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THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION IN $\mathbb{R}^3$

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Abstract. In the paper it is considered the regularization of the Cauchy problem for systems of elliptic type equations of the first order with constant coefficients factorisable Helmholtz operator in three-dimensional bounded domain. Using the results of [1-6; 19,20,21,22], we construct in explicit form Carleman matrix and, based on the regularized solution of the Cauchy problem.

1. Introduction

It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e. incorrect (example Hadamard, see for instance [10], p. 39). In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see for instance [3]).

The uniqueness of the solution follows from Holmgren's general theorem [14]. The conditional stability of the problem follows from the work of A.N. Tikhonov [13], if we restrict the class of possible solutions to a compactum.

In this paper we construct a family of vector-functions $U_{\sigma(\delta)}(x) = U(x, f_\delta)$ depending on a parameter σ and it is proved that, under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$; as $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at the point $x \in G$.

Following A.N. Tikhonov [13], a family of vector-functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem. For special domains, the problem

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of extending bounded analytic functions in the case when the data is specified exactly on a part of the boundary was considered by Carleman [4]. The researches of T. Carleman were continued by G.M. Goluzin and V.I. Krylov [12]. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in [11]. The use of the classical Green's formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by Academician M.M. Lavrent'ev (see for instance[5,6]), in his famous monograph. Extending Lavrent'ev idea, Yarmukhamedov constructed the Carleman function for the Cauchy problem for the Laplace and Helmholtz equations (see for instance [7,8,9]). The Cauchy problem for the multidimensional Lame system is considered by O.I. Makhmudov and I.E. Niyozov (see for instance [17,19]). The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, O.I. Makhmudov, I.E. Niyozov and others.

The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see for instance [1]). In many well-posed problems for a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz, the calculation of the value of the vector function on the whole boundary is inaccessible. Therefore, the problem of reconstructing, solving a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz (see for instance [19,20,21,22]), is one of the topical problems in the theory of differential equations.

Let $\mathbb{R}^3$ be the three-dimensional real Euclidean space,

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ y = (y_1, y_2, y_3) \in \mathbb{R}^3,\ x' = (x_1, x_2) \in \mathbb{R}^2, y' = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^3$ be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_3 = 0$ and of a smooth surface S lying in the half-space $y_3 > 0$, that i.s., $\partial G = S \bigcup T$.

We introduce the following notation:

$$x^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ transposed vector } x,\ r = |y - x|\,,\ \alpha = |y' - x'|,$$

$$U(x) = (U_1(x), \dots, U_n(x))^T,\, u^0 = (1, \dots, 1) \in \mathbb{R}^n,\, n = 2^m,\ m = 3,$$

$$E(z) = \left\| \begin{matrix} z_1 \dots 0 \\ \dots\dots. \\ 0 \dots z_n \end{matrix} \right\| \text{ diagonal matrix, } z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

Let $D(x^T)$ the $(n \times n)-$the matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(x^T)D(x^T) = E((|x|^2 + \lambda^2)u^0),$$

where $D^*(x^T)$ is the Hermitian conjugate matrix $D(x^T)$, $\lambda-$real number.

We consider in the domain G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right) U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of differential operators is of the first order.

We denote by $A(G)$ the class of vector-functions in a domain G of continuous on $\overline{G} = G \bigcup \partial G$ and satisfying the system (1).

2. Statement of the problem

The Cauchy problem. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y), \ y \in S. \tag{2}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G} M(y, x)U(y)ds_y, \quad x \in G, \tag{3}$$

where

$$M(y, x) = \left(E \left(-\frac{e^{i\lambda r}}{4\pi r} u^0 \right) D^* \left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Here $t = (t_1, t_2, t_3)$ is the unit exterior normal, drawn at a point y, the surface ∂G, $-\frac{e^{i\lambda r}}{4\pi r}$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$. [15].

We denote by $K(w)$ is an entire function taking real values for real w ($w = u + iv$; $u, v-$real numbers) and satisfying the following conditions:

$$\begin{gathered} K(u) \neq 0, \ \sup_{v \geq 1} |v^p K^p(w)| = \mathrm{M}(u, p) < \infty, \\ -\infty < \xi < \infty, \ p = 0, 1, 2, 3. \end{gathered} \tag{4}$$

We define a function $\Phi(y, x)$ when $y \neq x$ by the following equality:

$$\Phi(y, x) = -\frac{1}{2\pi^2 K(x_3)} \int\limits_0^\infty \mathrm{Im} \frac{K(w)}{w - x_3} \frac{\cos \lambda u}{\sqrt{u^2 + \alpha^2}} du, \tag{5}$$

In the formula (5), choosing

$$K(w) = \exp(\sigma w^2), \ K(x_3) = \exp(\sigma x_3^2), \ \sigma > 0,$$

we get

$$\Phi_\sigma(y, x) = -\frac{e^{-\sigma x_3^2}}{2\pi^2} \int\limits_0^\infty \mathrm{Im} \frac{\exp(\sigma w^2)}{w - x_3} \frac{\cos \lambda u}{\sqrt{u^2 + \alpha^2}} du, \tag{6}$$

The formula (3) is true if instead $-\frac{e^{i\lambda r}}{4\pi r}$ of substituting the function

$$\Phi_\sigma(y, x) = -\frac{e^{i\lambda r}}{4\pi r} + g_\sigma(y, x), \tag{7}$$

where $g_\sigma(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int\limits_{\partial G} N_\sigma(y, x)U(y)ds_y, \quad x \in G, \tag{8}$$

where

$$N_\sigma(y, x) = \left(E \left(\Phi_\sigma(y, x) u^0 \right) D^* \left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

3. Regularization of the Cauchy problem

Theorem 3.1. *Let $U(y) \in A(G)$ it satisfy the inequality*

$$|U(y)| \leq 1, \; y \in T. \tag{9}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y,x)U(y)ds_y, \;\; x \in G, \tag{10}$$

Then we have the estimate

$$|U(x) - U_\sigma(x)| \leq C(x)\sigma e^{-\sigma x_3^2}, \;\; \sigma > 1, \; x \in G. \tag{11}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(x)$.

Proof. Using the integral formula (8) and the equality (10), we obtain

$$U(x) = U_\sigma(x) + \int_T N_\sigma(y,x)U(y)ds_y, \; x \in G.$$

Taking into account the inequality (9), we estimate the following

$$\begin{aligned} |U(x) - U_\sigma(x)| \leq \left| \int_T N_\sigma(y,x)U(y)ds_y \right| \leq \\ \leq \int_T |N_\sigma(y,x)|\, ds_y, \;\; x \in G. \end{aligned} \tag{12}$$

To do this, we estimate the integrals $\int_T |\Phi_\sigma(y,x)|\, ds_y$, $\int_T \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_j} \right| ds_y$, $(j = 1, 2)$ and $\int_T \left| \frac{\partial \Phi_\sigma}{\partial y_3}(y,x) \right| ds_y$ on the part T of the plane $y_3 = 0$.

Separating the imaginary part of (6), we obtain

$$\begin{aligned} \Phi_\sigma(y,x) = \frac{e^{\sigma(y_3^2 - x_3^2)}}{2\pi^2} \Bigg[\int_0^\infty \frac{e^{-\sigma(u^2+\alpha^2)} \cos 2\sigma y_3 \sqrt{u^2+\alpha^2}}{u^2+r^2} \; \cos \lambda u \, du - \\ - \int_0^\infty \frac{e^{-\sigma(u^2+\alpha^2)} (y_3 - x_3) \sin 2\sigma y_3 \sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{\cos \lambda u}{\sqrt{u^2+\alpha^2}} du \Bigg], \;\; x_3 > 0. \end{aligned} \tag{13}$$

Taking into account equality (13), we have

$$\int_T |\Phi_\sigma(y,x)|\, ds_y \leq C(x)\sigma e^{-\sigma x_3^2}, \; \sigma > 1, \; x \in G, \tag{14}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y,x)}{\partial y_j} = \frac{\partial \Phi_\sigma(y,x)}{\partial s} \frac{\partial s}{\partial y_j} = 2(y_j - x_j) \frac{\partial \Phi_\sigma(y,x)}{\partial s}, \; j = 1, 2. \tag{15}$$

Where

$$\frac{\partial\Phi_\sigma(y,x)}{\partial s} = \frac{e^{\sigma(y_3^2-x_3^2)}}{2\pi^2}\left[\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{-\sigma\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{u^2+r^2} - \right.\right.$$

$$\left. - \frac{\sigma y_3\sin 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} - \frac{\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2}\right)\cos\lambda u\,du -$$

$$- \int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{-\sigma(y_3-x_3)\sin 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} + \right. \tag{16}$$

$$+\frac{\sigma y_3(y_3-x_3)\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)(u^2+\alpha^2)} - \int\limits_0^\infty \frac{(y_3-x_3)\sin 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} -$$

$$\left.\left. - \int\limits_0^\infty \frac{(y_3-x_3)\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)(u^2+\alpha^2)^{3/2}}\right)\cos\lambda u\,du\right],\ s=\alpha^2.$$

Taking into account (15) - (16), we obtain

$$\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{17}$$

Similarly we obtain

$$\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{18}$$

To estimate the integral $\int\limits_T \left|\frac{\partial\Phi_\sigma}{\partial y_3}(y,x)\right| ds_y$, we use the equality

$$\frac{\Phi_\sigma(y,x)}{\partial y_3} = \frac{e^{\sigma(y_3^2-x_3^2)}}{2\pi^2}\left[\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{2\sigma y_3\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{u^2+r^2} - \right.\right.$$

$$\left. - \frac{2\sigma\sqrt{u^2+\alpha^2}\sin 2\sigma y_3\sqrt{u^2+\alpha^2}}{u^2+r^2} - \frac{2(y_3-x_3)\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2}\right)\cos\lambda u\,du -$$

$$- \int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{2\sigma y_3(y_3-x_3)\sin 2\sigma y_3\sqrt{u^2+\alpha^2}+\sin 2\sigma y_3\sqrt{u^2+\alpha^2}}{u^2+r^2} - \right. \tag{19}$$

$$\left.\left. - \frac{2\sigma\sqrt{u^2+\alpha^2}(y_3-x_3)\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{u^2+r^2} - \frac{2(y_3-x_3)^2\cos 2\sigma y_3\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2}\right)\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}du\right].$$

Taking into account the equality (19), we obtain

$$\int\limits_T \left|\frac{\partial\Phi_\sigma}{\partial y_3}(y,x)\right| ds_y \leq C(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G, \tag{20}$$

From the inequalities (14), (17), (18), and (20), we obtain (11). □

Corollary 3.1. The limiting equality

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set in the domain G.

Theorem 3.2. *Let $U(y) \in A(G)$ it satisfy condition (9), and on a smooth surface S the inequality*

$$|U(y)| \le \delta,\ 0 < \delta < e^{-\sigma \bar{y}_3^2}, \tag{21}$$

where $\bar{y}_3^2 = \max\limits_{y\in S} y_3^2$. Then we have the estimate

$$|U(x)| \le C(x)\sigma\delta^{\frac{x_3^2}{\bar{y}_3^2}},\ \sigma > 1,\ x \in G. \tag{22}$$

Proof. Using the integral formula (8), we have

$$U(x) = \int\limits_S N_\sigma(y,x)U(y)ds_y + \int\limits_T N_\sigma(y,x)U(y)ds_y,\ x \in G.$$

We estimate the following

$$|U(x)| \le \left|\int\limits_S N_\sigma(y,x)U(y)ds_y\right| + \left|\int\limits_T N_\sigma(y,x)U(y)ds_y\right|,\ x \in G. \tag{23}$$

Taking inequality (21) into account, we estimate the first integral in (23).

$$\begin{aligned}\left|\int\limits_S N_\sigma(y,x)U(y)ds_y\right| &\le \int\limits_S |N_\sigma(y,x)|\,|U(y)|\,ds_y \le \\ &\le \delta\int\limits_S |N_\sigma(y,x)|\,ds_y,\ x \in G.\end{aligned} \tag{24}$$

To do this, we estimate the integrals $\delta\int\limits_S |\Phi_\sigma(y,x)|\,ds_y$, $\delta\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_j}\right| ds_y$, $(j = 1, 2)$ and $\delta\int\limits_S \left|\frac{\partial\Phi_\sigma}{\partial y_3}(y,x)\right| ds_y$ on a smooth surface S.

Taking into account the equality (13), we have

$$\delta\int\limits_S |\Phi_\sigma(y,x)|\,ds_y \le C(x)\sigma\delta\, e^{\sigma(y_3^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{25}$$

To estimate the second integral, we use equalities (15) and (16).

$$\delta\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \le C(x)\sigma\delta\, e^{\sigma(y_3^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{26}$$

Similarly, using equalities (15) and (16) we obtain

$$\delta\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \le C(x)\sigma\delta\, e^{\sigma(y_3^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{27}$$

Taking into account the equality (19), we obtain

$$\delta\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \le C(x)\sigma\delta\, e^{\sigma(y_3^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{28}$$

From (25) - (28), we obtain

$$\left|\int\limits_S N_\sigma(y,x)U(y)ds_y\right| \le C(x)\sigma\delta\, e^{\sigma(y_3^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{29}$$

The following is known

$$\left|\int\limits_T N_\sigma(y,x)U(y)ds_y\right| \le C(x)\sigma\, e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{30}$$

Now taking into account (29) - (30), we have

$$|U(x)| \le \frac{C(x)\sigma}{2}(\delta^{\sigma \bar{y}_3^2}+1)e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{31}$$

Choosing σ from the equality

$$\sigma = \frac{1}{\bar{y}_3^2}\ln\frac{1}{\delta}, \tag{32}$$

we obtain the inequality (22). □

Let $U(y)\in A(G)$ and together $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error, $0<\delta<e^{-\sigma\bar{y}_3^2}$, $\max\limits_S|U(y)-f_\delta(y)|\le\delta$.

We set

$$U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x)f_\delta(y)ds_y,\ x\in G. \tag{33}$$

The following theorem takes place

Theorem 3.3. *Let $U(y)\in A(G)$ on the part of the plane $y_3=0$ satisfy condition (9).*

Then we have the estimate

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \le C(x)\sigma\delta^{\frac{x_3^2}{\bar{y}_3^2}},\ \sigma>1,\ x\in G. \tag{34}$$

Proof. From the integral formulas (8) and (33), we have

$$U(x)-U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x)\left\{U(y)-f_\delta(y)\right\}ds_y + \int_T N_\sigma(y,x)U(y)ds_y.$$

Now, repeating the proof of Theorems 1 and 2, we obtain

$$|U(x)-U_{\sigma\delta}(x)| \le \frac{C(x)\sigma}{2}(\delta\, e^{\sigma\bar{y}_3^2}+1)\, e^{-\sigma x_3^2}.$$

Hence, choosing σ from (32), we obtain (34). □

Corollary 3.2. The limiting equality

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on each compact set in the domain G.

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