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A Remark on the Form of Accumulation Functions in Economic Growth Models

Ekonomik Büyüme Modelleri Birikim Denklemlerinin Formu Üzerine Bir Hatırlatma

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ABSTRACT

This study is a remark designed to underline the importance of using the right form of accumulation functions in economic growth models. After appearance of endogenous growth models in the growth literature, exclusion of a counter-force in the accumulation function, though required by the nature of the variable, is started to be made more frequently. This (mis-) behavior is due to now a common knowledge that a growth model must rely on non-diminishing returns to a factor of production in order to generate endogenous growth. This rule, however, can lead to the following potentially misleading assumption: diminishing marginal productivity to each factor of production-given that there is no other source of long run growth-is sufficient for generating (the true) steady state equilibrium (at levels). In this work, we heuristically argue that an accumulation function with no counter-moving force, though it is required by the nature of the variable, with or without diminishing returns, may bias the results of the model.

Keywords: Accumulation function, counter-moving force, steady state, differential equations, economic growth.

ÖZET

Bu çalışma ekonomik büyüme modellerinde değişkenin doğasının gerektirdiği doğru birikim denkleminin önemini göstermek için tasarlanmıştır. Ekonomik büyüme alanyazınında içsel büyüme modellerinin ortaya çıkmasından sonra birikim denklemlerinde, kuramsal olarak gerekiyor olsa bile, karşı-yönde hareket eden bir kuvvet daha çok ihmal edilmeye başlandı. Bu (yanlış) varsayım içsel büyüme elde etmek için en azından bir üretim faktöründe azalan marjinal verim kanununun ihmal edilmesi gerçeğine dayanıyordu. Ne var ki bu kural aşağıdaki potansiyel yanlış varsayıma yol açabilir: azalan marjinal verim doğru durağan durum dengesi sonucunu elde etmek için yeterlidir. Bu çalışmada, değişkenin doğası gereği gerekli iken, birikim denkleminde karşı-yönde hareket eden kuvveti ihmal etmenin, azalan verim kanunu sağlasın ya da sağlamasın, modelin sonuçlarını (yanıltıcı yönde) değiştirebileceğini iddia ediyoruz.

Anahtar Kelimeler: Birikim fonksiyonu, karşı-yönde hareket eden kuvvet, diferansiyel denklemler, ekonomik büyüme.

1. INTRODUCTION

It is interesting to document that there is established agreement on the form of production function in economic growth models but no significant agreement on the form of accumulation functions. However, the role played by the form of accumulation function is as important as the form of production function. In that respect, heuristically speaking, we argue that the form of accumulation function is underestimated, if not ignored, in many growth models. In this remark, we underline the importance of using the true form of accumulation function in an economic growth model through examples. By the phrase "the true form", we mean "the form that the nature of the accumulated stock variable requires". Two good examples will elaborate what we mean by the nature of a variable. Let us take the most common stock variable used in growth models, namely, physical capital stock. From the nature of the physical capital stock, we know that it is subject to depreciation. Hence, the true form of capital accumulation function should not include only how the capital is added to the stock, but also how the stock does depreciate. On the other hand, aggregate knowledge stock, by its nature, is not subject to depreciation or something similar to that. Hence, the form of knowledge accumulation function is qualitatively different than the physical capital function due to differences in the nature of the two variables: the former has a counter-force in its accumulation function and the latter does not.

Since Solow (1956), it is well-known that diminishing returns to each factor of production is necessary, but not in itself sufficient, for stationary equilibrium (at levels) in a dynamic model. In particular, an equation of motion must include 'counter-moving forces' in addition to diminishing returns in order to generate stationary equilibrium (see figures I-III and V in Solow (1956), emphasizing this point). We observe that the importance of this requirement is sometimes overlooked in the literature. We have a

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heuristic explanation for this tendency to overlook this factor: after the revival of economic growth theory in mid-1980s, the focus was on endogenous growth mechanisms, which, by and large, were based on non-diminishing returns to a factor of production. This fact may have led to the mis-conclusion that diminishing marginal productivity to each factor of production—given that there is no other source of long run growth—is sufficient for generating steady state equilibrium at levels. In this short paper, we first identify the generic forms of accumulation functions that yield steady state value or growth rate. Next, as a particular example, we use Lucas (1988) framework to show that the stationary equilibrium of a dynamic model necessarily requires a counter-moving force in equation(s) of motion, in addition to diminishing marginal return to individual factors of production. Choosing Lucas (1988) has no particular aim other than its fit to the purposes of this paper. That is, this paper is not a critique of Lucas (1988) paper. The organization of this work is as follows. Section 2 presents the generic form of accumulation functions that generate steady-state results and discusses several versions of the human capital accumulation function in Lucas (1988) to show that (i) relying on diminishing marginal returns to each factor of production is not sufficient to ensure long run equilibrium, (ii) it is important not to ignore incorporating a counter-moving force in the accumulation function, if the nature of the variable requires, even when there are no diminishing-returns. Section 3 concludes the note.

2. FEW DEMONSTRATIONS

An accumulation function in economic growth theory defines the dynamics of a stock variable. For example, the fundamental equation of growth, cf., Solow (1956), defines a rigorous physical capital accumulation rule, $\dot{K} = sY\text{-}\delta K$, where K is physical capital stock, s is exogenous saving rate, Y is production function, δ is constant depreciation rate and a dot on top of a variable defines its continuous time derivative.[1] Some other stock variables widely used in the growth literature are human capital and knowledge. The generic form of most accumulation functions in economic growth theory is as follows:[2,3]

$$\dot{X}=f(X)-g(X) \qquad (1)$$

In (1), accumulation of the stock of variable X, which is function of time, depends on two counter-moving forces, $f(X)$ and $g(X)$. We presume that the following conditions are satisfied for having a stable long run equilibrium value or growth rate:

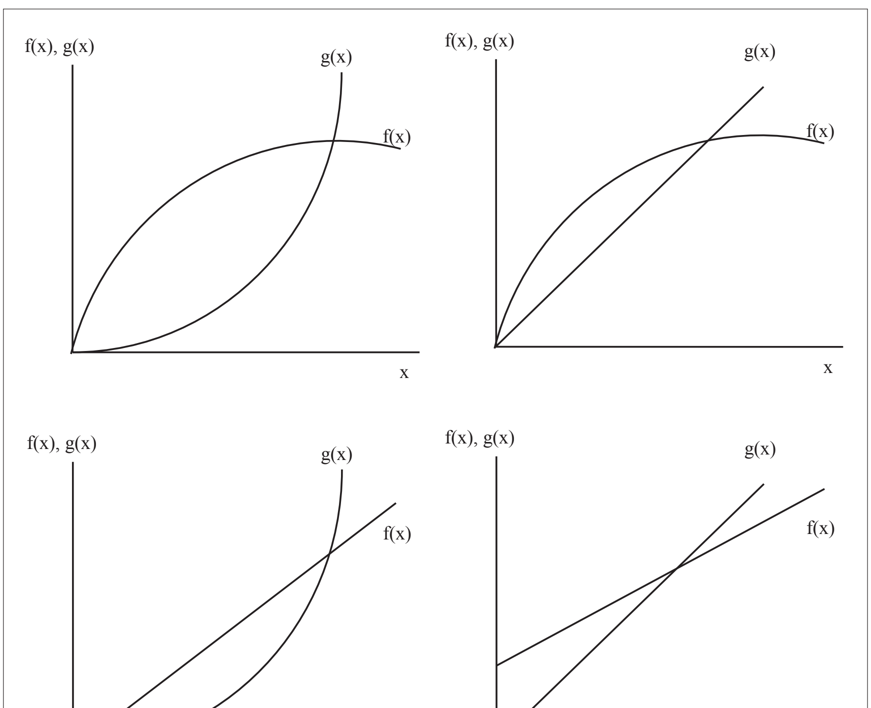

Figure 1: Possible Scenarios GeneratingLong-run Equilibrium

(i) $f'(X)>0$ and $f''(X)\leq 0$; $g'(X)>0$ and $g''(X)\geq 0$

(ii) $f(0)\geq g(0)$

(iii) $g'(X)>f'(X)$ if $f''(X)=0$ and $g''(X)=0$

Condition (i) implies that $f(X)$ is increasing at constant or decreasing rate and that $f(X)$ is increasing at constant or increasing rate. Condition (ii) implies that either both functions must start from origin or initial value of $f(\cdot)$ must be greater than the initial value of $g(\cdot)$. Conditions (i) and (ii) must be satisfied for a long run equilibrium. Condition (iii) is necessary for a solution only if $f''(X)=0$ and $g''(X)=0$ are true simultaneously. Hence, for a single variable accumulation function all the illustrations in figure 1 are possible scenarios that generate long run equilibrium (at level or growth rate):[4]

This paper argues that results of a growth model may change qualitatively, whenever the accumulation function is defined as

$$\dot{X}=f(X), \tag{2}$$

though the nature of the variable requires (1) as the true accumulation function.

Let us now show the critical role played by a counter-moving force in a plain accumulation function, which cannot be secured by the very existence of diminishing returns. To this end, let us assume that we have the following accumulation function, which you may consider the very first part of the fundamental equation of growth of Solow for a unitary saving rate:

$$\dot{h}=h^{\xi} \qquad 0<\xi<1 \tag{3}$$

The solution of this simple differential equation would yield

$$h(t)=[(1-\xi)\cdot t+h_0^{1-\xi}]^{\frac{1}{1-\xi}} \tag{4}$$

where h_o is the initial value of stock of human capital. Notably, $Lim_{t\to\infty}\ h(t)=\infty$, that is, there is no steady state at levels to the differential equation. One may easily show that this differential equation has a zero growth rate at steady state, though it approaches infinity at levels. To see this, write (3) in growth form: $\frac{\dot{h}}{h}=h^{\xi-1}$. Substituting the solution of $h(t)$ from (4) into this growth equation, one can easily show that $Lim_{t\to\infty}\frac{\dot{h}}{h}=0$. Hence, for single-equation accumulation functions, diminishing marginal returns ensures a stationary state, but not a long run equilibrium at level.

Let us now assume that we have a modified (3), in which there is a counter-moving force to h. For matter of illustration, if h is human capital per person, then $(n+\delta)$ is the effective depreciation of human capital (you may also consider this the complete Solovian fundamental equation of growth for a unitary saving rate):

$$\dot{h}=h^{\xi}-(n+\delta)h \tag{5}$$

where $0<\delta<1$ is the depreciation rate. The solution of the problem would yield

$$h(t)=\left[\frac{1}{n+\delta}+\left(h_0^{1-\xi}-\frac{1}{n+\delta}\right)e^{-(n+\delta)(1-\xi)t}\right]^{\frac{1}{1-\xi}} \tag{6}$$

which has a stationary state value at infinity, $Lim_{t\to\infty}h(t)=\left(\frac{1}{n+\delta}\right)^{\frac{1}{1-\xi}}\equiv h_{ss}$, where a subscript ss means steady-state. As one may note, the two results in (4) and (6) are *qualitatively different*. The latter has a steady-state at level and in growth rate; the former has only steady state growth rate.

Complexity increases when one works a growth model with more than one accumulation functions. In order to illustrate this, we again choose Lucas (1988) framework. In his paper, Lucas (1988) uses the following human capital accumulation function, which is the source of endogenous growth in his model:

$$\dot{h}=a_h(1-u)h \tag{7}$$

where h is human capital stock per person, a_h is productivity of education sector, and u is the share of human capital employed in final good production. Notably, there is no diminishing marginal returns to h in the accumulation function for generating endogenous growth. Suppose that social planner's problem is maximization of $U=\int_0^{\infty}e^{-(\rho-n)\cdot t}\cdot\frac{c^{1-\theta}-1}{1-\theta}dt$ subject to $\dot{K}=K^{\alpha}(uhL)^{1-\alpha}-c\cdot L$ and (7) and corresponding transversality conditions, where K is physical capital stock, L is labor stock and grows at rate n , and c is consumption per capita (all lower-case letters correspond to per capita versions of a variable). One may easily show that endogenous growth rate at steady state would be $g=\frac{1}{\theta}(a_h+n-\rho)$, and that $\hat{c}_{ss}=\hat{y}_{ss}=\hat{k}_{ss}=\hat{h}_{ss}=g$.

Instead, if the human capital accumulation function were defined with a counter-moving force, say,

$$\dot{h}=a_h(1-u)h-(n+\delta)h, \tag{8}$$

where δ is the decay rate of human capital, the endogenous growth rate at steady state would be

$g = \frac{1}{\theta}(a_h - \delta - \rho)$. As long as $a_h > \delta + \rho$, there is no qualitative difference between the original and modified models. Heuristically speaking, this may be the reason why the counter-moving force in the accumulation function has been ignored in many endogenous growth models.

However, whenever there is diminishing-returns to the accumulation function, it is of considerable importance whether there is a counter-moving force or not. For matter of illustration, let us continue with the Lucas (1988) framework. Suppose that human capital accumulation function is $\dot{h} = a_h (1-u) h^{\xi}$ versus $\dot{h} = a_h (1-u) h^{\xi} - (n+\delta) \cdot h$. In the first case, in which there is decreasing returns to human capital accumulation but no counter force in the accumulation function, we find the model implies that the human capital accumulation sector (i.e. the education sector) will disappear in the long run from the model and that all existing human capital will be employed in private production.[5] Given that human capital does not depreciate, and that the returns to the education sector diminish, it is indeed optimal for the social planner (decentralized solution would not be different) to employ all human capital factors of production in final good production. In the second scenario, it is very simple to show that there is stationary state of h and u and other variables of the model. In particular, one can show that $Lim_{t\to\infty} h(t) = \left(\frac{a_h \cdot (1-u_{SS})}{n+\delta}\right)^{\frac{1}{1-\xi}}$ and that $Lim_{t\to\infty} u(t) = \frac{(\rho+\delta)-\xi(n+\delta)}{(\rho+\delta)-(1-\xi)(n+\delta)}$ ($u>0$ implies $\xi > ½$). Notably, the two scenarios yield *qualitatively different* results.

We would like to give one more example, which has more serious implications, when a counter-force is ignored in the definition of accumulation function, though it is required by the nature of the variable. Our example will be the seminal paper by Dasgupta and Heal (1974). Those familiar with this paper would know that the social planner's solution to their problem leads to the famous $\frac{\partial F_R}{\partial t}\frac{1}{F_R} = F_K$ equation, where R is the extraction quantity of a non-renewable resource, K is physical capital, F_M indicates marginal physical productivity for $M=K,R$. This relationship leads to the differential equation $\dot{X} = X^{\alpha}$, where $X = \frac{K}{R}$ for the Cobb-Douglas technology. This differential equation behaves very much like (3). If there were depreciation in the model, however, the differential equation would be $\dot{X} = X^{\alpha} - \delta \cdot X$, which behaves very much like (5).[6] Clearly, the two results are again qualitatively different. To see this, and the significance of the *qualitative difference* in the results with and without a counter-force in Dasgupta and Heal (1974) framework, let us look at the decentralized solution of the same framework. If the original Dasgupta and Heal (1974) model were modeled in a decentralized solution, the reader would see that $\frac{\partial F_R}{\partial t}\frac{1}{F_R} = F_K$ is nothing but the (wrong) Hotelling's Rule: $\frac{\dot{q}}{q} = r$, where q is the price of non-renewable resource and r is the rental rate of capital.[7] If there is depreciation in the model, the Hotelling Rule becomes $\frac{\dot{q}}{q} = r - \delta$, where the right hand side is the real interest. In the first scenario, one finds q goes to infinity, as in (4). In the second scenario, q converges to a constant value, as in (6). Clearly, the two results are qualitatively different, and it must be true that one of them gives the wrong conclusions.

Our point is that an accumulation function must be modeled as the nature of the accumulated variable requires. If a model is an endogenous growth model, ignoring a counter-force in accumulation functions, though the nature of the variable requires, may not necessarily cause any distortion in results. If, however, a growth model relies on diminishing returns to the accumulating factor and if the counter-force is ignored, though the nature of the accumulated variable requires one, then there is a higher risk of misleading results. In conclusion, to ensure accurate results, one should pay attention to the accumulation nature of the variable, especially whenever there is no endogenous growth.

3. CONCLUDING REMARKS

Differential equations are very sensitive to changes. Accumulation functions are no exception. It is important to define an accumulation function of a variable as its nature requires. The general understanding in growth theory is that defining a diminishing marginal productivity is sufficient for generating steady state results. This perception however is not fully correct. If the nature of the variable requires, one should always add a counter-moving force into accumulation functions in order to ensure that the model has a steady state not only at growth rates but also at levels. Ignoring this fact risks the distortion of results.

END NOTES

[1] The Ramsey (1928) version of the same rule is $\dot{K} = Y - C - \delta K$, where C is consumption.

[2] All variables are function of time, unless otherwise stated.

[3] Two points deserve explanation in (1). First, the form of (1) is not necessarily the one that exhausts all types of accumulation functions but, heuristically speaking, it is the most common form. Second, we could have assumed that X is a vector of variables rather than a single variable. It is beyond the aims of this paper to work out a purely mathematical version of accumulation functions.

[4] Without loss of generality, we assume that $f(0) = g(0) = 0$ in all but the last illustration.

[5] See Annex A for a formal proof.

[6] In this work, we refrain to show the details of derivations. Interested readers may refer to Gaitan, Tol and Yetkiner (2004) for more details.

[7] Recall that Hotelling's rule is a non-arbitrage condition between non-renewable resource and financial assets, stating that the nonrenewable is also an asset and therefore its (real) price must grow at the real interest rate.

[8] I am grateful to Adriaan van Zon, Thomas Ziesemer, Ceyhun Elgin and the anonymous referee for their helpful comments. All remaining errors are mine.

REFERENCES

Dasgupta, P. ve Heal, G. (1974) "The Optimal Depletion of Exhaustible Resources" *Review of Economic Studies,* 41:3-28.

Gaitan, B., Tol, R.S.J ve Yetkiner, İ.H. (2004) "The Hotelling's Rule Revisited in a Dynamic General Equilibrium Model" Hamburg University Research Unit Sustainability and Global Change Working Papers, No: FNU-44.

Lucas, R.E.Jr. (1988) "On the Mechanics of Economic Development" *Journal of Monetary Economics,* 22:3-42.

Ramsey, F.P. (1928) "A Mathematical Theory of Saving" *Economic Journal,* 38(152):543-559.

Solow, R. (1956) "A Contribution to the Theory of Economic Growth" *Quarterly Journal of Economics,* 70:65-94.

Annex A

Solution of Lucas Model when $\dot{h} = a_h \cdot (1-u) \cdot h^{\xi}$

Suppose that human capital education sector is defined as $\dot{h} = a_h \cdot (1-u) \cdot h^{\xi}$, where a_h is productivity of education sector, $0 < \xi < 1$ and the rest of the model is same as Lucas (1988), except that we ignore externality for matter of simplicity. The social planner's solution of the model would imply the following Hamiltonian:

$$H = e^{-(\rho-n)\cdot t} \cdot \frac{C^{1-\theta}-1}{1-\theta} + \lambda \cdot \{Y - c \cdot L\} + \mu \cdot \{a_h \cdot (1-u) \cdot h^{\xi}\} \qquad \text{(A.0)}$$

First order conditions would be as follows:

$$\frac{\partial H}{\partial c} = 0 \Rightarrow e^{-(\rho-n)\cdot t} \cdot c^{-\theta} + \lambda \cdot \{-L\} \qquad \text{(A.1)}$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow \lambda \cdot \left\{(1-\alpha)\frac{Y}{u}\right\} + \mu \cdot \{a_h \cdot (-1) \cdot h^{\xi}\} = 0 \qquad \text{(A.2)}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial K} \Rightarrow \dot{\lambda} = -\lambda \cdot \left\{\alpha \frac{Y}{K}\right\} \qquad \text{(A.3)}$$

$$\dot{\mu} = -\frac{\partial H}{\partial h} \Rightarrow \dot{\mu} = -\left[\lambda \cdot \left\{(1-\alpha)\frac{Y}{h}\right\} + \mu \cdot \{\xi \cdot a_h \cdot (1-u) \cdot h^{\xi-1}\}\right] \qquad \text{(A.4)}$$

$$\dot{K} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{K} = K^{\alpha}(u \cdot h \cdot L)^{1-\alpha} - c \cdot L \qquad \text{(A.5)}$$

$$\dot{h} = \frac{\partial H}{\partial \mu} \Rightarrow \dot{h} = a_h \cdot (1-u) \cdot h^{\xi} \qquad \text{(A.6)}$$

(A.6) implies $\hat{h}_{ss} = a_h \cdot (1-u_{ss}) \cdot h_{ss}^{\xi-1}$ at steady state, if there is one. Time derivative of both sides implies $\frac{d}{dt}(\hat{h}_{ss}) \equiv 0$ and $\frac{d}{dt}\left(a_h \cdot (1-u_{ss}) \cdot h_{ss}^{\xi-1}\right) = 0 \Rightarrow \dot{u}_{ss} = (\xi-1)(1-u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}}$. (A.2) yields $\lambda \cdot (1-\alpha) \cdot Y = \mu \cdot a_h \cdot u \cdot h^{\xi}$. If we use this information in (A.4), we get $-\frac{\dot{\mu}}{\mu} = h^{\xi-1} \cdot a_h \cdot [(1-\xi) \cdot u + \xi]$. Time derivative of this equation at steady state, if there is one, implies $[(1-\xi) \cdot u_{ss} + \xi] \cdot \frac{\dot{h}_{ss}}{h_{ss}} = \dot{u}_{ss}$. Using $\dot{u}_{ss} = (\xi-1)(1-u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}}$ above implies,

$$[(1-\xi) \cdot u_{ss} + \xi] \cdot \frac{\dot{h}_{ss}}{h_{ss}} = (\xi-1)(1-u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}} \Rightarrow$$

$$\frac{\dot{h}_{ss}}{h_{ss}} = 0$$

Suppose that this is true. Then, $\frac{\dot{u}_{ss}}{u_{ss}} = 0$ must also hold. Under $\frac{\dot{h}_{ss}}{h_{ss}} = 0$, (A.6) implies $u_{ss} = 1$ or $h_{ss} = 0$. As $h_{ss} = 0$ is trivial, the model implies that education sector will disappear in the long run and all human capital will be employed in private production, that is, $u_{ss} = 1$. That would then imply $\hat{Y}_{ss} = \hat{K}_{ss} = n$ due to production function $Y = K^{\alpha}(uhL)^{1-\alpha}$. (A.5) implies $\hat{c}_{ss} = 0$ or $\hat{C}_{ss} = n$. (A.1) then implies $\hat{\lambda}_{ss} = -\rho$. (A.3) requires $\frac{Y_{ss}}{K_{ss}} = \frac{\rho}{\alpha}$. Log-differentiation of (A.2) implies $\hat{\mu}_{ss} = \hat{\lambda}_{ss} + n = -(\rho - n)$. This information used in (A.4) implies $h_{ss} = \left(\frac{a_h}{\rho - n}\right)^{\frac{1}{1-\xi}}$. Since boundedness from above of overall utility necessarily implies $\rho > n$, $h_{ss} > 0$. Using $-\hat{\lambda}_{ss} = \alpha \frac{Y_{ss}}{K_{ss}}$ from (A.3) in (A.5) yields $\frac{Y_{ss}}{K_{ss}} = \frac{C_{ss}}{K_{ss}}$. One can easily show that $k_{ss} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \left(\frac{a_h}{\rho - n}\right)^{\frac{1}{1-\xi}}$ from production function, where $k = \frac{K}{L}$. The rest follows straightforwardly.