer. Math. Soc. **9**, 276-282, 1986.

[37] Yancong, X., Fanwei, M. *New oscillation critera for second-order nonlinear matrix differential equations*, Appl. Math. J. Chinese Univ. Set. B, **21** (3), 313-319, 2006.

[38] Yang, Q., Cheng, S.S. *Kamenev-type oscillation criteria for second-order matrix differential systems with damping*, Ann. Polon. Math. **85** (2), 145-152, 2005.

[39] Yang, Q., Tang, Y. *Oscillation theorems for certain second-order self-adjoint matrix differential systems*, J. Math. Anal. Appl. **288**, 565-585, 2003.

[40] Yang, X.J. *Oscillation criteria for certain second-order matrix differential equations*, J. Math. Anal. Appl. **265**, 285-295, 2002.

[41] Yeh, C.C. *Oscillation theorems for nonlinear second-order differential equations with damped term*, Proc. Amer. Math. Soc. **84**, 397-402, 1982.

[42] Zheng, Z.W., Meng, F.W., Yu, Y.H. *On the oscillation of second order matrix differential systems*, Acta. Math. Sinica. **41** (6), 1231-1238, 1998.

[43] Zhuang, R.K. *Interval criteria for oscillatory of second order matrix differential systems*, Acta. Math. Sinica. **44** (6), 1037-1044, 2001.