EGE AKADEMİK BAKIŞ / *EGE ACADEMIC REVIEW*

Cilt: 15 • Sayı: 4 • Ekim 2015
ss. 445-448

Indivisible Goods, Core and Walrasian Equilibrium: A Survey*

Bölünmez Mallar, Çekirdek ve Walras Dengesi: İnceleme

Hakan İNAL[1]

ABSTRACT

This article surveys some of the developments in allocation of indivisible goods and Walrasian equilibrium using a common notation aiming to make it easier for the reader to comprehend various contributions on the subject.

Keywords: Discrete Goods, Walras equilibrium, Core, Many-to-one Matching.

ÖZET

Bölünmez malların paylaşımı ve Walras dengesi konusundaki bazı önemli gelişmeler okuyucuya kolaylık olması için ortak bir matematiksel dil kullanılarak incelenmiştir.

Anahtar Kelimeler: Bölünmez Mallar, Walras Dengesi, Çekirdek, Bire Çok Eşleşme.

1. INTRODUCTION

In this paper, I present some of the major results on allocation of indivisible goods, Walrasian equilibrium and the core using a common notation to make it easier for the reader to understand and compare these results. Early works laying the foundations range from the housing market in Shapley and Scarf (1974) and Roth and Postlewaite (1977) to those works on many-to-one matchings including, Gale and Shapley (1962), Kelso and Crawford (1982), Roth and Sotomayor (1990), and Echenique and Oviedo (2004). Recent works on the subject include Gul and Stacchetti (1999), and Sun and Yang (2006) who build on some of these aforementioned studies, see Sönmez and Ünver (2011) for a recent survey on these subjects.

Study of Walrasian equilibrium and the core in discrete goods markets dates back to the early works in the matching, specifically many-to-one matching, markets. Many-to-one matchings were introduced by Gale and Shapley (1962) in the context of college admission problem. Each college has preferences over individual students, and has a limited number of seats, quota, it can fill. Each student also has preferences over colleges. Gale and Shapley (1962) present an algorithm in which students propose to colleges they prefer among those who did not rejected them, and colleges reject all but those they prefer most within their quotas. They show that this algorithm leads to a matching of colleges and students such that it is pairwise stable and it is student optimal, i.e. it is preferred to all other stable matchings by all students. In Section 3, many-to-one matching models with and without money are presented. Shapley and Scarf (1974), the earliest study of core and Walrasian equilibrium in economies with discrete goods is presented in Section 2. In a recent work, Gul and Stacchetti (1999), building on the work of Kelso and Crawford (1982), presented an analysis of core and Walrasian equilibrium in discrete good economies with money. An extension of their work to include both substitutes and complements is developed by Sun and Yang (2006). These models are presented in Section 3.3.

2. HOUSING MARKET

There is a finite set of agents $N = \{1, \ldots, n\}$ and a set of houses X such that each agent is initially assigned to a distinct house, $|N| = |X|$. Initial allocation of houses is represented by a one-to-one and onto function $\mu_0 : N \to X$ such that for each agent $i \in N$, $\mu_0(i) \in X$ is the house initially owned by agent i. Each agent has strict preferences over houses and interested in owning at most one house. A group of agents $J \in 2^N \setminus \{\emptyset\}$ form a weakly blocking coalition for a matching μ if there exists a matching μ' such that $\cup_{i\in J}\, \mu_0(i) = \cup_{i\in J}\, \mu'(i)$, and for each $j \in J$, $\mu'(j) \succcurlyeq_j \mu(j)$, with at least one strict inequality. If all inequalities are strict, then the corresponding coalition is called a strongly blocking coalition. A matching μ is in the weak core if there is no weakly blocking coalition. A matching μ is in the strong core if there is no

[1] **Department of International Trade and Finance, Izmir University, Izmir, Turkey**
* **I am grateful to the Editor Prof. Dr. Özlem Önder and an anonymous referee for their helpful comments.**

strongly blocking coalition. Shapley and Scarf (1974) presented the **top trading cycle** (**TTC**) algorithm, attributed to David Gale, to show that in this problem there exists a strong core allocation of houses. Moreover they show that there exist Walrasian equilibrium prices supporting this allocation.

Gale's Top-Trading Cycles Algorithm

Initially, all houses and their owners are on the market. At step $k \in \{0,1,\ldots\}$ of the algorithm, each agent points at the owner of the house he prefers most among the houses on the market. As there are finite number of agents, there exists at least one cycle of agents. Moreover, as preferences are assumed to be strict, cycles of agents do not intersect. Assign prices $p_k = n-k$ to all of those houses in cycles, and transfer each house in the cycle from the owner to the agent who demands it. Remove those agents with their houses from the market. Repeat the procedure until there are no more houses left on the market.

Shapley and Scarf (1974) show that the matching resulting from the TTC algorithm is in the strong core, and the prices are Walrasian equilibrium prices. Roth and Postlewaite (1977) show that the TTC algorithm outcome is the unique weak core matching, and the unique Walrasian equilibrium allocation. Note that the weak core is defined by weak domination, and the strong core is defined by strong domination. Hence, the weak core is contained in the strong core.

3. LABOR MARKET

3.1 Many-to-one Matching Model without Money

There are a finite set of firms $N = \{1,\ldots,n\}$, and a finite set of workers $X = \{1,\ldots,m\}$. Each worker $i \in X$ has strict preferences P_i over firms, and each firm j has preferences over groups of workers defined by choice functions $C_j(\cdot)$ such that for each subgroup of workers $S \subseteq X$, $C_j(S) \subseteq S$.

A preference relation satisfies substitutability if for each worker $i,i' \in X$ with $i \neq i'$, for each set of workers $S \subseteq X$ with $i,i' \in S$, if $i \in C_j(S)$ where $C_j(\cdot)$ is the choice function representing the preference relation, then $i \in C_j(S \setminus \{i'\})$.

A matching μ is a mapping from the set $N \cup X$ to the set of all subsets of $N \cup X$ such that for each $j \in N$ and for each $i \in X$:

(i) $\mu(i) \in N \cup \{\varnothing\}$ and $|\mu(i)|= 1$,

(ii) $\mu(j) \in 2^X$,

(iii) $i \in \mu(j)$ if and only if $\mu(i) = j$.

A matching μ is acceptable for worker $i \in X$ if $\mu(i) \succ_i \varnothing$, and μ is acceptable for firm $j \in N$ if $\mu(j) = C_j(\mu(j))$.

A matching μ is individually rational if for each agent $k \in X \cup N$, $\mu(k)$ is acceptable for agent k.

A worker and a firm $(i,j) \in X \times N$ pairwise block a matching μ if $i \notin \mu(j)$, $j \succ_i \mu(i)$ and $i \in C_j(\mu \cup \{i\})$.

A matching is pairwise stable if it is individually rational and not pairwise blocked by any pair $(i,j) \in X \times N$ of agents.

A group of workers and a firm $(I,j) \in 2^X \setminus \{\varnothing\} \times N$ block a matching μ if $I \cap \mu(j) = \varnothing$, and for each $i \in I$, $j \succ_i \mu(i)$ and $I \subseteq C_j(\mu(j) \cup I)$.

A matching is stable if it is individually rational and not blocked by any pair $(I,j) \in 2^X \setminus \{\varnothing\} \times N$ of agents.

A group of workers and a group of firms $(I,J) \in 2^X \setminus \{\varnothing\} \times 2^N \setminus \{\varnothing\}$ is a group block for a matching μ if there exists a matching μ' such that for each $i \in I$, $\mu'(i) \in J$ and $\mu'(i) \succ_i \mu(i)$, and for each $j \in J$, $\mu'(j) \subseteq I \cup \mu(j)$ and $\mu'(j) \succ_j \mu(j)$.

A matching is group stable if it is individually rational and not blocked by any pair $(I,J) \in 2^X \setminus \{\varnothing\} \times 2^N \setminus \{\varnothing\}$ of agents.

A group of workers and a group of firms $(I,J) \in 2^X \setminus \{\varnothing\} \times 2^N \setminus \{\varnothing\}$ form a weakly blocking coalition for a matching μ if there exists a matching μ' such that for each $i \in I$, $\mu'(i) \in J$ and $\mu'(i) \succcurlyeq_i \mu(i)$, and for each $j \in J$, $\mu'(j) \subseteq I$ and $\mu'(j) \succcurlyeq_i \mu(j)$, with at least one strict inequality for a worker or a firm. A matching μ is in the weak core if there is no weakly blocking coalition. Let C_W stand for the weak core.

A group of workers and a group of firms $(I,J) \in 2^X \setminus \{\varnothing\} \times 2^N \setminus \{\varnothing\}$ form a strongly blocking coalition for a matching μ if there exists a matching μ' such that for each $i \in I$, $\mu'(i) \in J$ and $\mu'(i) \succ_i \mu(i)$, and for each $j \in J$, $\mu'(j) \subseteq I$ and $\mu'(j) \succ_j \mu(j)$. A matching μ is in the strong core if there is no strongly blocking coalition. Let C stand for the strong core. Observe that $\mathsf{C}_W \subseteq \mathsf{C}$. Echenique and Oviedo (2004) show that the set of stable matchings, which contains group stable matchings, is equal to weak core. Moreover, weak core is contained in the set of pairwise stable matchings.

Roth and Sotomayor (1990), and Echenique and Oviedo (2004) show that the set of stable matchings is nonempty when firms have substitute preferences.

A matching is worker (firm)-optimal if it is not worse than any other stable matching for any worker (firm). Echenique and Oviedo (2004) show that when firms have substitute preferences, then the set of stable S (weak core, C_W) matchings form a non-empty lattice, and the largest (smallest) element of the lattice is the worker (firm)-optimal matching.

3.2 Many-to-one Matching Model with Money

There are a finite set of firms $N=\{1,\ldots,n\}$, and a finite set of workers $X=\{1,\ldots,m\}$. Each worker $i\in X$ has utility $u_i(j;p_{ij})$ from working at firm $j\in N$ at wage p_{ij}. Firm $j\in N$, on the other hand, has value (gross product) $v_j(S)$ from hiring workers $S\subseteq X$, and has net value (net profit) of $v_j(S)-\Sigma_{i\in S}p_{ij}$. The following are the assumptions made in Kelso and Crawford (1982):

(Monotonicity) For each $j\in N$, $i\in X$ and $S\subseteq X\setminus\{i\}$, $v_j(S\cup\{i\})-v_j(S)\geq r_{ij}$ where r_{ij} is the reservation wage of worker i for firm j, i.e. $u_i(j;r_{ij})=u_i(\varnothing;0)$ ($u_i(\varnothing;0)$ is the utility of worker i when he is unemployed with wage 0).

(No Free Lunch) For each firm j, $v_j(\varnothing)=0$, i.e. firm with no employees produces nothing.

Define the demand of each firm $j\in N$ and for each wage vector p_j, j faces as follows: $D_j(p_j)=\{S\subseteq X:\forall S'\subseteq X, v_j(S)-\Sigma_{i\in S}p_{ij}\geq v_j(S')-\Sigma_{i\in S'}p_{ij}\}$.

(Gross Substitutes- GS) For each wage vector $q_j,p_j\in\mathsf{R}^m$ with $q_j\geq p_j$, for each $j\in N$, and for each $S\in D_j(p_j)$, there exists $S'\in D_j(p_j)$ such that $\{i\in S\mid q_{ij}=p_{ij}\}\subseteq S'$.

The Salary Adjustment Process

Step 0. Initially, all wages are set such that for each worker $i\in X$, and for each firm $j\in N$, $p_{ij}(0)=r_{ij}$.

Step t. Given the collection of possible wage offers $\{p_{ij}(t)\}_{i\in X}$, each firm $j\in N$ determines a set of workers maximizing its net profit provided that for each wage offer $p_{ij}(t)$ made by j not rejected by some worker i at Step $t-1$, worker i is among those receiving a wage offer in the current step. By the gross substitutes assumption, there exists such set of workers maximizing net profit of firm j at current wages containing those workers who received wage offers at Step $t-1$ not rejected by these workers. Each worker receiving wage offers reject all but the most preferred wage offer with an arbitrary tie-breaking rule. The next step's wage offers are determined such that $p_{ij}(t+1)=p_{ij}(t)$ for those who did not reject their offers, and $p_{ij}(t+1)=p_{ij}(t)+1$ for those who did reject their offers.

The algorithm ends when there are no more rejections of wage offers.

A matching μ together with a salary schedule $\{p_{i\mu(i)}\}_{i\in X}$ is individually rational if for each $i\in X$ and for each $j\in N$, $u_i(\mu(i);p_{i\mu(i)})\geq u_i(\varnothing;0)$ and $v_j(\mu(j)\ -\Sigma_{i'\in\mu(j)}p_{i'j}\geq 0$

An individually rational matching μ together with a salary schedule $\{p_{i\mu(i)}\}_{i\in X}$ is a discrete core allocation if there are no $(I,j)\in 2^X\setminus\{\varnothing\}\times N$ with a salary schedule $\{p'_{ij}\}_{i\in I}$ such that for each $i\in I$, $u_i(j;p'_{ij})>u_i(\mu(i);p_{i\mu(j)})$ and $v_j(I)-\Sigma_{i'\in I}p_{i'j}>v_j(\mu(j)\ -\Sigma_{i'\in\mu(j)}p_{i'j}$.

Kelso and Crawford (1982) show that when the monotonicity, no free luch and gross substitutes assumptions hold, the salary-adjustment process above reaches a discrete core allocation in finitely many steps.

3.3 Walrasian Equilibrium with Substitute and Complement Discrete Goods

Sun and Yang (2006) examined the Walrasian equilibrium in economies where there are complement and substitute indivisible goods. The set of goods X is partitioned into two sets X_1 and X_2 such that goods in the same set are substitutes whereas those in different sets are complements to each other. Formally, demand correspondence D_j satisfies the gross substitutes and complements (GSC) condition if for any p, $i\in X_k$, $\delta\geq 0$ and $S\in D_j(p)$, there exists $S'\in D_j(p+\delta 1_i)$ such that $[S\cap X_k]\setminus\{i\}\subseteq S'$ and, $S^c\cap X_k^c\subseteq S'^c$.

A Walrasian equilibrium is a price vector p and an allocation (matching) μ of objects such that for each agent $j\in N$, $\mu(j)\in D_j(p)$.

Sun and Yang (2006) show that if every agent's utility function satisfies GSC condition then there exists a Walrasian equilibrium.

In an earlier work, Gul and Stacchetti (1999) studied the allocation of discrete goods, and Walrasian equilibrium in such markets. Their model is based on the model in Kelso and Crawford (1982), described above. The set of workers X is now treated as the set of indivisible goods, and the set of firms N represent the set of agents in the economy. Hence, each agent $j\in N$ receives value $v_j(S)-\Sigma_{i\in S}p_i$ from consuming goods $S \subseteq X$ at prices $\{p_i\}_{i\in S}$. Their model is a special case of the model in Sun and Yang (2006), either $X_1 = X, X_2 = \varnothing$ or $X_2 = X, X_1 = \varnothing$.

(Monotonicity) For each $S \subseteq S' \subseteq X$, $v_j(S) \leq v_j(S')$.

For each $S, S' \subseteq X$, $S\Delta S'$ is the symmetric difference between sets S and S', and $\#(S\Delta S')$ is the Hausdorff Distance between S and S'.

(Single Improvement Property –SI) For each $p = \{p_i\}_{i\in X}$, and for each $S \notin D(p)$, there exists $S' \subseteq X$ such that $\#(S' \setminus S) \leq 1$, $\#(S \setminus S') \leq 1$, and $v_j(S') - \Sigma_{i\in S'} p_i > v_j(S) - \Sigma_{i\in S} p_i$.

Define $[S_1, S_2, S_3] = (S_1 \setminus S_2) \cup S_3$.

(No Complementarities Condition - NC) For each $p = \{p_i\}_{i\in X}$, for each $S_1, S_2 \in D(p)$, and $S_{1'} \subseteq S_1$, there exists $S_{2'} \subseteq S_2$ such that $[S_1, S_{1'}, S_{2'}] \in D(p)$.

Gul and Stacchetti (1999) show that there exists a Walrasian equilibrium when agents have substitute preferences. Moreover, they show that if every agent's utility function satisfies monotonicity then GS, SI, and NC are equivalent. Furthermore, using agents with unit-demand preferences, Gul and Stacchetti (1999) show that substitute preferences is almost necessary for the existence of a Walrasian equilibrium.

REFERENCES

Echenique, F. and Oviedo, J. (2004) "Core many-to-one matchings by Fixed-point methods" J. Econ. Theory, 115: 358-376.

Gale, D. and Shapley, L. S. (1962) "College Admissions and the Stability of Marriage" The American Mathematical Monthly, 69(1): pp. 9-15.

Gul, F. and Stacchetti, E. (1999) "Walrasian Equilibrium with Gross Substitutes" J. Econ. Theory, 87: 95-124.

Kelso, A. S. and Crawford, V. P. (1982) "Job Matching, Coalition Formation, and Gross Substitutes" Econometrica, 50: 1483-1504.

Roth, A. E. and Postlewaite, A. (1977) "Weak versus strong domination in a market with indivisible goods" Journal of Mathematical Economics, 4(2): 131-137.

Roth, A. E. and Sotomayor, M. (1990) *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Econometric Society Monograph Series, Cambridge University Press.

Shapley, L. and Scarf, H. (1974) "On cores and indivisibility" Journal of Mathematical Economics, 1(1): 23-37.

Sönmez, T. and Ünver, U. (2011) "Matching, Allocation, and Exchange of Discrete Resources", *J. Benhabib, A. Bisin, and M. Jackson (eds.), Handbook of Social Economics*, Vol. 1A: 781-852, The Netherlands: North-Holland.

Sun, N. and Yang, Z. (2006) "Equilibria and Indivisibilities: Gross Substitutes and Complements" Econometrica, 74(5): 1385-1402.