Cumhuriyet Sci. J., 47(2) (2026) 356-360
DOI: 10.17776/csj.1860028

Cumhuriyet Science Journal

http://csj.cumhuriyet.edu.tr Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Generating the Free Group of Rank Two with Dynnikov Coordinates

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Research Article

History
Received: 09.01.2026
Accepted: 02.04.2026

ABSTRACT

It is well known that if the geometric intersection number of two simple closed curves is at least two, then the Dehn twists about these curves generate a free group of rank two. In this paper, we consider a pair of intersecting standard curves in the three-punctured disk D3 and show that the corresponding Dehn twists generate a free group of rank two. This result is proved using a coordinate-based alternative approach formulated entirely in terms of Dynnikov coordinates, which allows the ping–pong dynamics providing a sufficient criterion for freeness to be seen explicitly.

Keywords: *Dynnikov coordinates, Free group, Mapping class group, Ping-Pong, Update rules.*

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1. Introduction

Let $\sum$ be a compact, connected, orientable surface of genus g possibly with finitely many punctures or boundary components. The mapping class group $Mod(\sum)$ is the group of isotopy classes of orientation preserving homeomorphisms of $\sum$. A classical result asserts that $Mod(\sum)$ is generated by Dehn twists [1-3] which have infinite order. In this paper we restrict to the case where $\sum$ is the $n-$puntured disk D_n where n = 3. Throughout the article, a curve c in D_n will mean the isotopy class of an essential simple closed curve (i.e. c is not homotopic to a point, a puncture or a boundary component). We denote by t_c the positive (right-handed) Dehn twist about the curve c.

Given k simple closed curves on a surface $\sum$, understanding the algebraic structure of the subgroups generated by the Dehn twists about these curves is an important problem [4-6]. This problem has been completely resolved for the case $k=2$ using the geometric intersection number of the curves [7]. More precisely, the geometric intersection number $i(c_1,c_2)$ of the curves c_1 and c_2 is defined as the minimal number of intersection points between representatives $\gamma_1 \in c_1$ and $\gamma_2 \in c_2$. Depending on the value of $i(c_1,c_2)$, many results have been obtained concerning the classification of the groups generated by two Dehn twists. In particular, it is shown in [8-10] that if $i(c_1,c_2) \geq 2$, then $\langle t_{c_1}, t_{c_2} \rangle$ is isomorphic to the free group F_2 of rank two. For further classifications of groups generated by two Dehn twists, see [5,7].

In the three-punctured disk D_3, elements of the mapping class group are represented by braids. The action of the braid group B_3 on curves is described by update rules in terms of Dynnikov coordinates [11,12]. In this paper, using the update rules for B_3, the action of Dehn twists is expressed in Dynnikov coordinates [11,13,14]. This coordinate-based alternative approach allows the ping–pong dynamics underlying the freeness of the subgroup generated by the Dehn twists along the given curves to be realized concretely on subsets of a coordinate space.

This paper is organized as follows. In Section 2, we briefly introduce the Dynnikov coordinate system and present the update rules for B_3. Using these update rules, we then describe the action of Dehn twists along standard curves in terms of Dynnikov coordinates. In Section 3, we prove Theorem 1.1.

Theorem 1.1

Let c_1 and c_2 be standard curves in D_3. Then $G = \langle t_{c_1}, t_{c_2} \rangle \cong F_2$.

2. Actions of Dehn Twists on Dynnikov Coordinates

In this section, we first briefly introduce the Dynnikov coordinate system [11] and the update rules describing the action of the braid group B_3 on curves in D_3 in terms of Dynnikov coordinates. We then use these rules to compute the action of Dehn twists along standard curves via the corresponding update rules.

2.1 Dynnikov Coordinates

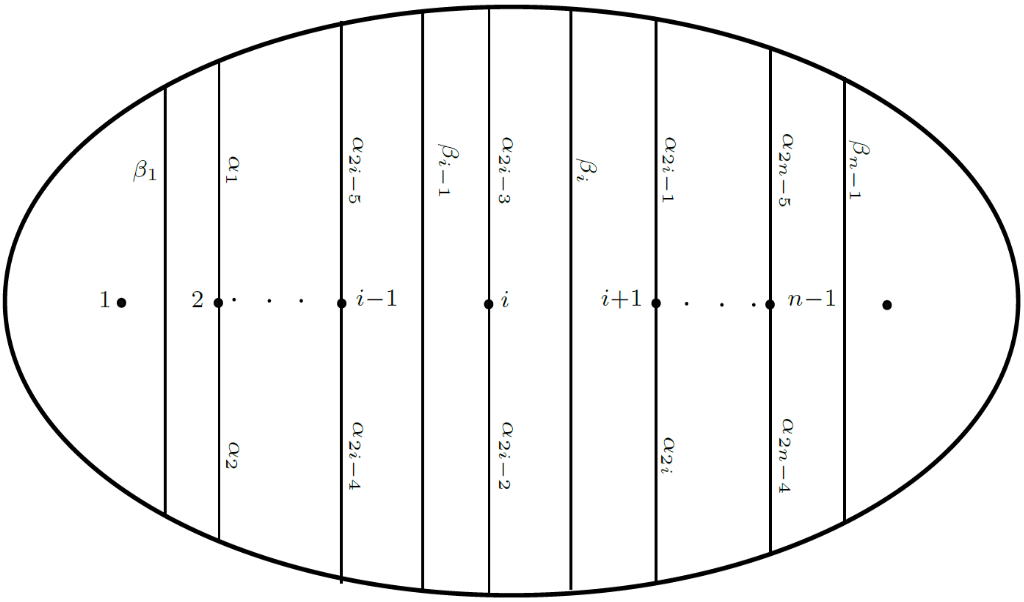

Figure 1. Dynnikov arcs

Dynnikov coordinates on D_n are defined using the system of Dynnikov arcs α_i $(1 \leq i \leq 2n-4)$ and β_i $(1 \leq i \leq n-1)$ shown in Figure 1. For each element $\mathcal{L}$ of the set $\mathcal{L}_n$of multicurves (finite families of curves) in D_n, there exists a representative $L \in \mathcal{L}$ that intersects each Dynnikov arc minimally. Let the intersection numbers with the arcs be denoted again by α_i and β_i. Then the Dynnikov coordinate function

$$\rho : \mathcal{L}_n \rightarrow \mathbb{Z}^{2n-4} \backslash \{0\}$$

is defined, for $1 \leq i \leq n-2$, by

$$a_i = \frac{(\alpha_{2i} - \alpha_{2i-1})}{2}, \quad b_i = \frac{(\beta_i - \beta_{i+1})}{2}.$$

Moreover, the Dynnikov coordinates of $\mathcal{L}$ are given by

$$\rho(\mathcal{L}) = (a;\ b) = (a_1, \dots, a_{n-2};\ b, \dots, b_{n-2}).$$

2.2 Update Rules

The mapping class group $Mod(D_n)$ acts well on $\mathcal{L}_n$ (each mapping class sends each curve to a curve). Moreover, since $Mod(D_n)$ is isomorphic to the Artin braid group B_n, the group B_n also acts well on $\mathcal{L}_n$. The action of B_n on $\mathcal{L}_n$ is described by update rules in terms of Dynnikov coordinates [11,12]. Since in this paper we work on the surface D_3, the update rules required are those for B_3, which are given in Theorem 2.1.

Theorem 2.1. Let (a ,b) $\in \mathbb{Z}^2 \setminus \{0\}$. For each $1 \leq i \leq 2$, write $\sigma_i(a,b) = (a', b')$ and $\sigma_i^{-1}(a,b) = (\tilde{a}, \tilde{b})$.

For i = 1:

$$a' = (a + b) - max(0, a, b)$$
$$b' = max(0, b) - a$$
$$\tilde{a} = max(0, a + max(0, b)) - b$$
$$\tilde{b} = max(0, b) + a$$

For i = 2:

$$a' = max(a + max(a, b), b)$$
$$b' = b - \mathrm{a} - max(0, b)$$
$$\tilde{a} = a - max(0, a + b, b)$$
$$\tilde{b} = a + b + max(0, b)$$

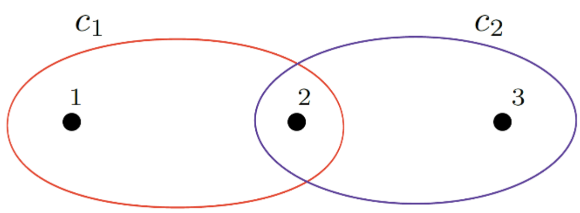

Figure 2. Standard curves on the three-punctures disk

Definition.
In D_3, the curves $c_1, c_2 \in \mathcal{L}_3$ with Dynnikov coordinates

$$\rho(c_1) = (0;\ 1)$$
$$\rho(c_2) = (0;\ -1)$$

are called *standard curves* (see Figure 2).

Using the identities $t_{c_1} = {\sigma_1}^2$ and $t_{c_2} = {\sigma_2}^2$ together with the update rules given in Theorem 2.1, one can derive explicit formulas describing the action of Dehn twists along the standard curves in D_3 in terms of Dynnikov coordinates. Similar computations apply to $t_{c_1}^{-1} = \sigma_1^{-2}$ and $t_{c_2}^{-1} = {\sigma_2}^{-2}$.

Lemma 1.1 (Action of t_{c_1})

Let $\mathcal{L} \in \mathcal{L}_3$ be a curve with Dynnikov coordinates $\rho(\mathcal{L}) = (a;\, b)$. Then the Dynnikov coordinates $\rho(t_{c_1}(\mathcal{L})) = (a'';\, b'')$ are given as follows.

Case 1: $a > 0,\ b > 0$

(i) $a > b$

$a'' = b - a,\quad b'' = -b$

(ii) $a < b$ and $2a > b$

$a'' = -a + b,\quad b'' = -2a + b$

(iii) $a < b$ and $2a < b$

$a'' = a,\quad b'' = -2a + b$

Case 2: $a < 0,\ b > 0$

$a'' = a,\quad b'' = -2a + b$

Case 3: $a < 0,\ b < 0$

$a'' = a + b,\quad b'' = -2a - b$

Case 4: $a > 0,\ b < 0$

$a'' = -a + b,\quad b'' = -b$

Proof.

We consider Case 1(i). By Theorem 2.1, writing $\sigma_1(a, b) = (a', b')$, we have

$$a' = (a + b) - max(0, a, b),$$
$$b' = max(0, b) - a.$$

Since $a > 0, b > 0$, and $a > b$, it follows that $a' = b$ and $b' = b - a$.

Similarly, writing ${\sigma_1}^2(a, b) = (a'', b'')$, we obtain

$$a'' = (a' + b') - max(0, a', b'),$$
$$b'' = max(0, b') - a'.$$

Thus we find $a'' = -a + b$ and $b'' = -b$. Consequently, the corresponding matrix is

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

The remaining cases are computed in a similar manner, yielding the matrices listed in Table 1.

Table 1. Action of the Dehn Twist t_{c_1} in Dynnikov Coordinates

$a > 0, b > 0$					
$a > b$	$a < b, 2a > b$	$a < b, 2a < b$	$2a < 0, b > 0$	$2a < 0, b < 0$	$2a > 0, b < 0$
$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

The following lemmas are proved in a similar manner.

Lemma 1.2 (Action of t_{c_2})

Let $\mathcal{L} \in \mathcal{L}_3$ be a curve with Dynnikov coordinates $\rho(\mathcal{L}) = (a;\, b)$. Then the Dynnikov coordinates $\rho(t_{c_2}(\mathcal{L})) = (a'';\, b'')$ are given as follows.

- Case 1: $a > 0,\ b > 0$

$a'' = a + b,\quad b'' = -2a - b,$

- Case 2: $a < 0,\ b > 0$

$a'' = -a + b,\quad b'' = -b,$

- Case 3: $a < 0,\ b < 0$

(i) $a > b$ and $2a > b$

$a'' = a,\quad b'' = -2a + b,$

(ii) $a > b$ and $2a < b$

$a'' = -a + b,\quad b'' = -2a + b,$

(iii) $a < b$

$a'' = -a + b,\quad b'' = -b.$

- Case 4: $a > 0,\ b < 0$

$a'' = a,\quad b'' = -2a + b.$

Lemma 1.3. (Action of $t_{c_1}^{-1}$)

Let $\mathcal{L} \in \mathcal{L}_3$ be a curve with Dynnikov coordinates $\rho(\mathcal{L}) = (a;\, b)$. Then $\rho(t_{c_1}^{-1}(\mathcal{L})) = (a'';\, b'')$ is given as follows.

- Case 1: $a > 0, b > 0$

$a'' = a,\quad b'' = 2a + b,$

- Case 2: $a < 0, b > 0$

(i) $-a > b$

$a'' = -a - b,\quad b'' = -b,$

(ii) $-a < b$ and $-2a > b$

$a'' = -a - b,\quad b'' = 2a + b,$

(iii) $-a < b$ and $-2a < b$

$a'' = a,\quad b'' = 2a + b.$

- Case 3: $a < 0, b < 0$

$a'' = -a - b,\quad b'' = -b,$

- Case 4: $a > 0, b < 0$

$a'' = a - b,\quad b'' = 2a - b.$

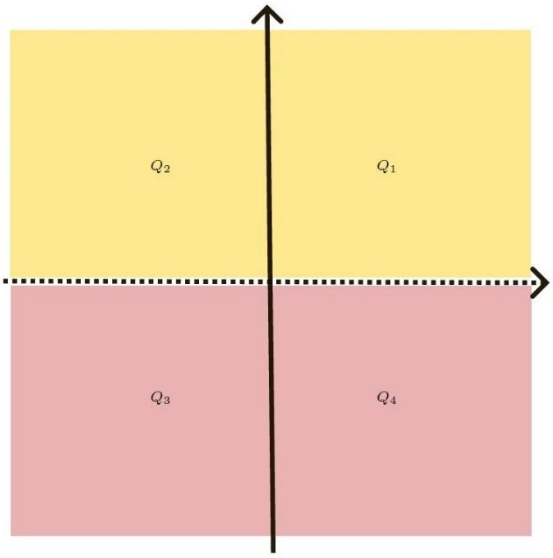

Figure 3: The regions Q_1 and Q_2.

Lemma 1.4. (Action of $t_{c_2}^{-1}$)

Let $\mathcal{L} \in \mathcal{L}_3$ be a curve with Dynnikov coordinates $\rho(\mathcal{L}) = (a;\, b)$. Then the Dynnikov coordinates $\rho(t_{c_2}^{-1}(\mathcal{L})) = (a'';\, b'')$ are given as follows.

- Case 1: $a > 0,\ b > 0$

$$a'' = -a - b,\quad b'' = -b,$$

- Case 2: $a < 0,\ b > 0$

$$a'' = a - b,\quad b'' = 2a - b,$$

- Case 3: $a < 0,\ b < 0$

$$a'' = a,\quad b'' = 2a + b.$$

- Case 4: $a > 0,\ b < 0$

(i) $-a < b$

$$a'' = -a - b,\quad b'' = -b,$$

$(ii)\quad -a > b \text{ and } -2a > b$

$$a'' = a,\quad b'' = 2a + b,$$

(iii) $\quad -a > b \text{ and } -2a < b$

$$a'' = -a - b,\quad b'' = 2a + b.$$

Remark.

Let the sets Q_i be defined by

$$Q_1 = \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, b > 0\}.$$

$$Q_2 = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b > 0\}.$$

$$Q_3 = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b < 0\}.$$

$$Q_4 = \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, b < 0\}.$$

Set $X_1 = Q_1 \cup Q_2$ and $X_2 = Q_3 \cup Q_4$ (see Figure 3).

Then, by Lemma 1.1 and Lemma 1.2, we observe that $t_{c_1}(X_2) \subset X_1$ and $t_{c_2}(X_1) \subset X_2$.

Corollary 1.5. (Action of $t_{c_1}^{p}$ on Q_3 and Q_4)

For every $p \in \mathbb{Z} \setminus \{0\}$, we have

$$t_{c_1}^{p}(Q_3 \cup Q_4) \subset Q_1 \cup Q_2.$$

Proof.

Let $\rho(t_{c_1}^{p}(\mathcal{L})) = (a', b')$. Repeating computations similar to those carried out in Lemma 1.1, we obtain the following cases:

Case 1: If $p > 0$ and $(a,b) \in Q_3$, then

$$a' = a + b,\quad b' = -(2p)a - (2p-1)b,$$

which implies $(a', b') \in Q_2$.

Case 2: If $p < 0$ and $(a,b) \in Q_3$, then

$$a' = -a - b,\quad b' = 2(p+1)a + (2p+1)b,$$

which implies $(a', b') \in Q_1$.

Case 3: If $p > 0$ and $(a,b) \in Q_4$, then

$$a' = -a + b,\quad b' = 2(p-1)a - (2p-1)b,$$

which implies $(a', b') \in Q_2$.

Case 4: If $p < 0$ and $(a,b) \in Q_4$, then

$$a' = a - b,\quad b' = -2p\,a + (2p+1)b,$$

which implies $(a', b') \in Q_1$.

Corollary 1.6. (Action of $t_{c_2}^{p}$ on Q_1 and Q_2)

For all $p \in \mathbb{Z} \setminus \{0\}$, we have

$$t_{c_2}^{p}(Q_1 \cup Q_2) \subset Q_3 \cup Q_4.$$

Proof.

Let $\rho(t_{c_2}^{p}(\mathcal{L})) = (a', b')$. Repeating computations similar to those carried out in Lemma 1.2, we obtain the following cases:

Case 1: If $p > 0$ and $(a,b) \in Q_1$, then

$$a' = a + b,\quad b' = -(2p)a - (2p-1)b,$$

which implies $(a', b') \in Q_3$.

Case 2: If $p < 0$ and $(a,b) \in Q_1$, then

$$a' = -a - b,\quad b' = 2(p+1)a + (2p+1)b,$$

which implies $(a', b') \in Q_4$.

Case 3: If $p > 0$ and $(a,b) \in Q_2$, then

$$a' = -a + b,\quad b' = 2(p-1)a - (2p-1)b,$$

which implies $(a', b') \in Q_3$.

Case 4: If $p < 0$ and $(a,b) \in Q_2$, then

$$a' = a - b,\quad b' = -2p\,a + (2p+1)b,$$

which implies $(a', b') \in Q_4$.

3. Proof of the Main Theorem

In this section, letting c_1 and c_2 be the standard curves shown in Figure 2, we prove that the group generated by t_{c_1} and t_{c_2} is free, using the results established in Section 2. The proof is based on an explicit ping–pong construction carried out in Dynnikov coordinates, where the update rules make it possible to identify the required dynamical properties directly at the level of coordinates. The main tool of the proof is the Ping–Pong Lemma, which appears in several variants in the literature and provides a sufficient criterion for freeness. In this paper, we use the version given in [15].

Ping–Pong Lemma.
Let G = ⟨g, h⟩ be a group generated by the elements g and h acting on a set X. Suppose that there exist nonempty, disjoint subsets $X_g, X_h \subset X$ such that, for every $m \in \mathbb{Z} \setminus \{0\}$, $g^m(X_h) \subset X_g$ and $h^m(X_g) \subset X_h$. Then $\langle g, h\rangle \cong F_2$.

Using the Ping–Pong Lemma together with the Corollaries in Section 2, we obtain Theorem 1.1.

Proof of Theorem 1.1.

Let $G = \langle t_{c_1}, t_{c_2}\rangle$ and let $X = \mathbb{R}^2 \setminus \{0\}$. Define subsets $X_1, X_2 \subset X$ by $X_1 = \{(a,b) \in \mathbb{R}^2 \mid b > 0\}$ and $X_2 = \{(a,b) \in \mathbb{R}^2 \mid b < 0\}$. As illustrated in Figure 3, we have $X_1 = Q_1 \cup Q_2$ and $X_2 = Q_3 \cup Q_4$, and clearly $X_1 \cap X_2 = \emptyset$. Moreover, by Corollary 1.5 and Corollary 1.6, for every $p \in \mathbb{Z} \setminus \{0\}$ we have $t_{c_2}^p(X_1) \subset X_2$ and $t_{c_1}^p(X_2) \subset X_1$. Therefore, by the Ping–Pong Lemma, $G = \langle t_{c_1}, t_{c_2}\rangle \cong F_2$.

Conflict of Interest

There are no conflicts of interest in this work.

Acknowledgments

This study was carried out within the scope of the project numbered 123F221, supported by TÜBİTAK 1002 - A Short-Term Support Module.

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