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Some remarks on completeness and compactness in G-metric spaces

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ABSTRACT

Complete metric spaces have great importance in functional analysis and its applications. The purpose of this paper is to introduce and study on some types of completeness in generalized metric spaces by the aid of Bourbaki Cauchy and cofinally Bourbaki-Cauchy sequences which are belonging to the class bigger than the class of Cauchy sequences. Moreover, by transporting some topological concepts to generalized metric spaces, the relations between these concepts and these new types of completeness properties are given.

Keywords: generalized metric spaces, compactness, completeness

1. INTRODUCTION

In mathematics and applied sciences, metric spaces play a central role. So several generalizations of the notion of a metric space have been proposed by many authors. Also, complete metric spaces have great importance to prove fundamental results in functional analysis which have many fascinating applications. For instance, Baire category theorem which is obtained when investigating the behavior of continuous functions is a very useful property to lighten the structure of complete metric spaces. Some applications of this theorem reveal various significant properties of complete metric spaces. Furthermore, Banach fixed point theorem is an important tool in the theory of complete metric spaces. By virtue of a great deal of applications in areas such as variational and linear inequalities, optimization and approximation theory, the progress of fixed point theory in metric spaces has drawn great interest. A large list of references can be found in the papers [1, 2, 3, 4, 6, 9, 10, 11, 12, 13, 14, 16] related to the fixed point results in generalized metric spaces. Many authors introduce some new concepts between compactness and completeness in metric spaces and they give a number of new characterizations of these properties. For example, Beer [5] give some characterizations of cofinally complete metric spaces which implies that every cofinally Cauchy sequence has a convergent subsequence. More recently, Garrido and Merono [7] define Bourbaki-Cauchy sequences and cofinally Bourbaki-Cauchy sequences in metric spaces and they introduce two new types of completeness by using these new classes of generalized Cauchy sequences. Hence, these new concepts of completeness become properties stronger than the usual completeness. In [15], the authors define a new structure called as generalized metric or briefly G-metric and carry many concepts from metric spaces to the G-metric spaces. A generalized metric is a real valued function G defined on $X \times X \times X$ for a non-empty set X satisfying the following conditions.

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $G(x,x,y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x, y, z \in X$ with $z \neq y$,

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(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables),

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

The pair (X,G) is called a G-metric space. By G, one can construct a metric on X as follows:
$$d_G(x,y) = G(x,y,y) + G(x,x,y)$$
for all $x,y \in X$. Also, the inequality
$$G(x,y,y) \leq 2G(y,x,x)$$
holds for every $x,y \in X$.

The G-open ball (resp., G-closed ball) with centred $x \in X$ and radius ε is defined as $S_G(x,\varepsilon) = \{y \in X : G(x,y,y) < \varepsilon\}$ (resp., $S_G[x,\varepsilon] = \{y \in X : G(x,y,y) \leq \varepsilon\}$). The collection of all G-open balls in X generates a topology $\tau(G)$ on X and this topology is called G-metric topology. The sets of $\tau(G)$ are called as G-open. In a G-metric space, a sequence (x_n) is said to be G-convergent to $x \in X$, if $G(x_n,x_n,x) \to 0$ or equivalently $G(x_n,x,x) \to 0$ as $n \to \infty$. A sequence (x_n) is said to be G-Cauchy if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$ or equivalently $G(x_n,x_m,x_m) \to 0$ as $n,m \to \infty$. If every G-Cauchy sequence in a G-metric space is G-convergent, then the space is called as G-complete. A subset S in a G-metric space X is G-totally bounded if for every $\varepsilon > 0$ there exist finitely many elements $x_1,x_2,\ldots,x_n$ in X such that
$$S \subset \bigcup_{i=1}^{n} S_G(x_i,\varepsilon).$$

A G-metric space is said to be G-compact if the space is a compact topological space with respect to the G-metric topology, that is every G -open cover of X has a finite subcover or equivalently every sequence in the space has a G -convergent subsequence. Further, a G -metric space is G -compact if and only if it is G -complete and G -totally bounded. For more concepts and some characteristic properties of G -metric spaces, one can see [8].

This paper is devoted to introduce and study on some new properties in generalized metric spaces which are stronger than completeness but weaker than compactness. For this purpose, we define some new classes of generalized Cauchy sequences. Also, we transport some concepts in metric spaces to generalized metric spaces and give the relations between these concepts with new properties.

2. MAIN RESULTS

For $m \in \mathbb{N}$, $S_G^m(\mathrm{x},\delta)$ consists of points y in X such that there exists $a_1,a_2,\ldots,a_{m-1} \in X$ satisfying $G(x,a_1,a_1) < \delta, G(a_1,a_2,a_2) < \delta, \ldots, G(a_{m-1},y,y) < \delta$. The open δ-enlargement of a subset S in a G-metric space is defined as
$$S^\delta = \bigcup_{x\in S} S_G(x,\delta).$$
Hence, it can be easily seen that $S_G^m(x,\delta) = (S_G^{m-1}(x,\delta))^\delta$.

In a G-metric space, we call a subset as G-closed if its complement is in $\tau(G)$. As in a usual metric space, a set S is G-closed if and only if $x \in S$ whenever (x_n) is a sequence in S which is G-convergent to x. Also G-closed subsets of a G-compact set is G -compact.

By the G-neighborhood of a point x, we mean any set U in $\tau(G)$ containing x. $G-\mathrm{Cl}(S)$ stands for the G-closure of S which consists of points x in X such that every G-neighborhood of x and S has a nonempty intersection.

Lemma 2.1. Let (X,G) be a G-metric space. Then for all $x \in X$, $\varepsilon > 0$ and $m \in \mathbb{N}$ we have (1) $S_G^m(x,\varepsilon) \subseteq S_G(x,m\varepsilon)$,

(2) $S_G^m(x,\varepsilon/3) \subseteq S_{d_G}^m(x,\varepsilon) \subseteq S_G^m(x,\varepsilon)$.

Proof.

(1) Let $y \in S_G^m(x,\varepsilon)$. Then there exist $z_1,z_2,\ldots z_{m-1} \in X$ such that $G(x,z_1,z_1) < \varepsilon$, $G(z_1,z_2,z_2) < \varepsilon,\ldots,G(z_{m-1},y,y) < \varepsilon$. From (G5), we obtain
$$\begin{aligned} G(x,y,y) &\leq G(x,z_1,z_1) + G(z_1,y,y) \\ &\leq G(x,z_1,z_1) + G(z_1,z_2,z_2) + G(z_2,y,y) \\ &\vdots \\ &\leq G(x,z_1,z_1) + G(z_1,z_2,z_2) + \cdots \\ &\quad + G(z_{m-1},y,y) \\ &< \varepsilon + \varepsilon + \cdots \varepsilon = m\varepsilon \end{aligned}$$
which implies $y \in S_G(x,m\varepsilon)$.

(2) Choose $y \in S_G^m\left(x,\frac{\varepsilon}{3}\right)$. For $i = 0,\ldots,m-1$ we have $z_{i+1} \in S_G\left(z_i,\frac{\varepsilon}{3}\right)$, where $z_0 =$

$x, z_1, \dots, z_m = y \in X$. From the definition of the metric d_G with (G4) and (G5), we obtain

$$\begin{aligned} d_G(z_i, z_{i+1}) = {} & G(z_i, z_{i+1}, z_{i+1}) + G(z_i, z_i, z_{i+1}) \\ \leq {} & G(z_i, z_{i+1}, z_{i+1}) \\ & + G(z_i, z_{i+1}, z_{i+1}) \\ & + G(z_{i+1}, z_i, z_{i+1}) \\ = {} & G(z_i, z_{i+1}, z_{i+1}) \\ & + G(z_i, z_{i+1}, z_{i+1}) \\ & + G(z_i, z_{i+1}, z_{i+1}) \\ < {} & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for $i = 0, \dots, m-1$. This proves the first part of the inclusion in (2). The second part can be easily seen in a similar way.

A set S in a G-metric space (X, G) is called as G-Bourbaki bounded if for every $\varepsilon > 0$ there exist a finite number of elements $x_1, x_2, \dots, x_n$ in X and $m \in \mathbb{N}$ such that

$$S \subset \bigcup_{i=1}^{n} S_G^m(x_i, \varepsilon).$$

A sequence (x_n) in (X, G) is said to be G-cofinally Cauchy if for every $\varepsilon > 0$ there exists an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that for every $i, j, k \in \mathbb{N}_\varepsilon$, $G(x_i, x_j, x_k) < \varepsilon$. It is called as G-Bourbaki-Cauchy if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $c \in S_G^m(x, \varepsilon)$ $(x \in X)$ and G-cofinally Bourbaki-Cauchy if for every $\varepsilon > 0$ there exist an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ and $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}_\varepsilon$, $x_n \in S_G^m(x, \varepsilon)$ $(x \in X)$. We have the following corollaries whose proof follow from (2) in Lemma 2.1.

Corollary 2.2. Let (X, G) be a G-metric space and S be a subset of X. The following statements are equivalent.

(1) S is G-Bourbaki bounded.

(2) S is Bourbaki bounded with respect to the metric d_G.

Corollary 2.3. Let (X, G) be a G-metric space. The following statements are equivalent.

(1) The sequence (x_n) is G-cofinally Cauchy.

(2) The sequence (x_n) is a cofinally Cauchy sequence with respect to the metric d_G.

Corollary 2.4. Let (X, G) be a G-metric space. The following statements are equivalent.

(1) The sequence (x_n) is G-Bourbaki-Cauchy.

(2) The sequence (x_n) is a Bourbaki-Cauchy sequence with respect to the metric d_G.

Corollary 2.5. Let (X, G) be a G-metric space. The following statements are equivalent.

(1) The sequence (x_n) is G-cofinally Bourbaki-Cauchy.

(2) The sequence (x_n) is a cofinally Bourbaki-Cauchy sequence with respect to the metric d_G.

Theorem 2.6. Let (X, G) be a G-metric space and S be a subset of X. Then the following statements are equivalent:

(1) S is G-Bourbaki bounded.

(2) Any countable subset of S is G-Bourbaki bounded in X.

(3) Any sequence in S has a G-Bourbaki-Cauchy subsequence in X.

(4) Any sequence in S is G-cofinally Bourbaki-Cauchy in X.

Proof. If S is G-Bourbaki bounded, then every subset of S is G-Bourbaki bounded in X. Also, if a sequence has a G-Bourbaki-Cauchy subsequence, then the sequence itself is G-cofinally Bourbaki-Cauchy. Hence it is sufficient to show that the statements (2) ⇒ (3) and (4) ⇒ (1) hold.

(2) ⇒ (3) Let (x_n) be a sequence in S. Then, the set $\{x_n : n \in \mathbb{N}\}$ is G-Bourbaki bounded from the second statement. Hence for $\varepsilon_0 = 1$ there exist $m_1 \in \mathbb{N}$ and $z_1^1, \dots, z_{l_1}^1 \in X$ such that

$$\{x_n : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{l_1} S_G^{m_1}(z_i^1, 1).$$

At least one of the G-open balls in this union, say $S_G^{m_1}(z_{i_1}^1, 1)$, contains infinitely many terms of the sequence (x_n) and so there is a subsequence (x_n^1) of the sequence (x_n) in $S_G^{m_1}(z_{i_1}^1, 1)$. For for $\varepsilon_0 = 1/2$ there exist $m_2 \in \mathbb{N}$ and $z_1^2, \dots, z_{l_2}^2 \in X$ such that

$$\{x_n^1 : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{l_2} S_G^{m_2}(z_i^2, 1/2)$$

since the set $\{x_n^1 : n \in \mathbb{N}\}$ is also G-Bourbaki bounded. Similarly, we say $S_G^{m_2}(z_{i_2}^2, 1/2)$ contains infinitely many terms of the sequence (x_n^1) and so there is a subsequence (x_n^2) of the sequence (x_n^1) in $S_G^{m_2}(z_{i_2}^2, 1/2)$. Given any $\varepsilon > 0$, there exists

$k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$. By continuing the above process for $\varepsilon_0 = \frac{1}{k_0}$, we obtain

$$\{x_n^{k_0-1} : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{l_{k_0}} S_G^{m_{k_0}}\left(z_i^{k_0}, \frac{1}{k_0}\right),$$

where $m_{k_0} \in \mathbb{N}$ and $z_1^{k_0}, \ldots, z_{l_{k_0}}^{k_0} \in X$ and there is a subsequence $(x_n^{k_0})$ of the sequence $(x_n^{k_0-1})$ in $S_G^{m_{k_0}}(z_{i_{k_0}}^{k_0}, 1/k_0)$. Hence for all $n \geq k_0$ we have $x_n^n \in S_G^{m_{k_0}}(z_{i_{k_0}}^{k_0}, \varepsilon)$ which means that the diagonal subsequence (x_n^n) is a G-Bourbaki-Cauchy subsequence of (x_n) in X.

(4) ⇒ (1) Now, let every sequence in S be G-cofinally Bourbaki-Cauchy and suppose that S is not G-Bourbaki bounded in X. Then, there is an $\varepsilon_0 > 0$ such that for any finite subset $\{z_1, \ldots, z_l\}$ of X and for all $m \in \mathbb{N}$, the union of G-open balls $S_G^m(z_1, \varepsilon_0), \ldots, S_G^m(z_l, \varepsilon_0)$ do not cover S. Construct a sequence (x_n) in X such that for every $m \in \mathbb{N}$ and fixed $x_0 \in X$, $x_m \in S \backslash S_G^m(x_i, \varepsilon_0)$, where $i = 0, \ldots, m-1$. By our assumption, this sequence is G-cofinally Bourbaki-Cauchy and therefore there exist $m_0 \in \mathbb{N}$ and an infinite subset $\mathbb{N}_{\varepsilon_0}$ of $\mathbb{N}$ such that for every $n \in \mathbb{N}_{\varepsilon_0}$, we have $x_n \in S_G^{m_0}(z_0, \varepsilon_0/2)$ ($z_0 \in X$). Choose $n_0 \in \mathbb{N}_{\varepsilon_0}$. Hence we obtain $x_n \in S_G^{2m_0}(x_{n_0}, \varepsilon_0)$ for all $n \in \mathbb{N}_{\varepsilon_0}$ which contradicts the construction of the sequence (x_n).

A G-metric space (X, G) is said to be G-cofinally complete, G-Bourbaki complete or G-cofinally Bourbaki complete if every G-cofinally Cauchy, G-Bourbaki-Cauchy or G-cofinally Bourbaki-Cauchy sequence, respectively has a G-convergent subsequence in the space.

A subset in (X, G) is said to be G-relatively compact if G-closure of that subset is G-compact. (X, G) is said to be G-uniformly locally compact if there is a $\delta > 0$ such that for every $x \in X$ the set G-Cl$(S_G(x, \delta))$ is G-compact. Also, we call a subset in (X, G) as G-locally compact, G-locally totally bounded or G-locally Bourbaki bounded if each element in this set has a G-compact, G-totally bounded or G-Bourbaki bounded neighborhood, respectively.

Since every sequence in a compact G-metric space has a G-convergent subsequence, this space is also G-Bourbaki complete. Further, a compact G-metric space is G-totally bounded and therefore it is G-Bourbaki bounded. On the contrary, if a G-metric space G-Bourbaki bounded and G-Bourbaki complete, then any sequence in this space has a G-Bourbaki-Cauchy subsequence from the preceding theorem and this subsequence has a G-convergent subsequence. Hence, this space is G-compact. Moreover, G-cofinally Bourbaki completeness is a stronger property than G-Bourbaki completeness since the class of G-cofinally Bourbaki Cauchy sequences is bigger than the class of G -Bourbaki-Cauchy sequences. Hence, we obtain the following results.

Theorem 2.7. A G -metric space is G -Bourbaki bounded and G -Bourbaki complete if and only if it is G-compact.

Theorem 2.8. A G-metric space is G-Bourbaki bounded and G-cofinally Bourbaki complete if and only if it is G-compact.

In the following theorem, a different characterization of G-Bourbaki completeness is given by G-relatively compactness of G-Bourbaki bounded subsets. Firstly, we prove a lemma which will be useful.

Lemma 2.9. The G-closure of a G-Bourbaki bounded subset in a G-metric space (X, G) is G-Bourbaki bounded.

Proof. Let S be a G-Bourbaki bounded subset in X and $\varepsilon > 0$. Then, we find some $m \in \mathbb{N}$ and $z_1, \ldots, z_l \in X$ such that

$$S \subset \bigcup_{i=1}^{l} S_G^m(z_i, \varepsilon/2).$$

Take $x \in G$-Cl(S). Then, we have $G(x, y, y) < \varepsilon/2$, where y belongs to $S_G^m(z_{i_0}, \varepsilon/2)$ for some $i_0 \in \{1, \ldots, l\}$. Hence, we can choose some points $a_1, \ldots, a_{m-1} \in X$ satisfying $G(z_{i_0}, a_1, a_1) < \varepsilon$, $G(a_1, a_2, 2) < \varepsilon, \ldots, G(a_{m-1}, y, y) < \varepsilon$. Put $a_m = y$. Hence we obtain that $x \in S_G^{m+1}(z_{i_0}, \varepsilon)$. Thus, the inclusion

$$G - Cl(S) \subset S_G^{m+1}(z_{i_0}, \varepsilon)$$

holds and therefore G-Cl(S) is G-Bourbaki bounded.

Theorem 2.10. A G-metric space (X, G) is G-Bourbaki complete if and only if every G-

Bourbaki bounded subset in X is G-relatively compact.

Proof. Let X be G-Bourbaki complete and S be a G-Bourbaki bounded subset in X. Take any sequence (x_n) in G-Cl(S). Since G-Cl(S) is also G-Bourbaki bounded, from Theorem 2.6, (x_n) has a G-Bourbaki-Cauchy subsequence. By G-Bourbaki completeness of X, it follows that this subsequence has a G-convergent subsequence. Since G-Cl(S) is G-closed, we conclude that G-Cl(S) is G-compact.

For the converse, let (x_n) be a G-Bourbaki Cauchy sequence in X and $S = \{x_n : n \in \mathbb{N}\}$. Then every sequence in S has a G-Bourbaki Cauchy subsequence and so from Theorem 2.6, S is G-Bourbaki bounded. By hypothesis, G-Cl(S) is G-compact. Hence (x_n) has a G-convergent subsequence. This implies that X is G-Bourbaki complete.

It is clear that G-compactness implies G-Bourbaki completeness and G-cofinally Bourbaki completeness. Moreover, we will prove that the property of G-uniform local compactness is stronger than these two types of G-completeness. To show that, we give the following lemma.

Lemma 2.11. Let (X, G) be a G-uniformly locally compact space. If S is a G-compact subset in X, then G-Cl$(S^{\delta/2})$ is G-compact for some $\delta > 0$.

Proof. Let X be G-uniformly locally compact space. Then there is a $\delta > 0$ such that for every $x \in X$ the set G-Cl$(S_G(x, \delta))$ is G-compact. Now, let S be a G-compact subset in X. The G-open cover $\{S_G(y, \delta/2) : y \in S\}$ of S has a finite G-subcover $\{S_G(y_i, \delta/2) : y_i \in S,\ i = 1, \dots, l\}$, that is

$$S \subset \bigcup_{i=1}^{l} S_G(y_i, \delta/2). \quad (1)$$

Now, suppose that $z \notin (S_G(y_i, \delta/2))^{\delta/2}$ for $i = 1, \dots, l$. Then for every $x \in S_G(y_i, \delta/2)$ $(i = 1, \dots, l)$, we have $z \notin S_G(x, \delta/2)$. From inclusion (1), we obtain $z \notin \bigcup\{S_G\left(x, \frac{\delta}{2}\right) : x \in S\}$ and this implies that

$$S^{\delta/2} \subset \bigcup_{i=1}^{l} (S_G(y_i, \delta/2))^{\delta/2}. \quad (2)$$

Choose $z \in (S_G(y_i, \delta/2))^{\delta/2}$ for some $i = 1, \dots, l$. Then there exists $x \in S_G(y_i, \delta/2)$ such that $z \in S_G(x, \delta/2)$. Thus, by using rectangle inequality, we have

$$G(y_i, z, z) \leq G(y_i, x, x) + G(x, z, z) < \delta,$$

that is $z \in S_G(y_i, \delta)$ and so $z \in G$-Cl$(S_G(y_i, \delta))$. Hence, we write

$$\bigcup_{i=1}^{l} (S_G(y_i, \delta/2))^{\delta/2} \subset \bigcup_{i=1}^{l} G - \mathrm{Cl}(S_G(y_i, \delta). \quad (3)$$

By combining inclusions (2) and (3), we obtain

$$G - \mathrm{Cl}\left(S^{\frac{\delta}{2}}\right) \subset \bigcup_{i=1}^{l} G - \mathrm{Cl}(S_G(y_i, \delta). \quad (4)$$

It is clear that the set in the right side of inclusion (4) is G-compact since it is the finite union of G-compact sets. We conclude that G-Cl$(S^{\delta/2})$ is G-compact since it is G-closed subset of a G-compact set.

Theorem 2.12. Let (X, G) be a G-metric space. If X is G-uniformly locally compact, then it is G-cofinally Bourbaki complete.

Proof. By hypothesis, there exits $\delta > 0$ such that the set G-Cl$(S_G(x, \delta/2))$ is G-compact for all $x \in X$ and hence we obtain from Lemma 2.11 that G-Cl$[(G\text{-Cl}(S_G(x, \frac{\delta}{2})))^{\delta/2}]$ is G-compact. The inclusion

$$G - \mathrm{Cl}[(S_G(x, \frac{\delta}{2}))^{\delta/2}] \subset G - \mathrm{Cl}[(G - \mathrm{Cl}(S_G\left(x, \frac{\delta}{2}\right)))^{\delta/2}]$$

implies that $G - \mathrm{Cl}[(S_G(x, \frac{\delta}{2}))^{\delta/2}] = G - \mathrm{Cl}(S_G^2(\mathrm{x}, \delta/2))$ is G-compact.

Again, from Lemma 2.11, the set $G - \mathrm{Cl}[(G - \mathrm{Cl}(S_G^2\left(\mathrm{x}, \frac{\delta}{2}\right)))^{\delta/2}]$ is G-compact and from the inclusion

$$G - \mathrm{Cl}[(S_G^2\left(\mathrm{x}, \frac{\delta}{2}\right))^{\delta/2}] \subset G - \mathrm{Cl}[(G - \mathrm{Cl}(S_G^2(x, \delta/2)))^{\delta/2}]$$

we have that $G - \mathrm{Cl}[\left(S_G^2\left(x, \frac{\delta}{2}\right)\right)^{\frac{\delta}{2}}] = G - \mathrm{Cl}(S_G^3(x, \frac{\delta}{2}))$ is G-compact.

Continuing this process, we observe that $G - \mathrm{Cl}(S_G^m(x, \frac{\delta}{2}))$ is G-compact for all $m \in \mathbb{N}$.

Now, let (x_n) be a cofinally Bourbaki-Cauchy sequence in X. Then, we find $m \in \mathbb{N}$ and a subset $\mathbb{N}_{\delta/2} = \{n_1 < n_2 < \dots\} \subset \mathbb{N}$ such that $x_n \in S_G^m(x, \delta/2)$ for all $n \in \mathbb{N}_{\delta/2}$ and some $x \in X$. Hence, (x_{n_k}) is a sequence in $G-\mathrm{Cl}(S_G^m(x,\delta/2))$ which is G-compact and so it has G-convergent subsequence. Thus, we conclude that X is G-cofinally Bourbaki complete.

Now, we have two lemmas to give some equivalent conditions for G-uniformly locally compactness of a generalized metric space.

Lemma 2.13. Let (X, G) be a G-metric space.

(1) If X is G-locally totally bounded and G-cofinally complete, then it is G-locally compact.

(2) If X is G-locally Bourbaki bounded and G-cofinally Bourbaki complete, then it is G-locally compact.

Proof. Let $x \in X$ and B be a G-totally bounded (G-Bourbaki bounded) neighborhood of x. Choose a G-closed ball $S_G[x, \varepsilon]$ contained in B. Since every subset of G-totally bounded (G-Bourbaki bounded) set is G-totally bounded (G-Bourbaki bounded), the G-closed ball $S_G[x,\varepsilon]$ is also G-totally bounded (G-Bourbaki bounded). Take a G-cofinally Cauchy (G-cofinally Bourbaki-Cauchy) sequence in $S_G[x,\varepsilon]$. By G-cofinally completeness (G-cofinally Bourbaki completeness) of X, we say that this sequence has a G-convergent subsequence and by G-closedness of $S_G[x,\varepsilon]$, we conclude that $S_G[x,\varepsilon]$ is G-cofinally complete (G-cofinally Bourbaki complete) and so G-complete. Hence we obtain a G-compact neighborhood of x which means X is G-locally compact.

Lemma 2.14. Let (X, G) be G-locally compact and (x_n) be a sequence in X such that G-$\mathrm{Cl}(S_G(x_n, 1/n))$ is not G-compact for all $n \in \mathbb{N}$. Then (x_n) has no G-convergent subsequence.

Proof. Suppose that there is a G-compact neighborhood of every point in X. Take a sequence (x_n) in X such that G-$\mathrm{Cl}(S_G(x_n, 1/n))$ is not G-compact for all $n \in \mathbb{N}$. We assume that the subsequence (x_{n_k}) is G-convergent to x. By hypothesis, x has a G-compact neighborhood B. Let $\varepsilon > 0$ such that $S_G(x,\varepsilon) \subset B$. We have for some $k_0 \in \mathbb{N}$ that $\frac{1}{k_0} < \frac{\varepsilon}{2}$. Also there exits $k_\varepsilon \in \mathbb{N}$ such that $G(x_{n_k}, x_{n_k}, x) < \varepsilon/2$ for all $k \geq k_\varepsilon$. Put $k' = max\{k_0, k_\varepsilon\}$. We choose y from $S_G(x_{n_{k'}}, 1/n_{k'})$. Then by using rectangle inequality, we obtain

$$G(x,y,y) \leq G\left(x, x_{n_{k'}}, x_{n_{k'}}\right) + G\left(x_{n_{k'}}, y, y\right) < \varepsilon$$

and so we have $y \in S_G(x,\varepsilon)$ which yields the inclusion

$$G-\mathrm{Cl}(S_G(x_{n_{k'}}, 1/n_{k'})) \subset S_G(x,\varepsilon) \subset B.$$

Since G-closed subset of a G-compact set is G-compact $G-\mathrm{Cl}(S_G(x_{n_{k'}}, 1/n_{k'}))$ is G-compact which is a contradiction. Hence the sequence (x_n) constructed in the above way has no G-convergent subsequence.

Theorem 2.15. Let (X, G) be a G-metric space. Then the following statements are equivalent:

(1) X is G-locally totally bounded and G-cofinally complete.

(2) X is G-locally Bourbaki bounded and G-cofinally Bourbaki complete.

(3) X is G-uniformly locally compact.

Proof. Firstly, let X be G-locally totally bounded and G-cofinally complete space. From Lemma 2.13, X is G-locally compact. Now, we assume that X is not G-uniformly locally compact. Then there is a point x_n in X such that $G-\mathrm{Cl}(S_G(x_n, 1/n))$ is not G-compact for every $n \in \mathbb{N}$. Hence from Lemma 2.14, we say that the sequence (x_n) has no G-convergent subsequence. We can choose a sequence (w_k^n) in $G-\mathrm{Cl}(S_G(x_n, 1/n))$ for each $n \in \mathbb{N}$ such that it has no G-convergent subsequence. Let $\mathbb{N} = \bigcup_{n=1}^{\infty} K_n$, where K_n is infinite subset of $\mathbb{N}$ and $K_i \cap K_j = \emptyset$ for distinct $i, j \in \mathbb{N}$. Construct a sequence (θ_k) in the way that $\theta_k = w_k^n$ if $k \in K_n$. Given any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \frac{\varepsilon}{9}$. Since $w_k^{n_\varepsilon} \in G-\mathrm{Cl}(S_G(x_{n_\varepsilon}, 1/n_\varepsilon))$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} G(w_k^{n_\varepsilon}, x_{n_\varepsilon}, x_{n_\varepsilon}) &\leq G(w_k^{n_\varepsilon}, z, z) + G(z, x_{n_\varepsilon}, x_{n_\varepsilon}) \\ &\leq G(w_k^{n_\varepsilon}, z, z) + 2G(x_{n_\varepsilon}, z, z) \\ &< \frac{3}{n_\varepsilon} \end{aligned}$$

for some $z \in S_G(w_k^{n_\varepsilon}, 1/n_\varepsilon) \cap S_G(x_{n_\varepsilon}, 1/n_\varepsilon)$. Then for $i, j, k \in K_{n_\varepsilon}$, the inequality

$$\begin{aligned} G(\theta_i,\theta_j,\theta_k) &= G(w_i^{n_\varepsilon},w_j^{n_\varepsilon},w_k^{n_\varepsilon}) \\ &\leq G(w_i^{n_\varepsilon},x_{n_\varepsilon},x_{n_\varepsilon}) \\ &+ G(x_{n_\varepsilon},w_j^{n_\varepsilon},w_k^{n_\varepsilon}) \\ &\leq G(w_i^{n_\varepsilon},x_{n_\varepsilon},x_{n_\varepsilon}) \\ &+ G(w_j^{n_\varepsilon},x_{n_\varepsilon},x_{n_\varepsilon}) \\ &+ G(x_{n_\varepsilon},x_{n_\varepsilon},w_k^{n_\varepsilon}) \\ &< \frac{3}{n_\varepsilon}+\frac{3}{n_\varepsilon}+\frac{3}{n_\varepsilon}< \varepsilon \end{aligned}$$

holds which means that (θ_k) is G-cofinally Cauchy sequence in X. However it has no G-convergent subsequence which contradicts the fact that X is G-cofinally complete. Thus, X must be G-uniformly locally compact. Secondly, let X be G-uniformly locally compact space. Then we have that $S_G(x,\delta)$ is G-Bourbaki bounded neighborhood of x owing to the fact that δ is a positive real number such that $G-\mathrm{Cl}(S_G(x,\delta))$ is G-compact and so it is G-Bourbaki bounded for every $x\in X$. Hence, it follows that G-uniformly locally compactness of X implies both G-locally Bourbaki boundedness of X and G-cofinally Bourbaki completeness of X which is proved in Theorem 2.12. Lastly, let X be G-locally Bourbaki bounded and G-cofinally Bourbaki complete space. Again, from Lemma 2.13, X is G-locally compact. Then every point in X has a G-compact neighborhood and thus G-totally bounded neighborhood which means X is G-locally totally bounded. Moreover, as we mention before that G-cofinally Bourbaki completeness implies G-cofinally completeness. Consequently, if X is G-locally Bourbaki bounded and G-cofinally Bourbaki complete, then it is G-locally totally bounded and G-cofinally complete.

REFERENCES

[1] M. Abbas, T. Nazir and D. Doric, "Common fixed point of mappings satisfying (E.A) property in generalized metric spaces," *Applied Mathematics and Computation,* vol.218, pp. 7665-7670, 2012.

[2] M. Abbas, T. Nazir and P. Vetro, "Common fixed point results for three maps in G-metric spaces", *Filomat*, vol.25, no.4, pp.1-17, 2011.

[3] M. Abbas and B. E. Rhoades, "Common fixed point results for noncommuting mappings without continuity in generalized metric spaces," *Applied Mathematics and Computation,* vol.215, pp.262-269, 2009.

[4] T. V. An, N. V. Dung and V. T. L. Hang, "A new approach to fixed point theorems on G-metric spaces," *Topology and its Applications*, vol.160, no.12, pp.1486-1493, 2013.

[5] G. Beer, "Between compactness and completeness," *Topology and its Applications,* vol.155, no.6, pp.503-514, 2008.

[6] B. S. Choudhury, and P. Maity, "Coupled fixed point results in generalized metric spaces," *Mathematical and Computer Modelling,* vol.54, pp.73-79, 2011.

[7] M. I. Garrido, and A. S. Merono, "New types of completeness in metric spaces," *Annales Academiæ Scientiarum Fennicæ Mathematica,* vol.39, pp.733–758, 2014.

[8] M. İlkhan and E.E. Kara, "Uniform continuity and Cauchy continuity in G-metric spaces," *Journal of Inequalities and Special Functions,* vol.8, no.3, pp.59-68, 2017.

[9] A. Kaewcharoen, "Common fixed point theorems for contractive mappings satisfying *Φ*-maps in G-metric spaces," *Banach Journal of Mathematical Analysis,* vol.6, no.1, pp.101-111, 2012.

[10] S. A. Mohiuddine and A. Alotaibi, "Some results on a triplet fixed point for nonlinear contractions in partially ordered G-metric spaces," *Fixed Point Theory and Applications,* vol.2012, no.179, 12 pages, 2012.

[11] F. Moradlou, P. Salimi, and P. Vetro, "Some new extensions of Edelstein-Suzuki-type fixed point theorem to G-metric and G-cone metric spaces," *Acta Mathematica Scientia,* vol.33, no.4, pp.1049- 1058, 2013.

[12] Z. Mustafa, F. Awawdeh, and W. Shatanawi, "Fixed Point Theorem for Expansive Mappings in G-Metric Spaces," *International Journal of Contemporary Mathematical Sciences,* vol.5, no.50, pp.2463-2472, 2010.

[13] Z. Mustafa, H. Aydi and E. Karapınar, "On common fixed points in G-metric spaces using (E.A) property," *Computers & Mathematics with Applications,* vol.64, pp.1944-1956, 2012.

[14] Z. Mustafa, M. Khandagji, and W. Shatanawi, "Fixed point results on complete G-metric spaces," *Studia Scientiarum Mathematicarum Hungarica,* vol.48, no.3, pp.304-319, 2011.

[15] Z. Mustafa, and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis,* vol.7, no.2, pp.289-297, 2006.

[16] M. Öztürk and M. Başarır, "On some common fixed point theorems with φ-maps on G-cone metric spaces," *Bulletin of Mathematical Analysis and Applications,* vol.3, no.1, pp.121-133, 2011.