



## Trace of multiplication operator restricted to invariant subspaces of some weighted Bergman space

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### ABSTRACT

Let  $\omega$  be a logarithmically subharmonic weight that is radial and reproducing at the origin, and  $L_a^2(\mathbb{D}, \omega dA)$  be the weighted Bergman space. Let  $f$  be a bounded holomorphic function on the open unit disc,  $I$  be a  $z$ -invariant subspace of  $L_a^2(\mathbb{D}, \omega dA)$ , and  $f(M_I)$  denotes the restriction of the multiplication operator  $M_f$  to  $I$ . This paper investigates the trace of the self-commutator of the operator  $f(M_I)$ . More precisely, we compute the trace of the commutator  $[f(M_I)^*, f(M_I)]$  and show that it equals to  $\dim(I \ominus zI) \int_{\mathbb{D}} |f'(z)|^2 dA(z)$ .

**Keywords:** Invariant subspaces, Multiplication operator, Weighted Bergman space.

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## 1. Introduction

The Berger-Shaw theorem [1] states that the self-commutator of an  $n$ -multicyclic hyponormal operator  $T$  is in trace class and

$$\text{tr}[T^*, T] \leq \frac{n}{\pi} \text{Area}(\sigma(T)) \quad (1)$$

Moreover, if  $M_f$  is the multiplication operator on the Hardy space  $H^2(\mathbb{T})$ , where  $\mathbb{T}$  is the unit circle in the complex plane and  $f$  is a bounded analytic function on the open unit disc  $\mathbb{D}$ , then they showed in [2] that

$$\text{tr}[M_f^*, M_f] = \int_{\mathbb{D}} |f'(z)|^2 dA(z) \quad (2)$$

Analogues of the trace identity (2) occur in several other analytic function spaces. For example, (2) holds on the Bergman space  $L_a^2(G)$  for a bounded region  $G$  [2], and analogous results for weighted Bergman spaces were obtained in [3]. Aleman [4] established (2) for the Hardy space  $H^2(G)$  of an arbitrary bounded region  $G$ , while Feldman [5] proved an extension in a more general framework. In addition, trace formulas for the shift operator on both Bergman and certain weighted Bergman spaces can be found in [6, 7].

Let  $I$  be a closed subspace of  $L_a^2(\mathbb{D})$  and the multiplication operator  $M_z$  maps  $I$  into  $I$ . Let  $M_I$  denotes the restriction of  $M_z$  to  $I$ . The idea of studying such a restriction operator originates from [8], and in [9], Zhu posed several questions about the restriction operator  $M_I$ , including the problem of determining when its self-commutator is trace class. In particular, Zhu [9] proved that

$$\text{tr}[M_I^*, M_I] = n,$$

where  $n = \dim(I \ominus zI)$ . Subsequently, Zhu extended this result in [10] and obtained the trace identity

$$\text{tr}[f(M_I)^*, f(M_I)] = n \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad (3)$$

where  $f(M_I)$  denotes the restriction of the multiplication operator  $M_f$  to  $I$ , with  $f$  a bounded analytic function on  $\mathbb{D}$ , and  $n = \dim(I \ominus zI)$ . Furthermore, Ni [11] extends Zhu's trace identity (3) to the Hardy space associated with a rotation-invariant Borel measure.

The purpose of this work is to generalize Zhu's theorem to a class of weighted Bergman spaces. The structure of the paper is organized in the following way. Section 2 introduces the relevant terminology and summarizes the background material needed for the main arguments. Section 3 focuses on the study of skewed projection and some basic properties of  $M_I$ . Finally, Section 4 contains the proof of the central theorem of the paper.

## 2. Preliminaries

Let  $\mathbb{D}$  represents the open unit disc,  $dA$  be the normalized area measure on it, and  $Hol(\mathbb{D})$  be the space of all functions analytic on  $\mathbb{D}$ . Let  $H^\infty(\mathbb{D})$  denotes the space of all bounded analytic functions  $\mathbb{D}$ . Suppose the function  $\omega: \mathbb{D} \rightarrow [0, \infty)$  satisfies the following conditions:

1.  $\omega$  is radial, meaning  $\omega(z) = \omega(|z|)$  for each  $z \in \mathbb{D}$ .
2.  $\log \omega$  is a subharmonic function on  $\mathbb{D}$ .
3. For every bounded harmonic function  $g$  on  $\mathbb{D}$ ,  $\int_{\mathbb{D}} g(z)\omega(z)dA(z) = g(0)$ .

Notice that, if one takes the constant function  $g \equiv 1$  in condition 3 above, then one gets  $\int_{\mathbb{D}} \omega(z) dA(z) = 1$ , which means  $\omega dA$  is a probability measure on  $\mathbb{D}$ . The corresponding weighted Bergman space is given by

$$L^2_{\alpha}(\mathbb{D}, \omega dA) = \{f \in Hol(\mathbb{D}) :$$

$$\int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) < \infty\}.$$

If  $f \in L^2_{\alpha}(\mathbb{D}, \omega dA)$ , then  $\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z)$  defines a norm on  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ .

Let  $P^2_{\alpha}(\mathbb{D}, \omega dA)$  denotes the closure in  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  of the set of all polynomials. Although the two spaces agree for a large class of weights, there exist certain choices of weights  $\omega$  for which  $P^2_{\alpha}(\mathbb{D}, \omega dA)$  is strictly smaller than  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  (see [12, p. 245]). But according to a result in [13, p. 343], when the weight  $\omega$  is radial, the family of analytic polynomials forms a dense subset of  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ . Hence, for our choice of weights  $\omega$ , we have  $P^2_{\alpha}(\mathbb{D}, \omega dA) = L^2_{\alpha}(\mathbb{D}, \omega dA)$ . By [14, Lemma 2.1], the space  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  forms a Hilbert space consisting of analytic functions on  $\mathbb{D}$ , and its point evaluation functionals are locally uniformly bounded. Let  $k(z, \lambda)$  denote the reproducing kernel of this space. Then for every  $f \in L^2_{\alpha}(\mathbb{D}, \omega dA)$  and each  $\lambda \in \mathbb{D}$ , the reproducing property is expressed as  $f(\lambda) = \langle f, k(\cdot, \lambda) \rangle$ . By condition 3 imposed on the weight  $\omega$ , one deduces that  $k(z, 0) = k(0, \lambda) = 1$  for all  $z, \lambda \in \mathbb{D}$ .

Consider the operator  $M_z$  acting on  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  by multiplication with the independent variable  $z$ . It is known that  $M_z$  is bounded on  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  with  $\|M_z\| = 1$  (see [12, p. 246]). A closed subspace  $I$  of  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  is called  $M_z$ -invariant, or  $z$ -invariant for short, if  $M_z$  maps  $I$  into itself. Given a function  $f \in L^2_{\alpha}(\mathbb{D}, \omega dA)$ , we denote by  $[f]$  the  $z$ -invariant subspace generated by  $f$ ; that is, the smallest closed subspace invariant under  $M_z$  that contains  $f$ . Explicitly,

$$[f] = \{pf : p \text{ polynomial}\}^{clos \text{ in } L^2_{\alpha}(\mathbb{D}, \omega dA)}.$$

Suppose  $I$  is a  $z$ -invariant subspace of  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ . Then, by [14, Proposition 6.4] and [15, Theorem 4.1], we have  $I = [I \ominus zI]$ . Let  $M_I$  denotes the restriction of  $M_z$  to  $I$ , and let  $f(M_I)$  represents the restriction of the multiplication operator  $M_f$  to  $I$ , where  $f$  is a bounded analytic function on  $\mathbb{D}$ . A function  $\varphi \in L^2_{\alpha}(\mathbb{D}, \omega dA)$  is called  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ -inner function if the measure  $|\varphi|^2 \omega dA$  satisfies

$$\int_{\mathbb{D}} g(z) |\varphi(z)|^2 \omega(z) dA(z) = g(0) \tag{4}$$

for all bounded harmonic functions  $g$  on  $\mathbb{D}$ . In particular, the condition (4) implies that  $\varphi$  has norm one in  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ . Theorem 10.4 in [14] shows that any inner function  $\varphi \in L^2_{\alpha}(\mathbb{D}, \omega dA)$  satisfies  $\|\varphi f\| \geq \|f\|$  for all polynomials  $f$ . In other words, inner functions serve as expansive multipliers.

**Example 1.** For each  $\alpha \in \mathbb{D} \setminus \{0\}$ , consider the function  $\varphi_{\alpha}$ , defined by

$$\varphi_{\alpha}(z) = \sqrt{\frac{k(\alpha, \alpha)}{k(\alpha, \alpha) - 1}} \left(1 - \frac{k(z, \alpha)}{k(\alpha, \alpha)}\right) \tag{5}$$

It is well known that  $\varphi_{\alpha}$  solves the extremal problem  $\sup\{Reg(0) : g(\alpha) = 0, \|g\| = 1\}$  [16, 17]. Any such solution is called an extremal function. Lemma 4 in [18] then implies that  $\varphi_{\alpha}$  is an  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ -inner function.

Let  $T$  be a bounded linear operator on a Hilbert space  $H$ . The self-commutator  $[T^*, T]$  of  $T$  is defined as  $[T^*, T] = T^*T - TT^*$ , where  $T^*$  denotes the adjoint of  $T$ . If  $[T^*, T]$  is a positive operator, then  $T$  is called hyponormal.  $T$  is said to be subnormal if there exists a Hilbert space  $K \supseteq H$  and a normal operator  $S$  on  $K$  such that  $S(H) \subseteq H$  and  $T$  is the restriction of  $S$  to  $H$ . It follows that subnormal operators are hyponormal, see [7].  $T$  is called essentially normal when its self-commutator is compact. The operator  $T$  is called  $n$ -multicyclic if there are precisely  $n$  vectors  $x_1, x_2, \dots, x_n \in H$  whose closed linear span

$$\overline{\text{span}}\{h(T)x_i : i = 1, \dots, n, h \in R(\sigma(T))\}$$

coincides with  $H$ . Here  $R(\sigma(T))$  denotes the collection of all analytic rational functions defined on the spectrum  $\sigma(T)$  of  $T$ . The operator  $T$  is Fredholm if its range is closed and both  $\ker T$  and  $\ker T^*$  are finite dimensional. If  $T$  is a Fredholm operator, then the Fredholm index of  $T$ , written  $\text{ind}(T)$ , is given by

$$\text{ind}(T) = \dim(\ker T) - \dim(\ker T^*).$$

The essential spectrum of a bounded linear operator  $T$  on  $H$  is given by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

### 3. A skewed projection and some properties of $M_I$

This section is devoted to the study of skewed projection and several fundamental properties of  $M_I$ . By [19, Example 2.8 and Remark 2.11],  $M_z - \lambda$  is bounded below on  $L^2_{\alpha}(\mathbb{D}, \omega dA)$  for every  $\lambda \in \mathbb{D}$ . Consequently, its restriction  $M_z - \lambda|_I$  to a  $z$ -invariant subspace  $I$  is also bounded below. It follows that the subspace

$$(z - \lambda)I = \{g \in I : g(z) = (z - \lambda)f(z), f \in I, z \in \mathbb{D}\}$$

is closed in  $I$ . Therefore, the  $z$ -invariant subspace  $I$  can be decomposed as a direct sum  $I = (I \ominus zI) \oplus (z - \lambda)I$ , and this decomposition allows for the definition of a skewed projection onto  $I \ominus zI$ . The following lemma provides this result, extending Lemma 3.1 of [20] to the weighted Bergman space  $L^2_{\alpha}(\mathbb{D}, \omega dA)$ . Since the proof goes without any major changes, we will only give the details when  $I$  is generated by a function  $f$ . The general case follows from this special case. For details see Lemma 3.1 in [20].

**Lemma 1.** For any  $\lambda \in \mathbb{D}$ , the  $z$ -invariant subspace  $I$  can be decomposed as a direct sum of  $I \ominus zI$  and  $(z - \lambda)I$ . Moreover, if  $Q_{\lambda} : I \rightarrow I \ominus zI$  is the skewed projection operator corresponding to this decomposition of  $I$ , then

$$\|Q_{\lambda}\| \leq \frac{\sqrt{k(\lambda, \lambda) - 1}}{|\lambda|}. \tag{6}$$

**Proof.** We assume  $\lambda \neq 0$ , as the case  $\lambda = 0$  is clear. We first consider the case  $I = [f]$  where  $f \in L^2_{\alpha}(\mathbb{D}, \omega dA)$  has a zero of order  $n$  at the origin. Let  $\varphi$  be the extremal function associated with  $I$ , that is,  $\varphi$  solves

$$\sup\{Reg^{(n)}(0): g \in I, \text{ and } \|g\| \leq 1\}. \tag{7}$$

It follows from Shimorin’s results (see [16, 21]) that for any  $g \in I$ ,  $\frac{g}{\varphi}$  is analytic on  $\mathbb{D}$ . To prove that  $Q_\lambda$  is bounded, it is enough to verify that if  $g$  is in  $I$ , then

$$\left| \frac{g}{\varphi}(\lambda) \right| \leq C(\lambda) \|g\|, \tag{8}$$

for some constant  $C(\lambda)$  depending only on  $\lambda$ . It suffices to prove the above inequality under the assumption that  $\varphi(\lambda) \neq 0$ . By (5), the extremal function related to the zero set  $\{\lambda\}$  is

$$\varphi_\lambda(z) = \sqrt{\frac{k(\lambda,\lambda)}{k(\lambda,\lambda)-1}} \left( 1 - \frac{k(z,\lambda)}{k(\lambda,\lambda)} \right),$$

where  $z \in \mathbb{D}$ . Notice that  $\varphi_\lambda(0) = \left( \frac{k(\lambda,\lambda)-1}{k(\lambda,\lambda)} \right)^{1/2}$ . We assume  $g(\lambda) \neq 0$ , otherwise there is nothing to prove. Let  $h(z) := \varphi(z) - \frac{\varphi(\lambda)}{\lambda g(\lambda)} z g(z)$ . One can easily see that  $\frac{h}{\varphi_\lambda} \in [f] = I$ . Thus, from  $\varphi$ ’s extremal property (7),

$$\left| \frac{h^{(n)}(0)}{\varphi_\lambda^{(n)}(0)} \right| \leq \varphi^{(n)}(0) \left\| \frac{h}{\varphi_\lambda} \right\|.$$

Notice that  $h^{(n)}(0) = \varphi^{(n)}(0)$ , and by Shimorin’s theorem  $\left\| \frac{h}{\varphi_\lambda} \right\| \leq \|h\|$ . Now  $\|h\|^2 \leq 1 + \left| \frac{\varphi(\lambda)}{\lambda g(\lambda)} \right|^2 \|g\|^2$ , and an easy computation shows that  $\left| \frac{g}{\varphi}(\lambda) \right| \leq \frac{\sqrt{k(\lambda,\lambda)-1}}{|\lambda|} \|g\|$ . Thus, (8) holds. Hence, (6) follows. This also shows that if  $g(z) = \alpha\varphi(z) + (z - \lambda)k(z)$  where  $k \in [f]$ , then

$$\|\alpha\varphi\| = |\alpha| = \left| \frac{g}{\varphi}(\lambda) \right| \leq \frac{\sqrt{k(\lambda,\lambda)-1}}{|\lambda|} \|g\|.$$

This means that the subspaces  $I \ominus zI$  and  $(z - \lambda)I$  are at a positive angle. Finally, for each  $g \in [f] = I$ , we have

$$g(z) = \frac{g}{\varphi}(\lambda)\varphi(z) - (z - \lambda) \frac{g(z) - \frac{g}{\varphi}(\lambda)\varphi(z)}{z - \lambda} \in I \ominus zI + (z - \lambda)I.$$

Hence, this proves the lemma in the case  $I = [f]$ .

The idea to prove the following corollaries comes from [10]. Since the proofs are short, we included them here for completeness.

**Corollary 1.** If  $I$  is a  $z$ -invariant subspace of  $L^2_a(\mathbb{D}, \omega dA)$ , then for all  $\lambda \in \mathbb{D}$ ,  $f \in H^\infty(\mathbb{D})$ , and  $g \in I$ ,

$$Q_\lambda(fg) = f(\lambda)Q_\lambda g. \tag{9}$$

**Proof.** We can write  $f(z) = f(\lambda) + (z - \lambda)\phi(z)$ , with  $\phi \in H^\infty(\mathbb{D})$  and  $z \in \mathbb{D}$ . For any  $g \in I$ , this gives  $(fg)(z) = f(\lambda)g(z) + (z - \lambda)(\phi g)(z)$ . Since  $(z - \lambda)\phi g \in (z - \lambda)I$ , it follows that  $Q_\lambda(fg) = f(\lambda)Q_\lambda g$ .

**Corollary 2.** Let  $I \subseteq L^2_a(\mathbb{D}, \omega dA)$  be a  $z$ -invariant subspace with  $\dim(I \ominus zI) = n < \infty$ . Then, for each  $\lambda \in \mathbb{D}$ , the operator satisfies

$$\|Q_\lambda\|_2 \leq \frac{\sqrt{n(k(\lambda,\lambda)-1)}}{|\lambda|}, \tag{10}$$

with  $\|\cdot\|_2$  denoting the Hilbert-Schmidt norm.

**Proof.** Let  $\{u_1, \dots, u_n\}$  denote an orthonormal basis of  $I \ominus zI$ , and let  $\{v_m\}$  be an orthonormal basis of  $I$ . For each  $m \geq 1$ , the projection  $Q_\lambda v_m$  can be expanded in terms of the  $u_i$  as  $Q_\lambda v_m = \sum_{i=1}^n \langle Q_\lambda v_m, u_i \rangle u_i$ , which immediately implies

$$\|Q_\lambda v_m\|^2 = \sum_{i=1}^n |\langle Q_\lambda v_m, u_i \rangle|^2. \text{ Thus}$$

$$\sum_{m=1}^\infty \|Q_\lambda v_m\|^2 = \sum_{i=1}^n \sum_{m=1}^\infty |\langle v_m, Q_\lambda^* u_i \rangle|^2 = \sum_{i=1}^n \|Q_\lambda^* u_i\|^2 \leq n \frac{k(\lambda,\lambda)-1}{|\lambda|^2},$$

which implies that (10) holds.

**Remark 1.** The reproducing kernel  $k(z, \zeta)$  is known to satisfy the following inequality (see, for example, [12, Corollary 9.7, p. 250]);

$$\frac{1-|z\zeta|}{|1-z\bar{\zeta}|^2} \leq |k(z, \zeta)| \leq \frac{1+|z\zeta|}{|1-z\bar{\zeta}|^2}. \tag{11}$$

From (11), one can easily get that  $\|Q_\lambda\|_2 \leq \frac{\sqrt{3n}}{(1-|\lambda|^2)}$ . (12)

The following proposition collects several basic properties of  $M_I$ . Since the argument parallels those in Proposition 2.1, Theorem 2.1, and Theorem 3.1 of [9], the proof is omitted.

**Proposition 1.** Let  $I \subseteq L^2_a(\mathbb{D}, \omega dA)$  be a  $z$ -invariant subspace. Then

1.  $\|M_I\| = 1$  and the spectrum of  $M_I$  coincides with the closed unit disc  $\mathbb{D}$ .
2. The operator  $M_I$  is Fredholm, with  $\text{ind}(M_I) = -n$  if and only if  $\dim(I \ominus zI) = n < \infty$ .
3. When  $M_I$  is Fredholm, its essential spectrum satisfies  $\sigma_e(M_I) = \partial\mathbb{D}$ ; otherwise,  $\sigma_e(M_I) = \mathbb{D}$ .
4.  $M_I$  has a compact self-commutator if and only if  $I \ominus zI = I \cap (zI)^\perp$  is finite dimensional.

#### 4. Proof of the Main Theorem

We first figure out the trace of the self-commutator of  $p(M_I)$ , where  $p$  is a polynomial. To do that, we need the following result.

**Theorem 1** ([22, p. 234]). Suppose  $H$  is a Hilbert space and  $T \in B(H)$  is a hyponormal operator whose self-commutator belongs to the trace class. For all polynomials  $p, q \in \mathbb{C}[x, y]$ ,  $\text{tr}[p(X, Y), q(X, Y)] =$

$$\frac{1}{2\pi i} \int_{\mathbb{C}} (\partial_x p \partial_y q - \partial_y p \partial_x q) g_T dx dy, \tag{13}$$

where  $g_T$  denotes the principal function associated with  $T = X + iY$ .

The function  $g_T$  additionally satisfies the properties listed below ([22, p. 243-244]).

1. The support of  $g_T$  coincides with the spectrum of  $T$ , that is,  $\text{supp}(g_T) = \sigma(T)$ . (14)
2.  $g_T(z) = -\text{ind}(T - z)$ ,  $z \notin \sigma_e(T)$ . (15)

Since the operator  $M_z$  on the weighted Bergman space  $L^2_a(\mathbb{D}, \omega dA)$  is subnormal, the operator  $M_I$  is likewise subnormal, and thus hyponormal. Moreover,  $I = [I \ominus zI]$ , and hence  $M_I$  is an  $n$ -cyclic operator whenever  $\dim(I \ominus zI) = n$ . By the Berger-Shaw theorem (1) and

Proposition 1, we have  $tr[M^*_I, M_I] \leq n$ . Applying Proposition 1 again together with Theorem 1 (13) and the fundamental properties (14) and (15) of the principal function  $g_T$  for  $T = M_I$  implies the result below.

**Lemma 2.** Suppose  $I \subseteq L^2_\alpha(\mathbb{D}, \omega dA)$  is a  $z$ -invariant subspace and  $\dim(I \ominus zI) < \infty$ . Then, for every polynomial  $p(z)$ ,

$$tr[p(M_I)^*, p(M_I)] = \dim(I \ominus zI) \int_{\mathbb{D}} |p'(z)|^2 dA(z). \quad (16)$$

**Proof.** Assume  $\dim(I \ominus zI) = n$ . By Proposition 1, we have  $\sigma(M_I) = \mathbb{D}$ , and  $\sigma_e(M_I) = \partial\mathbb{D}$ . Moreover, for each  $z \in \mathbb{D}$ , the Fredholm index satisfies  $ind(M_I - z) = -n$ . It follows from (15) that the principal function of  $M_I$  is given by  $g_{M_I}(z) = n$  for  $z \in \mathbb{D}$ , and  $g_{M_I}(z) = 0$  for  $|z| > 1$ , since by (14)  $supp(g_{M_I}) = \mathbb{D}$ . Consequently, (16) holds by Theorem 1 (13).

The next lemma extends Lemma 5 of [10] to the setting of the weighted Bergman space  $L^2_\alpha(\mathbb{D}, \omega dA)$ . Since the reasoning is essentially unchanged, its proof is omitted.

**Lemma 3.** Let  $I \subseteq L^2_\alpha(\mathbb{D}, \omega dA)$  be a  $z$ -invariant subspace and  $k_I(z, \zeta)$  denote the reproducing kernel associated with  $I$ . Then, for every bounded analytic function  $f \in H^\infty(\mathbb{D})$ , the following identity holds:

$$tr[f(M_I)^*, f(M_I)] = \int_{\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(\lambda)|^2 |k_I(z, \lambda)|^2 \omega(z) dA(z) \omega(\lambda) dA(\lambda). \quad (17)$$

**Corollary 3.** Let  $I$  be a subspace of  $L^2_\alpha(\mathbb{D}, \omega dA)$  that is  $z$ -invariant. Then, for every  $f \in H^\infty(\mathbb{D})$ , one has

$$tr[f(M_I)^*, f(M_I)] \leq \dim(I \ominus zI) \int_{\mathbb{D}} |f'(z)|^2 dA(z). \quad (18)$$

**Proof.** If  $\dim(I \ominus zI) = \infty$  or  $\int_{\mathbb{D}} |f'(z)|^2 dA(z) = \infty$ , then there is nothing to do. Suppose  $\dim(I \ominus zI) = n < \infty$  and  $\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$ . Now let  $p_n$  be the  $n$ -th Taylor polynomial of  $f$ . Then  $p_n \rightarrow f$  pointwise and in the Dirichlet norm, and hence by an application of Fatou's lemma and Lemma 3 (17), we get

$$\begin{aligned} &tr[f(M_I)^*, f(M_I)] \\ &\leq \liminf_n \int_{\mathbb{D}} \int_{\mathbb{D}} |p_n(z) - p_n(\lambda)|^2 |k_I(z, \lambda)|^2 \omega(z) dA(z) \omega(\lambda) dA(\lambda) \\ &= \liminf_n tr[p_n(M_I)^*, p_n(M_I)], \end{aligned}$$

where the above last equality follows again from Lemma 3 (17). Applying Lemma 2 (16), we obtain

$$tr[p_n(M_I)^*, p_n(M_I)] = \dim(I \ominus zI) \int_{\mathbb{D}} |p'_n(z)|^2 dA(z).$$

Thus

$$tr[f(M_I)^*, f(M_I)] \leq \liminf_n \dim(I \ominus zI) \int_{\mathbb{D}} |p'_n(z)|^2 dA(z) = \dim(I \ominus zI) \int_{\mathbb{D}} |f'(z)|^2 dA(z),$$

Therefore, (18) holds.

It suffices, for the proof of the main theorem, to establish the reverse implication of Corollary 3. This will rely on the following lemma due to Berger and Shaw [1, 2].

**Lemma 4.** Let  $T$  and  $A$  be hyponormal operators acting on Hilbert spaces  $H$  and  $K$ , respectively. Assume there exists an injective Hilbert-Schmidt operator  $S: H \rightarrow K$  with dense range such that  $ST = AS$ . Then the inequality  $tr[T^*, T] \geq tr[A^*, A]$  holds.

We are now in a position to establish the main theorem. The argument follows the approach used in Theorem 10 of [10].

**Theorem 2.** Let  $I \subseteq L^2_\alpha(\mathbb{D}, \omega dA)$  be a  $z$ -invariant subspace and let  $f \in H^\infty(\mathbb{D})$ . Then

$$tr[f(M_I)^*, f(M_I)] = \dim(I \ominus zI) \int_{\mathbb{D}} |f'(z)|^2 dA(z). \quad (19)$$

**Proof.** By Corollary 3 (18), we only need to show that

$$tr[f(M_I)^*, f(M_I)] \geq \dim(I \ominus zI) \int_{\mathbb{D}} |f'(z)|^2 dA(z). \quad (20)$$

This inequality clearly holds if  $tr[f(M_I)^*, f(M_I)] = \infty$ . Hence, we suppose that  $tr[f(M_I)^*, f(M_I)] < \infty$ . If  $f \in H^\infty(\mathbb{D})$  is constant, then one readily verifies that the commutator  $[f(M_I)^*, f(M_I)]$  vanishes, and therefore the above inequality is trivially satisfied. Consequently, we restrict our attention to the case where  $f$  is not constant.

If  $\dim(I \ominus zI) = \infty$ , then by a similar argument of Lemma 9 of [10], we can show that  $f(M_I)$  is essentially normal if and only if  $f$  is constant. In this case, unless  $f$  is constant, we have  $tr[f(M_I)^*, f(M_I)] = \infty$ . Therefore, the only case left is when  $\dim(I \ominus zI) < \infty$ . Set  $\dim(I \ominus zI) = n$ . Fix a parameter  $r \in (0, 1)$  and set  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ . Denote by  $L^2_\alpha(\mathbb{D}_r, \omega dA)$  the weighted Bergman space over the disc  $\mathbb{D}_r$ . Choose an orthonormal basis  $\{\varphi_i\}_{i=1}^n$  of the subspace  $I \ominus zI$ . We then introduce a linear map

$$S: I \rightarrow \bigoplus_{i=1}^n L^2_\alpha(\mathbb{D}_r, \omega dA)$$

defined by  $S(g) = (g_1, \dots, g_n)$ , where for each  $i = 1, \dots, n$  and  $z \in \mathbb{D}$ ,  $g_i(z) = \langle Q_z g, \varphi_i \rangle$ . Then by a similar argument of Proposition 4.9 of [20], we can show that  $g = g_1 \varphi_1 + \dots + g_n \varphi_n$ . Hence, we can use this to show that  $S$  is one-to-one, and by using Corollary 1 (9), we can further show that  $S$  has dense range. Let  $\{e_j\}_{j=1}^\infty$  be an orthonormal basis of  $I$ . Then, for each index  $j$ , one obtains

$$\begin{aligned} \|Se_j\|^2 &= \sum_{i=1}^n \int_{\mathbb{D}_r} |\langle Q_z e_j, \varphi_i \rangle|^2 \omega(z) dA(z) = \int_{\mathbb{D}_r} \|Q_z e_j\|^2 \omega(z) dA(z) \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{j=1}^\infty \|Se_j\|^2 &= \int_{\mathbb{D}_r} \sum_{j=1}^\infty \|Q_z e_j\|^2 \omega(z) dA(z) \leq \\ &\int_{\mathbb{D}_r} \frac{3n}{(1-|z|^2)^2} \omega(z) dA(z) \leq \frac{3n}{(1-r^2)^2}, \end{aligned}$$

where the first inequality is a consequence of Remark 1 (12). It follows that  $S$  belongs to the Hilbert-Schmidt class. We now consider the bounded linear operator

$$M_f^n: \bigoplus_{i=1}^n L^2_\alpha(\mathbb{D}_r, \omega dA) \rightarrow \bigoplus_{i=1}^n L^2_\alpha(\mathbb{D}_r, \omega dA)$$

defined by  $M_f^n(g_1, \dots, g_n) = (fg_1, \dots, fg_n)$ .

Notice that the operator  $f(M_I)$  is subnormal since it arises as the compression of the normal multiplication operator  $M_f$  on  $L^2(\mathbb{D}, \omega dA)$  to the invariant subspace  $I$ . Since the direct sum of subnormal operators is subnormal,

$M_f^n$  is subnormal as well. Thus, these operators are hyponormal. To use Lemma 4, we need to show that  $Sf(M_I) = M_f^n S$ .

In fact, let  $g \in I$ . Then  $Sf(M_I)g = S(fg) = (G_1, \dots, G_n)$ , where  $G_i(z) = \langle Q_z(fg), \varphi_i \rangle$ ,  $z \in \mathbb{D}$ ,  $1 \leq i \leq n$ . Observe that, by Corollary 1 (9), we have

$$G_i(z) = \langle Q_z(fg), \varphi_i \rangle = f(z)\langle Q_z g, \varphi_i \rangle.$$

Thus

$$Sf(M_I)g = (f(z)\langle Q_z g, \varphi_1 \rangle, \dots, f(z)\langle Q_z g, \varphi_n \rangle) = M_f^n Sg.$$

Hence, by Lemma 4,

$$\text{tr}[(M_f^n)^*, M_f^n] \leq \text{tr}[f(M_I)^*, f(M_I)].$$

It follows from [3] that  $\text{tr}[M_f^*, M_f] = \int_{\mathbb{D}_r} |f'(z)|^2 dA(z)$ .

Thus,

$$\begin{aligned} \text{tr}[f(M_I)^*, f(M_I)] &\geq \text{tr}[(M_f^n)^*, M_f^n] = \\ n \text{tr}[M_f^*, M_f] &= n \int_{\mathbb{D}_r} |f'(z)|^2 dA(z). \end{aligned} \quad (21)$$

Since  $r \in (0,1)$  is arbitrary, letting  $r \rightarrow 1$  in (21) gives (20). Therefore, (19) holds.

## Conflict of Interest

There are no conflicts of interest in this work.

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