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Tanβ Bounds in Renormalizable SU(5) Grand Unified Theories

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Abstract − In this article non-supersymmetric, renormalizable grand unified theories based on the Special Unitary Group of degree 5 (SU(5)) gauge group are considered. These models extend the Georgi-Glashow paradigm by the presence of a Higgs field transforming as the 45 representation. The presence of a second Higgs doublet in this representation ensures realistic mass matrices for the charged fermions of the Standard Model. Perturbativity of the Yukawa couplings of down-type quarks and charged leptons up to the unification scale require the tanβ parameter, which is proportional to the ratio of the vacuum expectation values of the neutral components of the two Higgs doublets, to be constrained as follows: $3.4 \times 10^{-3} \leq \tan\beta \leq 1.2 \times 10^{2}$. A unification scenario in which the physical second Higgs doublet is at 1 TeV, the weak doublet scalar gluons are decoupled from the theory below the unification scale, and the unification is achieved at 1.9×10^{14} GeV is presented, too. Whereas the weak and color triplet scalar leptoquark must have slightly suppressed couplings to up-type quarks, because it has a mass of 1.5×10^{10} GeV.

Keywords – *Grand unified theories, beyond standard model, renormalizable SU(5) unification*

1. Introduction

Half a century after its conception the simplest grand unified theory (GUT) which is based on the gauge group special unitary group of degree 5 (SU(5)) [1] is still relevant in the exploration of physics beyond the standard model (SM). Not only does it unify the SM interactions, but also the matter particles, namely quarks and leptons, that interact through these interactions. It provides the basic framework in which the SM could be extended.

The mass spectrum of charged fermions, and, in particular, the equality of the masses at the unification scale of the down-type quark and charged lepton of each generation predicted by the original model fails to reproduce the observed ratios even for the third family. The minimal extension of the Georgi-Glashow (GG) model that leads to a realistic mass spectrum of charged fermions and maintains the renormalizability of the theory requires addition of scalars transforming as the 45 representation of SU(5) [2]. This enlarged spectrum of particles also ensures the unification of gauge couplings which is in agreement with the experimental values of physical quantities at the weak scale.

A second Higgs doublet with the same quantum numbers as those of the Higgs doublet of the SM resides in this additional representation. The neutral component of this Higgs doublet acquires a vacuum expectation value (VEV) after electroweak (EW) symmetry breaking as well. Hence, a new parameter related to the ratio of the VEVs of the neutral components of the Higgs doublets in the 5 and 45 representations and which is represented as $\tan\beta \propto v_{45}/v_5$ is introduced to the theory.

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Any GUT theory that claims to give a correct description of nature must be able to account for neutrino oscillations and thus accommodate nonzero neutrino masses. Neutrino mass matrices calculated in renormalizable SU(5) GUTs as, for example, in the ultraviolet completion of the Zee model [3] will inevitably involve the tanβ parameter. Furthermore, there are results in the literature for the mass of the lightest neutrino [4-6] which predict it to scale with $\tan^4\beta$. These facts motivate us to constrain the values this otherwise arbitrary parameter can take. This is the major objective of the present article.

Renormalizable SU(5) GUTs are reviewed in Section II, then, it is shown that unification at a sufficiently high scale is possible in this model, and lower and upper bounds for the tanβ parameter are obtained in Section III. Conclusions are drawn in Section IV.

2. Renormalizable SU(5) GUTs

In this section the Higgs sector of non-supersymmetric, renormalizable SU(5) grand unified theories is reviewed paying particular attention to their Higgs sector, unification and proton decay constraints in such models, mass matrices of charged fermions, and how they can be extended to accommodate massive neutrinos.

2.1 Higgs Sector

In SU(5) GUT, the left-handed matter particles of the SM fit in the two smallest irreducible representations of SU(5) as shown below in their decomposition under the SM subgroup.

$$\bar{5}_{F,a} \equiv (d^C, L^T)_a = \left(\bar{3}, 1, \frac{1}{3}\right)_a \oplus \left(1,2,-\frac{1}{2}\right)_a$$

$$10_{F,a} \equiv (u^C, Q, e^C)_a = \left(\bar{3}, 1, -\frac{2}{3}\right)_a \oplus \left(3,2,\frac{1}{6}\right)_a \oplus (1,1,1)_a$$

In the above equations a=1,2,3 is the generation (or family) index. The right-handed anti-particles of the above, then, reside in the conjugate representations, i.e., $5_{F,a}$ and $\overline{10}_{F,a}$.

Spontaneous Symmetry Breaking (SSB) at the super-massive scale is accomplished through the acquisition of a Vacuum Expectation Value (VEV) by the Higgs multiplet Φ_{24} among those residing in the (real) adjoint representation of SU(5), that is, 24_H. The scalar multiplets comprising the adjoint representation are given below.

$$24_H \equiv \left(\Phi_8, \Phi_3, \Phi_{(3,2)}, \Phi_{(\bar{3},2)}, \Phi_{24}\right) = (8,1,0) \oplus (1,3,0) \oplus \left(3,2,-\frac{5}{6}\right) \oplus \left(\bar{3},2,\frac{5}{6}\right) \oplus (1,1,0)$$

The VEV of the adjoint Higgs field which lies in the SM singlet direction is as follows:

$$\langle 24_H \rangle = \frac{v_{24}}{\sqrt{30}} \mathrm{diag}(2,2,2,-3,-3)$$

The above VEV breaks the SU(5) gauge symmetry down to that of the SM subgroup, that is, $SU(5) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y$. As a result of this GUT scale symmetry breaking, the gauge bosons corresponding to broken symmetry generators become massive with a mass M_{GUT} related to the energy of the GUT scale symmetry breaking, v_{24}, as in the equation below:

$$M_{\text{GUT}} = v_{24}\sqrt{\frac{5\pi\alpha_{\text{GUT}}}{3}}$$

This mass is identified with the GUT scale, M_V , i.e., $M_V = M_{\text{GUT}}$. $\alpha_{\text{GUT}} \equiv \alpha(M_{\text{GUT}})$ is the common value of the gauge couplings at the unification or GUT scale. However, at this stage, charged fermions of the SM are still massless. They acquire mass only after the subsequent SSB that occurs at the Electroweak (EW) scale. This second stage SSB proceeds via the acquisition of a VEV by the neutral components of the Higgs doublets present in the fundamental, 5_H, and 45_H representations of SU(5).

In addition to the first Higgs doublet, H_1, the scalar leptoquark, $S_{1,5}^*$, which mediates proton decay at tree level lives in the fundamental representation as shown in its decomposition below.

$$5_H \equiv \left(S_{1,5}^*, H_1\right) = \left(3,1,-\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$$

The VEV acquired by the neutral component of the first Higgs doublet at the EW SSB is as follows:

$$\langle 5_H \rangle^i = \delta^{i5}\frac{v_5}{\sqrt{2}}$$

Unlike the SM, in renormalizable SU(5) GUTs there's also a second Higgs doublet, H_2. The decomposition under the SM subgroup of the scalar 45_H that comprises the second Higgs doublet is given below.

$$45_H \equiv \left(\Sigma_1, \Sigma_2, S_3^*, R_2^*, S_{1,45}^*, \Sigma_6, H_2\right) = \left(8,2,\frac{1}{2}\right) \oplus \left(\bar{6},1,-\frac{1}{3}\right) \oplus \left(3,3,-\frac{1}{3}\right) \oplus \left(\bar{3},2,-\frac{7}{6}\right) \oplus \left(3,1,-\frac{1}{3}\right) \oplus \left(\bar{3},1,\frac{4}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$$

$S_3^*, R_2^*, S_{1,45}^*$ are scalar leptoquarks and we have followed the notation of [7] to denote them, while Σ_1 are scalar gluons that comprise a weak doublet. 45 tensor representation is traceless and has two upper indices that are anti-symmetric under interchange and a lower index. That is, $45_\gamma^{\alpha\beta} = -45_\gamma^{\beta\alpha}$ and $\sum_{\beta=1}^{5} 45_\beta^{\alpha\beta} = 0$. The neutral component of the second Higgs doublet acquires a VEV which conserves color gauge symmetry at the EW SSB as follows:

$$\langle 45_H \rangle_1^{15} = \langle 45_H \rangle_2^{25} = \langle 45_H \rangle_3^{35} = \frac{v_{45}}{\sqrt{2}}$$

The aforementioned property of tracelessness, of course, implies that

$$\langle 45_H \rangle_4^{45} = -3\frac{v_{45}}{\sqrt{2}}$$

It's convenient to do an orthogonal rotation that takes the basis spanned by H_1 and H_2 to the basis spanned by the mass eigenstates of H and H' in the following manner:

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} H \\ H' \end{pmatrix}$$

In this basis, $\langle H \rangle = v/\sqrt{2}$ where $v = 246$ GeV, whereas the VEV of H' vanishes. The angle of rotation satisfies $\tan\beta = 2v_{45}\sqrt{6}/v_5$, while $|v|^2 = |v_5|^2 + 24|v_{45}|^2$ as in [8].

2.2 Unification of Gauge Couplings

At the mass of the Z boson, m_Z, three physical observables, i.e., square of the sine of the Weinberg angle, $\sin^2\theta_W$, inverse of the electromagnetic fine structure constant, $\alpha_{\text{e.m.}}^{-1}$, and strong fine structure constant, α_s, in the minimal subtraction ($\overline{\text{MS}}$) scheme have the values $\sin^2\theta_W(m_Z) = 0.23120$, $\alpha_{\text{e.m.}}^{-1}(m_Z) = 127.906$, $\alpha_s(m_Z) = 0.1187$, respectively [9]. These values, in turn, determine through the following equations

$$\alpha_1^{-1} = \left(\frac{3}{5}\right)(1 - \sin^2\theta_W)\alpha_{\text{e.m.}}^{-1}$$

$$\alpha_2^{-1} = \sin^2\theta_W \alpha_{\text{e.m.}}^{-1}$$

$$\alpha_3^{-1} = \alpha_s^{-1}$$

the input values, that is, the values of the three fine structure constants at the weak scale, $\alpha_i(m_Z)$, that appear in the one-loop beta functions for $i = 1,2,3$ given below.

$$\alpha_i^{-1}(m_Z) = \alpha_{\text{GUT}}^{-1} + \frac{1}{2\pi} b_i \log\left(\frac{M_{\text{GUT}}}{m_Z}\right) \tag{2.1}$$

The SM values of the one-loop beta function coefficients, b_i, are given below:

$$b_1 = \frac{41}{10}, b_2 = -\frac{19}{6}, b_3 = -7$$

The above expressions for the b_i coefficients do not include the contribution of the second Higgs doublet. As is well known, SM particles do not lead to unification of coupling constants. Therefore, particles beyond those of the SM with masses in the range $m_Z \leq M_I \leq M_{\text{GUT}}$, and beta function coefficients, b_{iI}, must be present to ensure unification. As a result of these hypothetical particles, as shown below, one-loop beta function coefficients are replaced by effective coefficients B_i.

$$B_i = b_i + b_{iI} \frac{\log(M_{\text{GUT}}/M_I)}{\log(M_{\text{GUT}}/m_Z)}$$

It's more convenient to reduce the number of equations which have to be solved to two by taking the differences of those in (2.1) and defining new variables $B_{ij} \equiv B_i - B_j$. It was realized decades ago [10] that irrespective of the particle content of the GUT under consideration successful unification requires that the ratio of the effective coefficients and the GUT scale be given in terms of the physical observables at the weak scale as in the equations below.

$$\frac{B_{23}}{B_{12}} = \frac{5}{8}\left(\frac{\sin^2\theta_W(m_Z) - \alpha_{\text{e.m.}}(m_Z)/\alpha_s(m_Z)}{3/8 - \sin^2\theta_W(m_Z)}\right)$$

$$\log\left(\frac{M_{\text{GUT}}}{m_Z}\right) = \frac{16\pi}{5\alpha_{\text{e.m.}}(m_Z)}\left(\frac{3/8 - \sin^2\theta_W(m_Z)}{B_{12}}\right)$$

When the aforementioned values of the physical observables at the weak scale are substituted in the above formulas we obtain the ones given below.

$$\frac{B_{23}}{B_{12}} = 0.719$$

$$\log\left(\frac{M_{GUT}}{m_Z}\right) = \frac{184.9}{B_{12}} \tag{2.2}$$

As will be discussed later in this section, the basis in which proton does not decay into a positively charged meson and an antineutrino pair provides, in GeV, the lowest value of [11, 12]

$$M_{\text{GUT}} = 1 \times 10^{14}\sqrt{40\alpha_{\text{GUT}}}$$

for the GUT scale. The prefactor of 8×10^{13} given in [12] has been corrected as its decay rates are erroneously down by a factor of 4 which leads to an erroneous decrease by a factor of $\sqrt{2}$ in the GUT scale. A comparison of (A6-A13) in [12] with (9-11) and (13-17) in [13] reveals this error. Even for a GUT scale as low as this, the (highest) required value of $B_{12} \cong 6.7$ according to (2.2) is too small compared to that of the SM $B_{12} = \frac{41}{10} - \left(-\frac{19}{6}\right) \cong 7.3$. Thus, new scalars or fermions that have a negative value of this coefficient, that is, $b_{1I} - b_{2I}$, must populate the region between the weak and GUT scales.

Table 1. $b_{iI} - b_{jI}$ coefficients of the split multiplets in the $5_{\text{H}}, 24_{\text{H}}$, and 45_{H} Higgs representations. The contribution of H_1 is already included among those of the SM particles

	5_H		24_H		45_H						
$b_{iI} - b_{jI}$	H_1	$S_{1,5}^*$	Φ_8	Φ_3	Σ_1	Σ_2	S_3^*	R_2^*	$S_{1,45}^*$	Σ_6	H_2
$b_{1I} - b_{2I}$	$-\frac{1}{15}$	$\frac{1}{15}$	0	$-\frac{1}{3}$	$-\frac{8}{15}$	$\frac{2}{15}$	$-\frac{9}{5}$	$\frac{17}{15}$	$\frac{1}{15}$	$\frac{16}{15}$	$-\frac{1}{15}$
$b_{2I} - b_{3I}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{5}{6}$	$\frac{3}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$

By looking up in Table 1 it is seen that there are only three split multiplets –Φ_3 is decoupled- which have this property among those of the Higgs fields $5_H, 24_H$, and 45_H: H_2, Σ_1, and S_3^*. The two equations, that involve the masses of these three particles as well as the GUT scale, given below

$$2\pi[\alpha_1^{-1}(m_Z) - \alpha_2^{-1}(m_Z)] = \frac{109}{15}\log\left(\frac{M_{\text{GUT}}}{m_Z}\right) - \frac{1}{15}\log\left(\frac{M_{\text{GUT}}}{m_{\text{H}_2}}\right) - \frac{8}{15}\log\left(\frac{M_{\text{GUT}}}{m_{\Sigma_1}}\right) - \frac{9}{5}\log\left(\frac{M_{\text{GUT}}}{m_{S_3}}\right)$$

$$2\pi[\alpha_2^{-1}(m_Z) - \alpha_3^{-1}(m_Z)] = \frac{23}{6}\log\left(\frac{M_{\text{GUT}}}{m_Z}\right) + \frac{1}{6}\log\left(\frac{M_{\text{GUT}}}{m_{\text{H}_2}}\right) - \frac{2}{3}\log\left(\frac{M_{\text{GUT}}}{m_{\Sigma_1}}\right) + \frac{3}{2}\log\left(\frac{M_{\text{GUT}}}{m_{S_3}}\right)$$

can be solved to find the GUT scale and the mass of the scalar leptoquark in terms of the masses of the other particles. In writing down the above equations it has been assumed that all the other split multiplets are decoupled from the theory below the unification scale, that is, they have masses greater than the GUT scale. The result is as follows:

$$M_{\text{GUT}} = m_Z\exp\left\{\frac{3\pi}{80}[5\alpha_1^{-1}(m_Z) + \alpha_2^{-1}(m_Z) - 6\alpha_3^{-1}(m_Z)]\right\}\left(\frac{m_{\text{H}_2}m_Z^9}{m_{\Sigma_1}{}^{10}}\right)^{(1/80)} \tag{2.3}$$

$$m_{S_3} = m_Z\exp\left\{\frac{\pi}{240}[145\alpha_1^{-1}(m_Z) - 291\alpha_2^{-1}(m_Z) + 146\alpha_3^{-1}(m_Z)]\right\}\left(\frac{m_{\Sigma_1}{}^{10}m_Z^7}{m_{\text{H}_2}{}^{17}}\right)^{(1/240)} \tag{2.4}$$

Turning to the determination of the minimum value of the unification scale, for this purpose both the experimental lower bounds and the theoretical expressions for the lifetime of the proton in the various decay channels are needed. The experimental lower bounds, τ_{exp}, for the lifetime of the proton across the 8 best studied decay channels are given in Table 2,

Table 2. Bounds on the lifetime of the proton in different decay channels. Table is taken from [12]

Channel	Lifetime [10^{33} years]
$p \to e^+\pi^0$	16
$p \to \mu^+\pi^0$	7.7
$p \to \bar{\nu}\pi^+$	0.39
$p \to e^+K^0$	1.0
$p \to \mu^+K^0$	1.6
$p \to \bar{\nu}K^+$	5.9
$p \to e^+\eta$	10
$p \to \mu^+\eta$	4.7

whereas the theoretical lifetime of the proton is related to the decay rates of the different channels [13] given below through the equation $\tau_{theory} = \Gamma^{-1}$. Needless to say, $\tau_{theory} \geq \tau_{exp}$ for the theory to be valid and this enables one to place a lower bound on the GUT scale.

$$\Gamma(p \to \bar{\nu}K^+) = \frac{\left(m_p^2 - m_K^2\right)^2}{8\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 \sum_{i=1}^{3} \left| \frac{2m_p D}{3m_B} c(\nu_i, d, s^C) + \left[1 + \frac{m_p}{3m_B}(D + 3F)\right] c(\nu_i, s, d^C) \right|^2$$

$$\Gamma(p \to \bar{\nu}\pi^+) = \frac{m_p}{8\pi f_\pi^2} A_L^2 |\alpha|^2 (1 + D + F)^2 \sum_{i=1}^{3} |c(\nu_i, d, d^C)|^2$$

$$\Gamma(p \to \eta e_\alpha^+) = \frac{\left(m_p^2 - m_\eta^2\right)^2}{48\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 (1 + D - 3F)^2 \{|c(e_\alpha, d^C)|^2 + k_1^4 |c(e_\alpha^C, d)|^2\}$$

$$\Gamma(p \to K^0 e_\alpha^+) = \frac{\left(m_p^2 - m_K^2\right)^2}{8\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 \left[1 + \frac{m_p}{m_B}(D - F)\right]^2 \{|c(e_\alpha, s^C)|^2 + k_1^4 |c(e_\alpha^C, s)|^2\}$$

$$\Gamma(p \to \pi^0 e_\alpha^+) = \frac{m_p}{16\pi f_\pi^2} A_L^2 |\alpha|^2 (1 + D + F)^2 \{|c(e_\alpha, d^C)|^2 + k_1^4 |c(e_\alpha^C, d)|^2\}$$

where m_p=938.3 MeV, m_K=493.7 MeV, m_η=547.9 MeV, D=0.8, F=0.47, m_B=1.15 GeV, f_π=139 MeV, A_L=1.4×3.4, α=0.012 GeV^3 [12]. The variable k_1 is defined as $k_1^2 \equiv 4\pi\alpha_{\text{GUT}}/M_{\text{GUT}}^2$.

$$c\left(e_\alpha^C, d_\beta\right) = V_1^{11} V_2^{\alpha\beta} + (V_1 V_{UD})^{1\beta} \left(V_2 V_{UD}^\dagger\right)^{\alpha 1}$$

$$c\left(e_\alpha, d_\beta^C\right) = k_1^2 V_1^{11} V_3^{\beta\alpha}$$

$$c\left(\nu_l, d_\alpha, d_\beta^C\right) = k_1^2 (V_1 V_{UD})^{1\alpha} (V_3 V_{EN})^{\beta l}, \alpha = 1 \text{ or } \beta = 1$$

where $V_1 \equiv U_C^\dagger U, V_2 \equiv E_C^\dagger D, V_3 \equiv D_C^\dagger E, V_{\text{UD}} \equiv U^\dagger D, V_{\text{EN}} \equiv E^\dagger N$. Quark mixings are given by $V_{\text{UD}} \equiv U^\dagger D = K_1 V_{\text{CKM}} K_2$, whereas $V_{\text{EN}} = K_3 V_l^M$ denotes mixing in the leptonic sector in the Majorana case at low energies. K_1 and K_2 represent diagonal matrices with three and two phases, respectively. According to the convention used for the diagonalization of mass matrices of charged fermions in this article,

$$U_C^T Y_U U = Y_U^{diag}$$

$$D_C^T Y_D D = Y_D^{diag} \tag{2.5}$$

$$E_C^T Y_E E = Y_E^{diag} \tag{2.6}$$

Down-type quark and charged lepton generation indices run over $\alpha, \beta = 1, 2$, while there are 3 neutrino flavors, so $l = 1, 2, 3$.

The precise value of the GUT scale depends on the basis chosen, i.e., the choice of the unitary matrices that diagonalize the mass matrices of charged fermions. There's a special basis in which the GUT scale attains its minimum value of $1 \times 10^{14}\sqrt{40\alpha_{\text{GUT}}}$ in GeV quoted above. This basis is defined by the absence of proton decays into charged meson anti-neutrino pairs assuming neutrinos are Majorana. This is achieved by setting

$$(V_1 V_{UD})^{1\alpha} = 0$$

for $\alpha = 1$ and 2. The above condition implies that

$$\left|V_1^{11}\right| = \left|V_{\text{CKM}}^{13}\right|$$

and thus, substantially suppresses the remaining decays into meson charged lepton pairs. If, in addition, one chooses [12]

$$|V_2| = \left|V_3^T\right| = \begin{pmatrix} 0.11 & 0.77 \\ 0.16 & 0.61 \end{pmatrix} \tag{2.7}$$

for $\alpha, \beta = 1,2$, then as per the figures listed in Table 2 the four decay channels $p \to e^+\pi^0$, $p \to \mu^+\pi^0$; $p \to e^+K^0$, $p \to \mu^+K^0$ become equally constraining and one obtains the minimum value of the unification scale mentioned above.

The Higgs potential shall be listed below which leads to the SSB at the GUT and EW scales, as well as the mixing between the first, H_1, and second Higgs doublets, H_2. The terms that involve only the adjoint scalars [14] are responsible for the GUT scale symmetry breaking. Since the Higgs potential preserves the $24_H \to -24_H$ symmetry of the GG model, the mass relation of $m_{(1,3)} = 2m_{(8,1)}$ between the weak triplet scalars and scalar gluons is intact. This relation suggests that both will be

$$\begin{aligned}
\mathcal{L}_{Higgs} = {} & \frac{\mu_{24}^2}{2} 24_{Hj}^i 24_{Hi}^j - \frac{a_{24}}{4}\left(24_{Hj}^i 24_{Hi}^j\right)^2 - \frac{b_{24}}{2} 24_{Hj}^i 24_{Hk}^j 24_{Hl}^k 24_{Hi}^l + \mu_5^2 5_H^i 5_{Hi}^* - \frac{\lambda_5}{4}\left(5_H^i 5_{Hi}^*\right)^2 \\
& + \lambda_{\mu,5} 5_H^i 5_{Hi}^* 24_{Hk}^j 24_{Hj}^k - \lambda_{g,5} 5_H^i 5_{Hk}^* 24_{Hi}^j 24_{Hj}^k + \mu_{45}^2 45_{Hk}^{ij} 45_{Hij}^{*k} \\
& + \lambda_{\mu,45} 45_{Hk}^{ij} 45_{Hij}^{*k} 24_{Hm}^l 24_{Hl}^m - \beta_1 45_{Hk}^{ij} 45_{lm}^{*k} 24_{Hi}^l 24_{Hj}^m - \beta_2 45_{Hk}^{ij} 45_{Hil}^{*m} 24_{Hj}^k 24_{Hm}^l \\
& - \beta_3 45_{Hk}^{ij} 45_{Hil}^{*m} 24_{Hj}^l 24_{Hm}^k - \beta_4 45_{Hk}^{ij} 45_{Hil}^{*k} 24_{Hm}^l 24_{Hj}^m - \beta_5 45_{Hk}^{ij} 45_{Hij}^{*l} 24_{Hl}^m 24_{Hm}^k \\
& - \lambda_{5-45} 5_H^i 5_{Hi}^* 45_{Hk}^{jl} 45_{Hjl}^{*k} - \bar{\lambda}_{5-45} 5_H^i 5_{Hj}^* 45_{Hi}^{kl} 45_{Hkl}^{*j} - \tilde{\lambda}_{5-45} 5_H^i 5_{Hj}^* 45_{Hk}^{jl} 45_{Hil}^{*k} \\
& - \lambda_{45} 45_{Hm}^{ij} 45_{Hn}^{kl} 45_{Hij}^{*m} 45_{Hkl}^{*n} - \bar{\lambda}_{45} 45_{Hm}^{ij} 45_{Hn}^{kl} 45_{Hij}^{*n} 45_{Hkl}^{*m} - \tilde{\lambda}_{45} 45_{Hm}^{ij} 45_{Hn}^{kl} 45_{Hik}^{*m} 45_{Hjl}^{*n} \\
& - \lambda_{45,1} 45_{Hm}^{ij} 45_{Hi}^{kl} 45_{Hjn}^{*m} 45_{Hkl}^{*n} - \bar{\lambda}_{45,1} 45_{Hm}^{ij} 45_{Hi}^{kl} 45_{Hkn}^{*m} 45_{Hjl}^{*n} \\
& - \lambda_{45,2} 45_{Hk}^{ij} 45_{Hi}^{kl} 45_{Hjm}^{*n} 45_{Hln}^{*m} - \lambda_\varepsilon \varepsilon^{ijklm} \varepsilon_{nprst} 45_{Hn}^{ij} 45_{Hp}^{kl} 45_{Hru}^{*m} 45_{Hst}^{*u} \\
& - \bar{\lambda}_\varepsilon \varepsilon^{ijklm} \varepsilon_{nprst} 45_{Hn}^{ij} 45_{Hu}^{kl} 45_{Hpr}^{*m} 45_{Hst}^{*u} - \tilde{\lambda}_\varepsilon \varepsilon^{ijklm} \varepsilon_{nprst} 45_{Hn}^{ij} 45_{Hp}^{ku} 45_{Hrs}^{*l} 45_{Htu}^{*m}
\end{aligned}$$

decoupled when the scalar gluons are chosen to lie at the GUT scale, i.e., $m_{(8,1)} = M_{\text{GUT}}$. Then, there are the terms which involve only the fundamental Higgs and the mixing between the fundamental and adjoint Higgses.

These terms push the scalar leptoquark multiplet, $S_{1,5}^*$, to the GUT scale while keeping the first Higgs, H_1, light, that is, accomplish the doublet-triplet splitting and determine the VEV of the first Higgs doublet.

Likewise, the terms in which the 45_H interact among themselves and with the adjoint scalars lead to a mass spectrum of the multiplets as given below [15]

$$m_{H_2}^2 = -M^2 - \frac{v_{24}^2}{24}(6\beta_1 + \beta_2 + 4\beta_3 + \beta_4 + 2\beta_5)$$

$$m_{\Sigma_1}^2 = -M^2 - \frac{v_{24}^2}{8}(6\beta_1 + 5\beta_2 + 4\beta_3 + \beta_4 + 2\beta_5)$$

$$m_{\Sigma_2}^2 = -M^2 - \frac{v_{24}^2}{24}(2\beta_1 + 15\beta_2 + 4\beta_3 + 7\beta_4 + 6\beta_5)$$

$$m_{S_3}^2 = -M^2 - \frac{v_{24}^2}{24}(18\beta_1 + 15\beta_2 + 8\beta_3 + 3\beta_4 - 2\beta_5)$$

$$m_{R_2}^2 = -M^2 - \frac{v_{24}^2}{24}(2\beta_1 + 15\beta_2 + 20\beta_3 + 7\beta_4 - 2\beta_5)$$

$$m_{S_{1,45}}^2 = -M^2 - \frac{v_{24}^2}{24}(-6\beta_1 + 15\beta_2 + 20\beta_3 - \beta_4 + 6\beta_5)$$

$$m_{\Sigma_6}^2 = -M^2 - \frac{v_{24}^2}{24}(10\beta_1 - 5\beta_2 + 6\beta_3 + 5\beta_4 + 2\beta_5)$$

and determine the VEV of the second Higgs doublet. In the equations above a new variable has been defined $M^2 \equiv \mu_{45}^2 + v_{24}^2\lambda_{\mu,45}$. In order for SSB to occur at the EW scale, $m_{H_2}^2$ must both be negative and fine-tuned to be $\mathcal{O}(v^2)$. As the equations below explicitly demonstrate R_2^*, $S_{1,45}^*$, and Σ_6 whose masses are above the GUT scale must cancel each other to give a light second Higgs doublet and, if necessary, light weak doublet scalar gluons. This is always possible.

$$m_{H_2}^2 = \frac{1}{5}\left(m_{S_3}^2 - m_{R_2}^2 + m_{S_{1,45}}^2 + 2m_{\Sigma_6}^2\right) - \frac{2}{5}M^2$$

$$m_{\Sigma_1}^2 = \frac{3}{10}\left(m_{\Sigma_2}^2 + 3m_{S_3}^2 - 2m_{R_2}^2 + 2m_{S_{1,45}}^2 + 2m_{\Sigma_6}^2\right) + \frac{4}{5}M^2$$

Finally, a word is in order for the weak and color triplet scalar leptoquark, S_3^*, because it might cause the proton to decay too rapidly. Naively, it would have to have a mass exceeding $m_{S_3} \geq 10^{11}$ GeV in order for this not to happen [16]. However, if its coupling to the up-type quark is somewhat suppressed such as it would have to be in the scenario considered in this manuscript, then this bound could be evaded.

2.3 Charged Fermion Masses

The following Yukawa lagrangian

$$\mathcal{L}_Y = -Y_1\bar{5}_F 10_F 5_H^* - Y_2\bar{5}_F 10_F 45_H^* - \overline{10_F^C} 10_F(Y_3 5_H + Y_4 45_H) + h.c.$$

is responsible for generating masses for charged fermions in renormalizable SU(5) theories. The mass matrices of down-type quarks, M_D, charged leptons, M_E, and up-type quarks, M_U, which follow from this lagrangian are given below.

$$M_D = \frac{1}{2} Y_1 v_5^* + Y_2 v_{45}^* \tag{2.8}$$

$$M_E^T = \frac{1}{2} Y_1 v_5^* - 3Y_2 v_{45}^* \tag{2.9}$$

$$M_U = \sqrt{2}\,(Y_3 + Y_3^T) v_5 - 2\sqrt{2}\,(Y_4 - Y_4^T) v_{45}$$

These mass matrices are connected to the Yukawa coupling matrices mentioned in the previous subsection by the formulae

$$M_D = Y_D \frac{v^*}{\sqrt{2}} \tag{2.10}$$

$$M_E = Y_E \frac{v^*}{\sqrt{2}} \tag{2.11}$$

$$M_U = Y_U \frac{v}{\sqrt{2}}$$

Therefore, the diagonalization of the Yukawa couplings mentioned in the previous subsection apply without alteration to the mass matrices as well.

2.4 Neutrino Masses

Today, it is known that neutrinos oscillate from one flavor to another as they move in space. Explanation of this phenomenon requires that at least two neutrinos be massive. There are different mechanisms by which neutrino masses can be generated in grand unified theories. The simplest one, albeit not predictive, is to add two singlet right-handed neutrinos, 1_F, to the particle spectrum of the SM. In this case, the smallness of neutrio masses is still a mystery, because the masses of the right-handed singlets are not determined by unification constraints and are instead arbitrary, and the Dirac mass of neutrinos are adjusted to their physical values by hand.

One could also add a fermionic representation which includes a SM singlet as in the Type-I see-saw mechanism [17-19]. Of course, two of these multiplets such as those in 24_F or 75_F or one of each would be necessary. A variation of this mechanism involves a fermionic multiplet which contains particles whose quantum numbers are $(1,3,0)$. This is dubbed the Type-III see-saw [20] and it can be implemented by extending the particle content of the GG model by two fermionic 24_F representations. There are hybrid non-renormalizable [21] and renormalizable [16] models, too, which involve a single fermionic 24_F. In these models it's possible to give mass to two neutrinos at once by the presence of the singlet, $(1,1,0)$, and the weak triplet, $(1,3,0)$, multiplets in the fermionic adjoint representation.

It's possible to extend the particle spectrum of the SM only by scalars, too. The first mechanism utilizes a scalar with quantum numbers $(1,3,1)$ and is called the Type-II see-saw [22]. For example, 15_S representation of SU(5) contains the aforementioned multiplet. These are the only mechanisms of mass generation at tree-level. However, neutrinos could acquire mass at loop-level as well. In fact, a model proposed recently [23] makes use only of scalars in the 10_S representation to generate mass for neutrinos at the one-loop level.

3. Results and Discussion

(2.8) and (2.9) in the previous section are inverted to obtain the Yukawa couplings in terms of the mass matrices as follows:

$$Y_1 = \frac{1}{2v_5^*}\left[3M_D(Q) + M_E^T(Q)\right]$$

$$Y_2 = \frac{1}{4v_{45}^*}\left[M_D(Q) - M_E^T(Q)\right]$$

In the above equations it has been explicitly indicated that mass matrices are functions of Q to emphasize their renormalization scale dependence. Mass matrices are related to down-type quark and charged lepton Yukawa couplings through (2.10) and (2.11), combining this with (2.5) and (2.6) leads in a general basis to

$$Y_1 = \frac{\left(v^*/\sqrt{2}\right)}{2v_5^*}\left(3D_C^* Y_D^{\text{diag}} D^\dagger + E^* Y_E^{\text{diag}} E_C^\dagger\right)$$

$$Y_2 = \frac{\left(v^*/\sqrt{2}\right)}{4v_{45}^*}\left(D_C^* Y_D^{\text{diag}} D^\dagger - E^* Y_E^{\text{diag}} E_C^\dagger\right)$$

Substituting in $v_5^* = v^* cos\beta$ and $v_{45}^* = v^* sin\beta/\sqrt{24}$ we get

$$Y_1 = \frac{1}{2\sqrt{2}\cos\beta}\left(3D_C^* Y_D^{\text{diag}} D^\dagger + E^* Y_E^{\text{diag}} E_C^\dagger\right)$$

$$Y_2 = \frac{\sqrt{3}}{2\sin\beta}\left(D_C^* Y_D^{\text{diag}} D^\dagger - E^* Y_E^{\text{diag}} E_C^\dagger\right)$$

These are the general equations which enable one to constrain the values of the $\tan\beta$ parameter.

The renormalization scale dependence of the charged fermion mass matrices and hence the unitary transformations which diagonalize them were utilized to use different bases at low and high energies. In evaluating the neutrino mass which is a low energy observable the PMNS basis is utilized, while in the case of proton decay matrix elements that must be calculated at the GUT scale and then run down to the scale of the proton mass the scale mentioned in the Unification of Gauge Couplings subsection of Section II which minimizes the GUT scale is employed. It is demanded, of course, that the elements of the Yukawa coupling matrices remain perturbative throughout the whole energy range.

First, specialize to the PMNS (or flavor) basis, taking K_1, K_2 to be unit matrices for simplicity, to arrive at the expression given below.

$$Y_1 = \frac{1}{2\sqrt{2}\cos\beta}\left(3Y_D^{\text{diag}} V_{\text{CKM}}^\dagger + Y_E^{\text{diag}}\right)$$

$$Y_2 = \frac{\sqrt{3}}{2\sin\beta}\left(Y_D^{\text{diag}} V_{\text{CKM}}^\dagger - Y_E^{\text{diag}}\right)$$

All of the elements of these matrices must be perturbative in order for our calculations to be consistent. Therefore,

$$|Y_{1,ij}| = \frac{1}{2\sqrt{2}\cos\beta}\left|\left(3Y_D^{\text{diag}}V_{\text{CKM}}^\dagger + Y_E^{\text{diag}}\right)_{ij}\right| \leq \sqrt{4\pi}$$

$$|Y_{2,ij}| = \frac{\sqrt{3}}{2\sin\beta}\left|\left(Y_D^{\text{diag}}V_{\text{CKM}}^\dagger - Y_E^{\text{diag}}\right)_{ij}\right| \leq \sqrt{4\pi}$$

Obviously, for both cases it's the 33 element which will give the largest absolute value. Thus,

$$\cos\beta \geq \frac{1}{4\sqrt{2\pi}}\left(3Y_b V_{\text{CKM},33} + Y_\tau\right)$$

$$\sin\beta \geq \frac{\sqrt{3}}{4\sqrt{\pi}}\left(Y_b V_{\text{CKM},33} - Y_\tau\right)$$

Evaluating the above Yukawa couplings at the bottom quark and tau lepton masses, respectively, the limits given below for the $\tan\beta$ parameter are obtained.

$$3.4 \times 10^{-3} \leq \tan\beta \leq 1.2 \times 10^{2}$$

Now, the bounds are evaluated in the basis that minimizes the GUT scale. This is the high energy basis, so the figures used for the bottom quark and tau lepton Yukawa couplings are the ones at the GUT scale. Multiplying the first and second equations on the left by D_C^T and E^T, respectively, and on the right by D and E_C, respectively, the equations given below are obtained.

$$D_C^T Y_1 D = \frac{1}{2\cos\beta\sqrt{2}}\left(3Y_D^{\text{diag}} + V_3^* Y_E^{\text{diag}} V_2\right)$$

$$E^T Y_2 E_C = \frac{\sqrt{3}}{2\sin\beta}\left(V_3^T Y_D^{\text{diag}} V_2^\dagger - Y_E^{\text{diag}}\right)$$

$E^T Y_1 E_C$ and $D_C^T Y_2 D$ on the left-hand side of the equations are not considered, because it has been checked that these lead to less stringent bounds on the value of $\tan\beta$. Perturbativity of Yukawa couplings implies

$$|(D_C^T Y_1 D)_{ij}| = \frac{1}{2\cos\beta\sqrt{2}}\left|\left(3Y_D^{\text{diag}} + V_3^* Y_E^{\text{diag}} V_2\right)_{ij}\right| \leq \sqrt{4\pi}$$

$$|(E^T Y_2 E_C)_{ij}| = \frac{\sqrt{3}}{2\sin\beta}\left|\left(V_3^T Y_D^{\text{diag}} V_2^\dagger - Y_E^{\text{diag}}\right)_{ij}\right| \leq \sqrt{4\pi}$$

for each of the nine matrix elements separately. Hence, the full matrix whose two-by-two subspace was given in (2.7) has to be found assuming that all of its elements are real. Here is the result:

$$V_2 = V_3^T = \begin{pmatrix} 0.11 & 0.77 & 0.629 \\ -0.16 & -0.61 & 0.776 \\ 0.981 & -0.186 & 0.0561 \end{pmatrix}$$

As it turns out, in each case the absolute value of the 33 element is the largest and thus gives the most demanding bounds. Therefore,

$$|(D_C^T Y_1 D)_{33}| = \frac{1}{2\cos\beta\sqrt{2}}\left|\left(3Y_D^{\text{diag}} + V_3^* Y_E^{\text{diag}} V_2\right)_{33}\right| \leq \sqrt{4\pi}$$

$$|(E^T Y_2 E_C)_{33}| = \frac{\sqrt{3}}{2\sin\beta}\left|\left(V_3^T Y_D^{\text{diag}} V_2^\dagger - Y_E^{\text{diag}}\right)_{33}\right| \leq \sqrt{4\pi}$$

Finally,

$$\cos\beta \geq \frac{1}{4\sqrt{2\pi}}[3Y_b + |(V_3^* Y_\tau V_2)_{33}|]$$

$$\sin\beta \geq \frac{\sqrt{3}}{4\sqrt{\pi}}|(V_3^* Y_b V_2)_{33} - Y_\tau|$$

At this point, the values of the bottom quark and tau lepton Yukawa couplings evaluated at the GUT scale from [24] are substituted in to obtain the following limits for the tanβ parameter.

$$2.4 \times 10^{-3} \leq \tan\beta \leq 5.7 \times 10^2$$

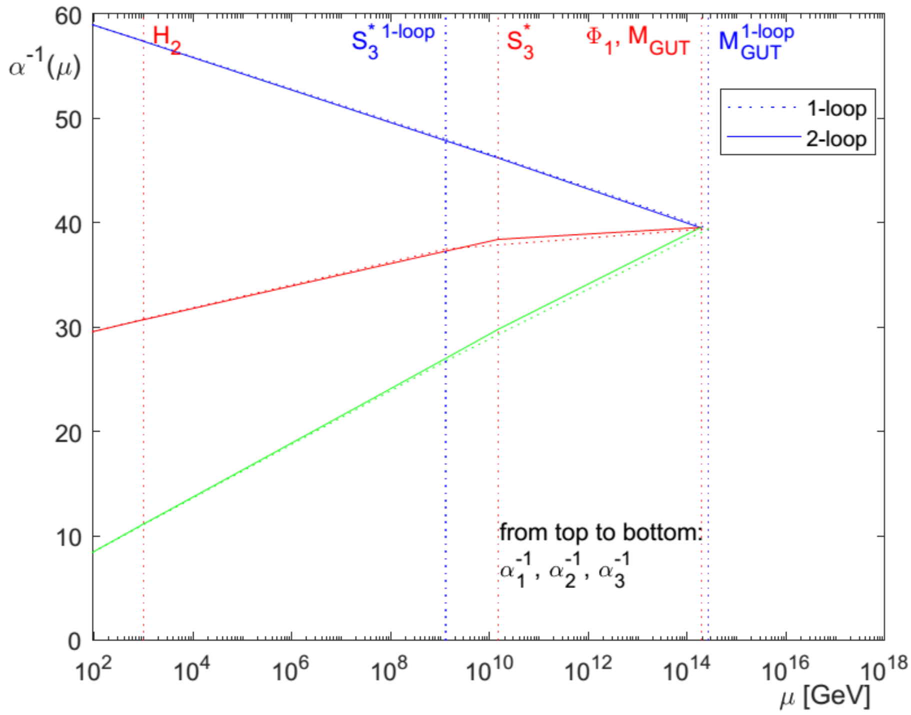

Figure 1. Inverse Gauge Couplings vs. Renormalization Scale in GeV in the case in which the second Higgs doublet has a mass of 1 TeV and the scalar gluons are at the unification scale. Solid lines indicate the 2-loop solution, whereas the 1-loop solution is shown as dotted lines

For completeness, a scenario is presented in which the physical second Higgs doublet, H_2, has a mass of 1 TeV, the weak and color triplet scalar leptoquark, S_3^*, has a mass of 1.5×10^{10} GeV, and the scalar gluons, Σ_1, are decoupled from the theory below the unification scale as their mass is equal to the GUT scale of 1.9×10^{14} GeV. The decoupling of the scalar gluons ensures that the weak and color triplet scalar leptoquark mass is maximized according to (2.4) for the given mass of the second Higgs doublet. The evolution of the inverses of the three fine-structure constants as a function of the renormalization scale for this scenario is shown in Figure 1.

A comparison of the GUT scale of 2.7×10^{14} GeV given by (2.3) obtained using the 1-loop running of gauge couplings with that of the 2-loop result of 1.9×10^{14} GeV shows it to be larger almost by 50%. On the other hand, there's more than an order of magnitude decrease in the mass of the weak and color triplet scalar leptoquark, S_3^*, of 1.3×10^9 GeV computed using (2.4) in the 1-loop approximation compared to the mass of 1.5×10^{10} GeV found at 2-loops. The common value of the inverse of the fine-structure constants at the unification scale in the 1-loop approximation is 39.4 and compares favorably with the figure of 39.7 obtained at 2-loops. The 1- and 2-loop results are shown as dotted and solid lines in Figure 1, respectively.

4. Conclusion

Renormalizable grand unified theories based on the SU(5) gauge group provide a realistic mass spectrum for the charged fermions of the SM with the extension of the GG model by 45_H [2]. The additional degrees of freedom in the Higgs sector combined with the freedom to choose a different basis for the mass matrices of charged fermions at low and high energies enables the unification of gauge coupling constants at as low a unification scale as 1.9×10^{14} GeV without violating the experimental constraints regarding proton decay due to gauge boson exchange.

Consistency of the model requires that the Yukawa couplings of the down-type quarks and charged leptons be perturbative throughout the whole range of energies. The more stringent condition on the perturbativity of the Yukawa couplings is obtained at low energies and the $\tan\beta$ parameter, provided that β lies in the first quadrant, is found to fall in the following range: $3.4 \times 10^{-3} \leq \tan\beta \leq 1.2 \times 10^2$. Even though these bounds do not impact the angle itself by much $0.195° \leq \beta \leq 89.5°$ as opposed to $0° \leq \beta \leq 90°$, they make a huge difference in a physical quantity such as the lightest neutrino mass which scales as $\tan^4\beta$ [4-6].

In the simplest scenario considered Dirac neutrino masses are generated for two of the neutrinos through the addition of two right-handed singlet neutrinos. Being singlets these fermions do not affect the running of gauge couplings and thus do not alter the analysis presented in this manuscript. There are several other possibilities [16-23] as was explained in the subsection on Neutrino Masses in Section II; however, the extra particles they introduce to the theory must be accounted for in the running of gauge couplings, too.

There are also supersymmetric grand unified theories with SU(5) gauge symmetry [25] and non-supersymmetric grand unified theories based on the 10-dimensional orthogonal group SO(10) [26] that involve only renormalizable couplings, which are investigated intensely. When the gauge group is SO(10), and the theory is renormalizable, 10_H and 126_H Higgses, which transform as the vector and complex self-dual five-index tensor representations, respectively, must be present to reproduce the observed mass matrices of charged leptons. Therefore, there are again two types of coupling, and both have to remain perturbative. However, the breaking of SO(10) gauge group down to that of the SM is more complicated and, in general, proceeds in multiple steps. Investigating whether this more complicated chain of symmetry breaking affects the results herein is worth studying. Therefore, in future studies, it would be interesting to establish similar bounds in those theories.

Author Contributions

The author read and approved the final version of the paper.

Declaration

This paper is not derived from any thesis or dissertation.

Conflict of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

Artificial Intelligence (AI) Use Statement

The author declares that no artificial intelligence (AI) tools were used in any part of the study or in the preparation of this manuscript.

References

[1] H. Georgi, S. L. Glashow, *Unity of all elementary-particle forces*, Physical Review Letters 32 (8) (1974) 438–441. [link]

[2] H. Georgi, C. Jarlskog, *A new lepton-quark mass relation in a unified theory*, Physics Letters B 86 (3-4) (1979) 297–300. [link]

[3] P. Fileviez Perez, C. Murgui, *Renormalizable SU(5) unification*, Physical Review D 94 (7) (2016) 075014. [link]

[4] S. Davidson, G. Isidori, A. Strumia, *The smallest neutrino mass*, Physics Letters B 646 (2-3) (2007) 100–104. [link]

[5] Z. Xing, D. Zhang, *On the two-loop radiative origin of the smallest neutrino mass and the associated Majorana CP phase*, Physics Letters B 807 (2020) 135598. [link]

[6] S. Zhou, *Smallest neutrino mass revisited*, Journal of High Energy Physics 2021 (11) (2021) 101. [link]

[7] I. Doršner, S. Fajfer, A. Greljo, J. F. Kamenik, N. Košňik, *Physics of leptoquarks in precision experiments and at particle colliders*, Physics Reports 641 (2016) 1–68. [link]

[8] I. Doršner, S. Fajfer, J. F. Kamenik, N. Košňik, *Light colored scalar as messenger of up-quark flavor dynamics in grand unified theories*, Physical Review D 82 (9) (2010) 094015 . [link]

[9] Particle Data Group, R. L. Workman, V. D. Burkert, V. Crede, E. Klempt, U. Thoma, L. Tiator, K. Agashe, G. Aielli, …, P. A. Zyla, *Review of particle physics*, Progress of Theoretical and Experimental Physics 2022 (8) (2022) 083C01. [link]

[10] A. Giveon, L. J. Hall, U. Sarid, *SU(5) unification revisited*, Physics Letters B 271 (1-2) (1991) 138–144. [link]

[11] I. Doršner, P. Fileviez Perez, *How long could we live?*, Physics Letters B 625 (1-2) (2005) 88–95. [link]

[12] G. Senjanović, M. Zantedeschi, *Minimal SU(5) theory on the edge: The importance of being effective*, Physical Review D 109 (9) (2024) 095009. [link]

[13] P. F. Perez, *Fermion mixings vs. d=6 proton decay*, Physics Letters B 595 (1-4) (2004) 476–483. [link]

[14] A. J. Buras, J. R. Ellis, M. K. Gaillard, D. V. Nanopoulos, *Aspects of the grand unification of strong, weak and electromagnetic interactions*, Nuclear Physics B 135 (1) (1978) 66–92. [link]

[15] K. S. Babu, E. Ma, *Suppression of proton decay in SU(5) grand unification*, Physics Letters B 144 (5-6) (1984) 381–385. [link]

[16] P. Fileviez Perez, *Renormalizable adjoint SU(5)*, Physics Letters B 654 (5-6) (2007) 189–193. [link]

[17] R. N. Mohapatra, G. Senjanović, *Neutrino masses and mixings in gauge models with spontaneous parity violation*, Physical Review D 23 (1) (1981) 165–180. [link]

[18] T. Yanagida, *Horizontal gauge symmetry and masses of neutrinos*, Proceedings of the International Conference on the Seesaw Mechanism25, Paris, 2005, pp. 261–264. [link]

[19] M. Gell-Mann, P. Ramond, R. Slansky, Complex spinors and unified theories, World Scientific Series in 20th Century Physics, pp. 266–272. [link]

[20] R. Foot, H. Lew, X. -G. He, G. C. Joshi, *See-saw neutrino masses induced by a triplet of leptons*, Zeitschrift für Physik C Particles and Fields 44 (3) (1989) 441–444. [link]

[21] B. Bajc, G. Senjanović, *Seesaw at LHC*, Journal of High Energy Physics 2007 (08) (2007) JHEP08(2007)014. [link]

[22] P. Minkowski, *$\mu \to e\gamma$ at a rate of one out of 10^9 muon decays?*, Physics Letters B 67 (4) (1977) 421–428. [link]

[23] Ç. Doğan, *Non-renormalizable grand unification utilizing the leptoquark mechanism of neutrino mass*, Progress of Theoretical and Experimental Physics 2025 (11) (2025) 113B04. [link]

[24] K. S. Babu, B. Bajc, S. Saad, *Yukawa sector of minimal SO(10) unification*, Journal of High Energy Physics 2017 (2) (2017) 136. [link]

[25] P. F. Perez, *Supersymmetric adjoint SU(5),* Physical Review D 76 (7) (2007) 071701. [link]

[26] K. S. Babu, R. N. Mohapatra, *Predictive neutrino spectrum in minimal SO(10) grand unification,* Physical Review Letters 70 (19) (1993) 2845–2848. [link]