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CUBIC B-SPLINE QUASI-INTERPOLATION METHOD FOR REGULARIZED LONG WAVE EQUATION

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Abstract

In this study, we present a numerical method to solve the Regularized Long Wave (RLW) equation, based on cubic B-spline quasi-interpolation for the space integration and Crank-Nicolson method for the time integration. The method is tested on the problems of propagation of a solitary wave and interaction of two solitary waves. The three conservation quantities of the motion are calculated to determine the conservation properties of the proposed algorithm.

Keywords: Finite element method, B-spline, solitary wave, regularized long wave equation, Crank Nicolson

1. Introduction

Various phenomena in disciplines could be described by nonlinear partial differential equations (NPDEs). Numerical solution of NPDEs is very important due to just limited classes of these equation are solved analitically. One of the nonlinear evolution equations which we deal with is the Regularized Long Wave (RLW) equation. This equation was originally introduced to describe the behavior of the undular bore by Peregrine [1] who developed the first numerical method of the RLW equation using the finite difference method. Benjamin et al. [2] showed the similarity of wave solutions of the RLW equation to the wave solutions of the more widely known Korteweg-de Vries (KdV) equation. RLW equation has been solved by various numerical method including

finite difference method [3-5], collocation method [7-12], Galerkin method [13-22], and Quadrature method [23,24].

In this paper, Crank-Nicolson method for time integration, and quasi cubic B-spline functions for space integration are used to obtain numerical solution of the RLW equation. In the test problems section, error norms and conservation quantities are calculated for the accuracy of the solution. This study is a part of the master thesis of Mersin [25]. In this thesis the numerical solution of some partial differential equations including RLW, EW, MRLW and MEW equations were solved by using quasi spline interpolation.

2. Governing equation

We consider the following RLW equation

$$u_t + u_x + \varepsilon uu_x - \mu u_{xxt} = 0 \quad (1)$$

with the boundary conditions

$$u(a, t) = u(b, t) = 0, \quad t > 0$$

and the initial condition

$$u(x, 0) = f(x).$$

We denoted the space of univariate splines of degree d which has C^{d-1} property by $S_d(X_N)$ on the uniform partition

$$X_N = x_i = a + ih, \quad i = 0, \dots, N$$

with the meshlength $h = \frac{b-a}{N}$, where $b = x_N$. Let the B-spline basis of $S_d(X_N)$ be $\{B_j, j \in J\}$ with $J = \{1, 2, \dots, N + d\}$, which can be computed by the de Boor-Cox formula [26]. Using the Boor-Cox formula, B_j is obtained as

$$B_j(x) = \frac{1}{6h^3} \begin{cases} (x - x_j)^3 & , x \in [x_j, x_{j+1}) \\ (x - x_j)^2(x_{j+2} - x) + \\ (x - x_j)(x_{j+3} - x)(x - x_{j+1}) + \\ (x_{j+4} - x)(x - x_{j+1})^2 & , x \in [x_{j+1}, x_{j+2}) \\ (x - x_j)(x_{j+3} - x)^2 + \\ (x - x_{j+1})(x_{j+3} - x)(x_{j+4} - x) + \\ (x_{j+4} - x)^2(x - x_{j+2}) & , x \in [x_{j+2}, x_{j+3}) \\ (x_{j+4} - x)^3 & , x \in [x_{j+3}, x_{j+4}) \\ 0 & \text{else} \end{cases} \quad (2)$$

for $j \in J$. Univariate B-spline quasi-interpolants can be defined as operators of the form

$$Q_d f = \sum_{j \in I} \mu_j B_j \quad (3)$$

[27-29]. For the cubic B-spline quasi interpolation

$$Q_3 f = \sum_{j=1}^{N+3} \mu_j(f) B_j \quad (4)$$

the coefficients are listed as follows:

$$\begin{aligned} \mu_1(f) &= f_0, \\ \mu_2(f) &= \frac{1}{18}(7f_0 + 18f_1 - 9f_2 + 2f_3), \\ \mu_j(f) &= \frac{1}{6}(-f_{j-3} + 8f_{j-2} - f_{j-1}), j = 3, \dots, N+1, \\ \mu_{N+2}(f) &= \frac{1}{18}(2f_{N-3} + 18f_1 - 9f_2 + 2f_3), \\ \mu_{N+3}(f) &= f_N. \end{aligned} \quad (5)$$

The main advantage of quasi interpolation is having a direct construction without solving any system of linear equations.

$$Q_3 u = \sum_{j=1}^{N+3} \mu_j(u) B_j, \quad (Q_3 u)' = \sum_{j=1}^{N+3} \mu_j(u) B_j', \quad (Q_3 u)'' = \sum_{j=1}^{N+3} \mu_j(u) B_j'' \quad (6)$$

are the approximations of the first and the second derivatives of the unknown function u . After using these approximations

$$\begin{aligned} Qu'(x_0) &= \frac{1}{h} \left(-\frac{11}{6}u_0 + 3u_1 - \frac{3}{2}u_2 + \frac{1}{3}u_3 \right), \\ Qu'(x_1) &= \frac{1}{h} \left(-\frac{1}{3}u_0 - \frac{1}{2}u_1 + u_2 - \frac{1}{6}u_3 \right), \\ Qu'(x_j) &= \frac{1}{h} \left(\frac{1}{12}u_{j-2} - \frac{2}{3}u_{j-1} + \frac{2}{3}u_{j+1} - \frac{1}{12}u_{j+2} \right), j = 2, \dots, N-2 \\ Qu'(x_{N-1}) &= \frac{1}{h} \left(\frac{1}{6}u_{N-3} - u_{N-2} - \frac{1}{2}u_{N-1} + \frac{1}{3}u_N \right), \\ Qu'(x_N) &= \frac{1}{h} \left(-\frac{1}{3}u_{N-3} + \frac{3}{2}u_{N-2} - 3u_{N-1} + \frac{11}{6}u_N \right) \end{aligned} \quad (7)$$

are obtained for the first derivation and,

$$\begin{aligned} Qu''(x_0) &= \frac{1}{h^2} (2u_0 - 5u_1 + 4u_2 - u_3), \\ Qu''(x_1) &= \frac{1}{h^2} (u_0 - 2u_1 + u_2), \\ Qu''(x_j) &= \frac{1}{h^2} \left(-\frac{1}{6}u_{j-2} + \frac{5}{3}u_{j-1} - 3u_j + \frac{5}{3}u_{j+1} - \frac{1}{6}u_{j+2} \right), j = 2, \dots, N-2 \\ Qu''(x_{N-1}) &= \frac{1}{h^2} (u_{N-2} - 2u_{N-1} + u_N), \\ Qu''(x_N) &= \frac{1}{h^2} (-u_{N-3} + 4u_{N-2} - 5u_{N-1} + 2u_N) \end{aligned} \quad (8)$$

are obtained for the second derivation [29].

3. Numerical scheme

Applying Crank Nicolson method to the equation

$$u_t + u_x + \varepsilon uu_x - \mu u_{xxt} = 0$$

we have

$$\frac{u^{n+1}-u^n}{\Delta t} + \frac{u_x^{n+1}+u_x^n}{2} + \varepsilon \frac{(uu_x)^{n+1}+(uu_x)^n}{2} - \mu \frac{u_{xx}^{n+1}-u_{xx}^n}{\Delta t} = 0. \quad (9)$$

Linearizing the nonlinear term $(uu_x)^{n+1}$ by using following approximation [30]:

$$(uu_x)^{n+1} \approx u^n u_x^{n+1} + u_x^n u^{n+1} - u^n u_x^n \quad (10)$$

Eq. (9) can be written as

$$\left(1 + \varepsilon \frac{\Delta t}{2} u_x^n\right) u^{n+1} + \frac{\Delta t}{2} (1 + \varepsilon(u^n)) u_x^{n+1} - \mu u_{xx}^{n+1} = u^n - \frac{\Delta t}{2} u_x^n - \mu u_{xx}^n. \quad (11)$$

After using quasi spline approximations for the first and the second derivatives (7-8), the Eq. (11) is written as

$$\begin{aligned} & U_0^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_0^n - \frac{11 \Delta t}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_0^n) - \mu \frac{2}{h^2} \right] + \\ & U_1^{n+1} \left[\frac{3 \Delta t}{h} \frac{\Delta t}{2} (1 + \varepsilon U_0^n) + \mu \frac{5}{h^2} \right] + \\ & U_2^{n+1} \left[-\frac{3 \Delta t}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_0^n) - \mu \frac{4}{h^2} \right] + \\ & U_3^{n+1} \left[\frac{1 \Delta t}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_0^n) + \mu \frac{1}{h^2} \right] = U_0^n - \frac{\Delta t}{2} (U_x)_0^n - \mu (U_{xx})_0^n \end{aligned}$$

$$\begin{aligned} & U_0^{n+1} \left[-\frac{1 \Delta t}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) - \mu \frac{1}{h^2} \right] + \\ & U_1^{n+1} \left[1 - \frac{1 \Delta t}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) + \mu \frac{2}{h^2} \right] + \\ & U_2^{n+1} \left[\frac{1 \Delta t}{h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) - \mu \frac{1}{h^2} \right] + \\ & U_3^{n+1} \left[-\frac{1 \Delta t}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) \right] = U_1^n - \frac{\Delta t}{2} (U_x)_1^n + -\mu (U_{xx})_1^n \end{aligned}$$

$$\begin{aligned}
& U_{m-2}^{n+1} \left[\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) + \mu \frac{1}{6h^2} \right] + \\
& U_{m-1}^{n+1} \left[-\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) - \mu \frac{5}{3h^2} \right] + \\
& U_m^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_m^n + \mu \frac{3}{h^2} \right] + \\
& U_{m+1}^{n+1} \left[\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) - \mu \frac{5}{3h^2} \right] + \\
& U_{m+2}^{n+1} \left[-\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) + \mu \frac{1}{6h^2} \right] = U_m^n - \frac{\Delta t}{2} (U_x)_m^n - \mu (U_{xx})_m^n
\end{aligned} \tag{12}$$

$$\begin{aligned}
& U_{N-3}^{n+1} \left[\frac{1}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) \right] + \\
& U_{N-2}^{n+1} \left[-\frac{1}{h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) - \mu \frac{1}{h^2} \right] + \\
& U_{N-1}^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_{N-1}^n - \frac{1}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) + \mu \frac{2}{h^2} \right] + \\
& U_N^{n+1} \left[\frac{1}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) - \mu \frac{1}{h^2} \right] = U_{N-1}^n - \frac{\Delta t}{2} (U_x)_{N-1}^n - \mu (U_{xx})_{N-1}^n
\end{aligned}$$

$$\begin{aligned}
& U_{N-3}^{n+1} \left[-\frac{1}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_N^n) + \mu \frac{1}{h^2} \right] + \\
& U_{N-2}^{n+1} \left[\frac{3}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_N^n) - \mu \frac{4}{h^2} \right] + \\
& U_{N-1}^{n+1} \left[-\frac{3}{h} \frac{\Delta t}{2} (1 + \varepsilon U_N^n) + \mu \frac{5}{h^2} \right] + \\
& U_N^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_N^n + \frac{11}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_N^n) - \mu \frac{2}{h^2} \right] = U_N^n - \frac{\Delta t}{2} (U_x)_N^n - \mu (U_{xx})_N^n
\end{aligned}$$

where $m = 2, \dots, N - 2$. The system (12) contains $N + 1$ unknowns and $N + 1$ equations. The first and the last equations are deleted to apply the boundary conditions to the system. Then our system turns to $N + 1$ unknown and $N - 1$ equations.

After the first and the last equations are deleted and the boundary conditions $U(a, t) = U(b, t) = 0$ are implemented, then the system (12) is transformed into

$$\begin{aligned}
& U_1^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_1^n - \frac{1}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) + \mu \frac{2}{h^2} \right] + \\
& U_2^{n+1} \left[\frac{1}{h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) - \mu \frac{1}{h^2} \right] + \\
& U_3^{n+1} \left[-\frac{1}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_1^n) \right] = U_1^n - \frac{\Delta t}{2} (U_x)_1^n - \mu (U_{xx})_1^n
\end{aligned}$$

$$\begin{aligned}
& U_1^{n+1} \left[-\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_2^n) - \mu \frac{5}{3h^2} \right] + \\
& U_2^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_2^n + \mu \frac{3}{h^2} \right] + \\
& U_3^{n+1} \left[\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_2^n) - \mu \frac{5}{3h^2} \right] + \\
& U_4^{n+1} \left[-\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_2^n) + \mu \frac{1}{6h^2} \right] = U_2^n - \frac{\Delta t}{2} (U_x)_2^n - \mu (U_{xx})_2^n \\
\\
& U_{m-2}^{n+1} \left[\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) + \mu \frac{1}{6h^2} \right] + \\
& U_{m-1}^{n+1} \left[-\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) - \mu \frac{5}{3h^2} \right] + \\
& U_m^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_m^n + \mu \frac{3}{h^2} \right] + \\
& U_{m+1}^{n+1} \left[\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) - \mu \frac{5}{3h^2} \right] + \\
& U_{m+2}^{n+1} \left[-\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_m^n) + \mu \frac{1}{6h^2} \right] = U_m^n - \frac{\Delta t}{2} (U_x)_m^n - \mu (U_{xx})_m^n
\end{aligned} \tag{13}$$

$$\begin{aligned}
& U_{N-4}^{n+1} \left[\frac{1}{12h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-2}^n) + \mu \frac{1}{6h^2} \right] + \\
& U_{N-3}^{n+1} \left[-\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-2}^n) - \mu \frac{5}{3h^2} \right] + \\
& U_{N-2}^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_{N-2}^n + \mu \frac{3}{h^2} \right] + \\
& U_{N-1}^{n+1} \left[\frac{2}{3h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-2}^n) - \mu \frac{5}{3h^2} \right] = U_{N-2}^n - \frac{\Delta t}{2} (U_x)_{N-2}^n - \mu (U_{xx})_{N-2}^n
\end{aligned}$$

$$\begin{aligned}
& U_{N-3}^{n+1} \left[\frac{1}{6h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) \right] + \\
& U_{N-2}^{n+1} \left[-\frac{1}{h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) - \mu \frac{1}{h^2} \right] + \\
& U_{N-1}^{n+1} \left[1 + \varepsilon \frac{\Delta t}{2} (U_x)_{N-1}^n - \frac{1}{2h} \frac{\Delta t}{2} (1 + \varepsilon U_{N-1}^n) + \mu \frac{2}{h^2} \right] = U_{N-1}^n - \frac{\Delta t}{2} (U_x)_{N-1}^n - \mu (U_{xx})_{N-1}^n
\end{aligned}$$

where $m = 3, \dots, N - 3$. It's clearly seen that the equation system consist of $N - 1$ equations and $N - 1$ unknowns where $U_1^{n+1}, U_2^{n+1}, \dots, U_{N-1}^{n+1}$ are unknowns. Initial unknowns $U_0^0, U_1^0, \dots, U_N^0$ are calculating by using initial condition $U(x_m, 0) = f(x_m)$ $m = 0, 1, \dots, N$ to solve system (13) iteratively. The local truncation error for the Eq. (13) is computed as

$\frac{h^2}{2} (U_{xxxx})_m^n + \frac{k^2}{2} [\mu (U_{xx})_m^n (U_{xt})_m^n - \mu \varepsilon^2 \mu U_m^n ((U_x)_m^n)^2 (U_{xx})_m^n + \varepsilon \mu U_m^n (U_{xt})_m^n (U_{xx})_m^n + \varepsilon \mu^2 (U_x)_m^n (U_{xx})_m^n (U_{xxt})_m^n + \mu (U_{xxtt})_m^n (U_x)_m^n - \mu^2 (U_{xx})_m^n (U_{xxtt})_m^n - \varepsilon \mu ((U_x)_m^n)^2 (U_{xx})_m^n] + \dots$ by neglecting the terms of high order. Since the local truncation error vanishes as the time and space steps become smaller, the one step difference equation (13) is consistent with RLW equation.

4. The test problems

4.1 Motion of single solitary wave

The exact solution of the RLW equation is given by

$$u(x, t) = 3c \operatorname{sech}^2(k[x - \tilde{x}_0 - (1 + \varepsilon c)t]) \quad (14)$$

which describes a single bell-shape solitary wave of amplitude $3c$, travelling with velocity $v = 1 + c$ in the positive x -direction over the space interval $[a, b]$. The initial condition for single solitary wave solution of the RLW equation can be written as

$$u(x, 0) = 3c \operatorname{sech}^2(k[x - \tilde{x}_0]) \quad (15)$$

where $k = \sqrt{\frac{\varepsilon c}{4\mu v}}$. The RLW equation possess three conservation constants,

$$C_1 = \int_{-\infty}^{\infty} u dx, \quad C_2 = \int_{-\infty}^{\infty} (u^2 + \mu(u_x)^2) dx, \quad C_3 = \int_{-\infty}^{\infty} (u^3 + 3u^2) dx \quad (16)$$

corresponding to mass, momentum and energy, respectively [31]. Exact values of these conservation constants can be calculated by Maple as:

$$C_1 = \frac{6c}{k}, \quad C_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, \quad C_3 = \frac{36c^2}{5k}(4c + 5). \quad (17)$$

To compare the analytical and the numerical solutions, error norm $L_\infty = \max_m |U_m - u(x_m, t)|$ is used, where $u(x_m, t)$ is corresponding to the exact solution on (x_m, t_n) and U_m is corresponding to the approximate solution on (x_m, t_n) .

Initial solution (15) and position of solitary waves at time $t = 20$ over the space interval $-40 \leq x \leq 60$ are shown at the Figure 1 for $c = 0.1$, $\tilde{x}_0 = 0$ and parameters $\varepsilon = \mu = 1$.

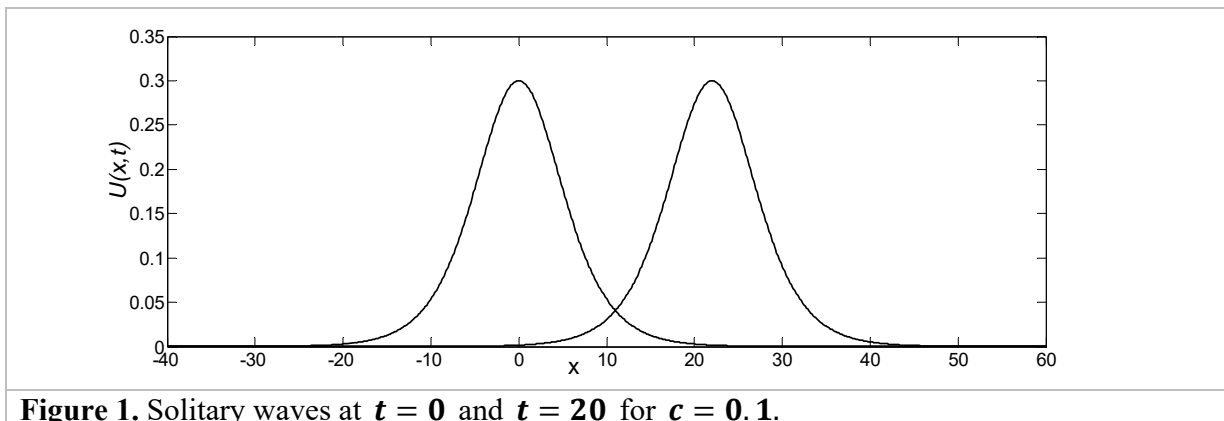


Figure 1. Solitary waves at $t = 0$ and $t = 20$ for $c = 0.1$.

On the first test problem parameters are chosen as space step $h = 0.125$, time step $\Delta t = 0.1$ and amplitude $3c = 0.3$. Program was run up to time $t = 20$ and the conservation quantities C_1, C_2, C_3 and L_∞ error norm are given at the Table 1 in different times. Its easy to see that the results are acceptable and in good aggrement with the exact solution according to the the error norms L_∞ and conservation quantities.

Table 1. Conservation quantities and the error norms for $h = 0.125, \Delta t = 0.1, c = 0.1$ and $-40 \leq x \leq 60$.

Time	$L_\infty \times 10^5$	C_1	C_2	C_3
0	0	3.9799267	0.8104625	2.5790074
4	2.12	3.9799300	0.8104625	2.5790074
8	4.26	3.9799267	0.8104625	2.5790074
12	6.22	3.9799257	0.8104625	2.5790074
16	8.00	3.9799171	0.8104625	2.5790074
20	9.62	3.9798828	0.8104625	2.5790074
Exact	0	3.9799497	0.8104625	2.5790074

The absolute value of the difference between the numerical solution and the analytical solution at $t = 20$ is plotted in the Figure 2. It is clear to see that the maximum error is occurred in the middle of the space interval.

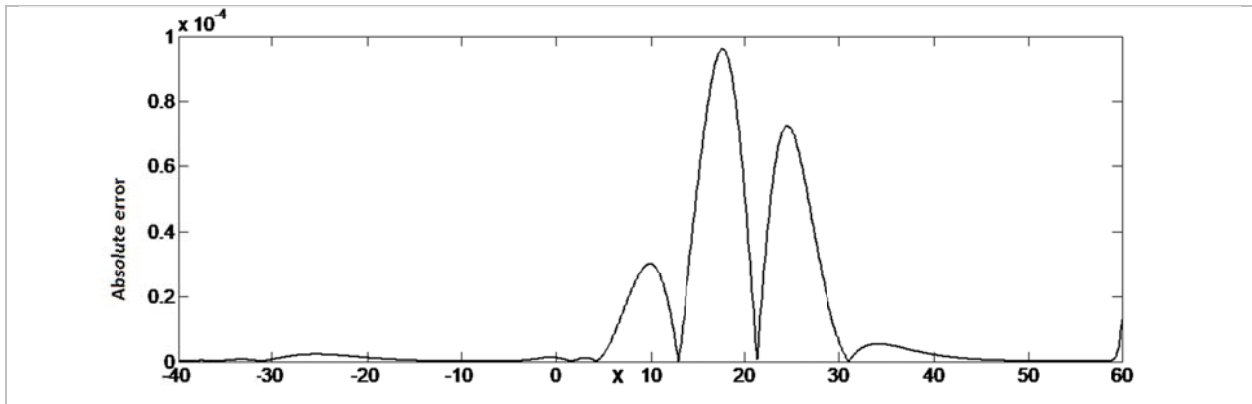


Figure 2. Absolute Error = |Analytical Solution – Numerical Solution| for $h = 0.125, \Delta t = 0.1, c = 0.1$ and $-40 \leq x \leq 60$ at the time $t = 20$.

The solitary wave problem is investigated by changing the amplitude of the solitary waves. Therefore, initial solution (15) and position of solitary waves at time $t = 20$ over the space interval $40 \leq x \leq 60$ are shown at the Figure 3 for $c = 0.03, \tilde{x}_0 = 0$ and parameters $\varepsilon = \mu = 1$.

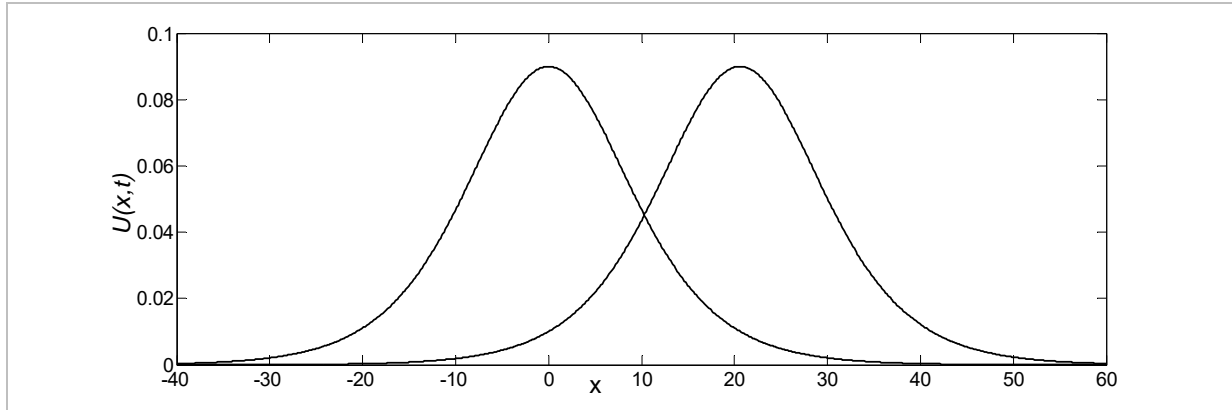


Figure 3. Solitary waves at $t = 0$ and $t = 20$ for $c = 0.03$.

On this problem, parameters are chosen as space step $h = 0.125$, time step $\Delta t = 0.1$ and amplitude $3c = 0.09$. The program was run up to time $t = 20$ and the conservation quantities C_1, C_2, C_3 and error norm L_∞ are given at the Table 2 by different times. According to the Table 2 its easy to see the absolute error has increased according to $c = 0.1$.

Table 2. Conservation quantities and the error norm for $h = 0.125$, $\Delta t = 0.1$, $c = 0.03$ and $-40 \leq x \leq 60$.

Time	$L_\infty \times 10^4$	C_1	C_2	C_3
0	0	2.10704672	0.12730126	0.38880465
4	2.30	2.10709771	0.12730112	0.38880407
8	2.21	2.10689617	0.12730112	0.38880405
12	2.12	2.10654963	0.12730111	0.38880397
16	2.14	2.10592816	0.12730109	0.38880365
20	4.32	2.10461363	0.12730104	0.38880235

Figure 4 shows the absolute value of the difference between the analytical and numerical solutions of the program. The program was run up to time $t = 20$ with $h = 0.125$, $\Delta t = 0.1$, $c = 0.03$ over the space interval $-40 \leq x \leq 60$. The maximum error can be seen at the end of the space interval. It means that there is a problem with the boundary conditions. The reason for the error seen at the end of the space domain is due to the space interval of the wave is not chosen close enough to the zero.

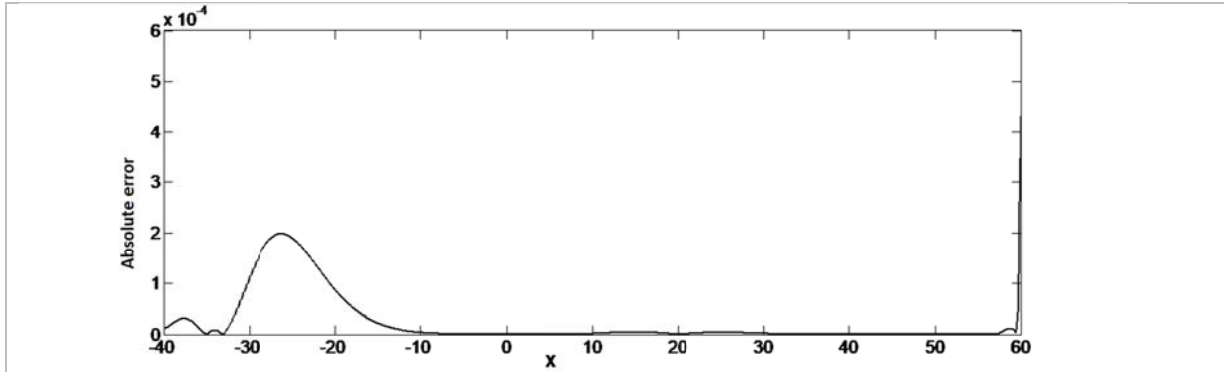


Figure 4. Absolute Error = |Analytical Solution – Numerical Solution| $h = 0.125, \Delta t = 0.1, c = 0.03$ and $-40 \leq x \leq 60$ at the time $t = 20$.

In order to reduce the error, the program was run again by extending the space interval from $-40 \leq x \leq 60$ to $-80 \leq x \leq 120$. The obtained results are given in the Table 3. It can be seen from the table that the error is decreased when the larger space interval $-80 \leq x \leq 120$ is used.

Table 3. Conservation quantities and the error norms for $h = 0.125, \Delta t = 0.1, c = 0.03$ and $-80 \leq x \leq 120$.

Time	$L_{\infty} \times 10^6$	C_1	C_2	C_3
0	0	2.10940502	0.12730172	0.38880599
4	0.88	2.10940516	0.12730172	0.38880599
8	1.79	2.10940511	0.12730172	0.38880599
12	2.71	2.10940507	0.12730172	0.38880599
16	3.65	2.10940508	0.12730172	0.38880599
20	4.57	2.10940502	0.12730172	0.38880599
Exact		2.10940750	0.12730172	0.38880599

The Figure 5 shows the absolute value of the difference between the analytical and numerical solutions at time $t = 20$ over the interval of $-80 \leq x \leq 120$ and it is seen that the error is decreased and the maximum error is also occurred in the middle of the interval. As a result, the selection of boundary conditions is very important in the solitary wave test problems.

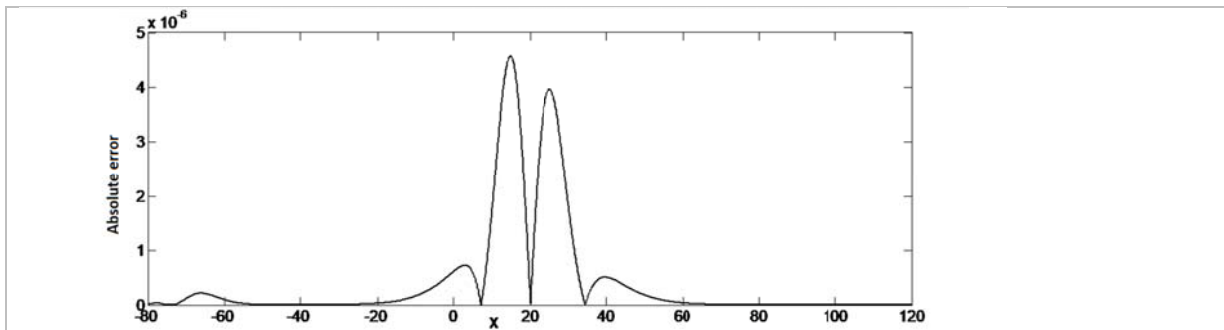


Figure 5. Absolute Error = |Analytical Solution – Numerical Solution| for $h = 0.125, \Delta t = 0.1, c = 0.03$ and $-80 \leq x \leq 120$ at the time $t = 20$.

Table 4 displays a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other methods. The results of the proposed method in good agreement with previous studies results.

Method	$L_\infty \times 10^5$	C_1	C_2	C_3
Present	9.62	3.9798828	0.8104625	2.5790074
[13]	8.6	3.97988	0.810465	2.57901
[18]	175.5	3.98203	0.808650	2.57302
[19]	156.6	3.96160	0.804185	2.55829
[21]	19.8	3.98206	0.811164	2.58133
[22]	7.34	3.9798879	0.8104622	2.5790063
Exact	7	3.979949	0.8104625	2.5790074

4.2 Interaction of two solitary waves

The collision problem of two solitary waves has the following initial condition

$$u(x, 0) = 3c_1 \operatorname{sech}^2(k_1[x - \tilde{x}_1]) + 3c_2 \operatorname{sech}^2(k_2[x - \tilde{x}_2]) \quad (18)$$

where $k_i = \sqrt{\frac{\varepsilon c_i}{4\mu(1+\varepsilon c_i)}}$, $i = 1, 2$. In (18), the solitary waves have $3c_1$ and $3c_2$ amplitudes, and each peak points of them are located at \tilde{x}_1 and \tilde{x}_2 , respectively over the problem domain $[a, b]$. If the parameters are chosen as $c_1 > c_2$ and $\tilde{x}_2 > \tilde{x}_1$ in the initial solution (18), the bigger wave which has bigger amplitude will stand on the left. Hence, if the parameters are chosen properly, the bigger wave will reach and pass the smaller wave as its faster than the smaller wave. So the collision will be occurred.

Exact values of the conservation constants for this problem can be calculated by Maple as

$$\begin{aligned} C_1 &= 6 \left(\frac{c_1}{k_1} + \frac{c_2}{k_2} \right), \\ C_2 &= 12 \left(\frac{c_1^2}{k_1} + \frac{c_2^2}{k_2} \right) + \frac{48}{5} \mu (k_1 c_1^2 + k_2 c_2^2), \\ C_3 &= \frac{36c_1^2}{5k_1} (4c_1 + 5) + \frac{36c_2^2}{5k_2} (4c_2 + 5). \end{aligned}$$

When the parameters are chosen as $\tilde{x}_1 = 20$, $\tilde{x}_2 = 65$, $c_1 = 2/3$ and $c_2 = 0.1$, the initial condition of RLW equation for the interecation of two solitary waves test problem is

$$u(x, 0) = 2 \operatorname{sech}^2 \left(\frac{\sqrt{10}}{10} [x - 20] \right) + 0.3 \operatorname{sech}^2 \left(\frac{1}{\sqrt{44}} [x - 65] \right). \quad (19)$$

The program was run up to time $t = 150$ with the parameters $h = \Delta t = 0.1$ over the space interval $[0, 300]$. The solitary waves' at times $t = 0$, $t = 65$ and $t = 150$ are plotted in the Figure 6. It is observed from the Figure 6 that the time of the collision is around $t = 65$ and the

solitary waves retain their shape after the collision.

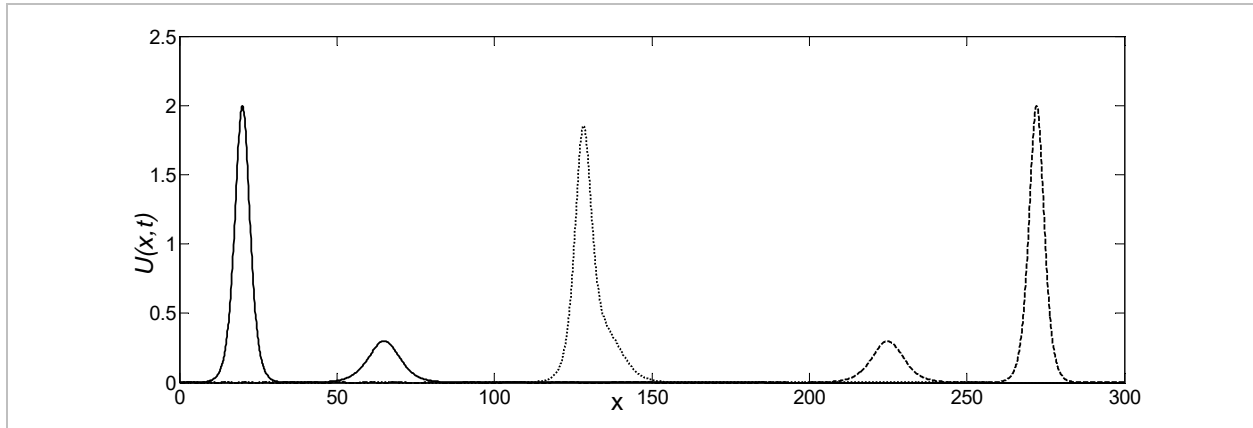


Figure 6. Collision of two solitary waves at times $t = 0$, $t = 65$, $t = 150$.

The numerical values of the conservation constants are given in Table 5 for various times.

Table 5. Conservation quantities for the collision of two solitary waves.

Time	C_1	C_2	C_3
0	16.62901975	19.02523346	80.16048477
30	16.62903736	19.02519099	80.16021819
60	16.62903725	19.02299037	80.14604732
90	16.62903719	19.02393162	80.15214565
120	16.62903854	19.02519877	80.16026810
150	16.62903891	19.02520778	80.16032521

Absolute errors for conservation constants are given in the Figure 7. In this figure, the largest error occurs for C_3 , then for C_2 , and the smallest error for C_1 . It shows that the absolute error of the conservation constants is increasing and then decreasing between $t = 50$ and $t = 100$ because of the collision.

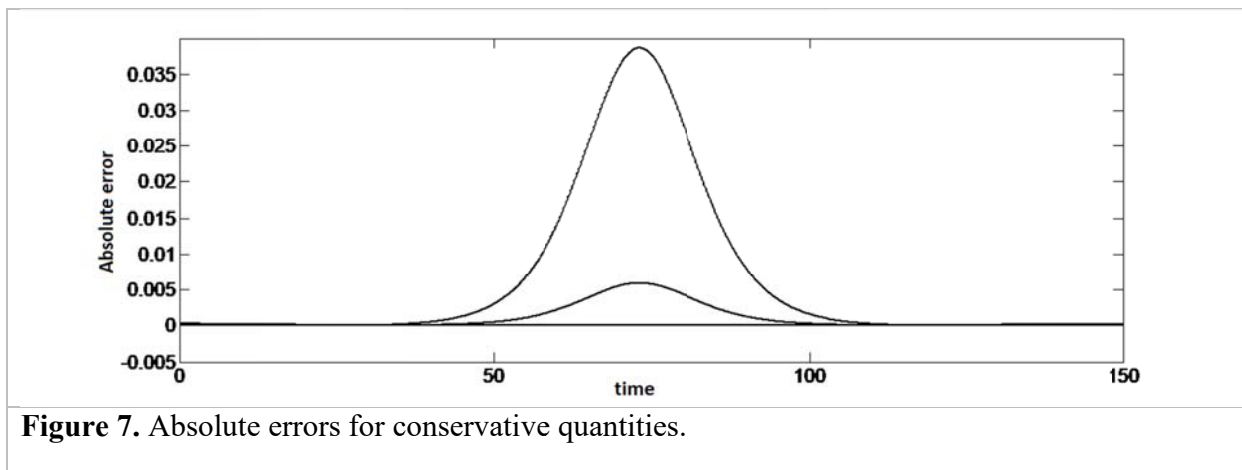


Figure 7. Absolute errors for conservative quantities.

5. Conclusion

In this study, quasi cubic B-splines for space discretization and Crank Nicolson method for time discretization are proposed for numerical solution of the RLW equation. The advantage of the proposed method is the unknown function can be directly calculated by approaching the derivatives as in the finite difference method. The proposed numerical method for the RLW equation is investigated by two test problems. After the numerical solutions are examined, it is seen that the proposed method gives good results. Consequently, quasi cubic B-spline method which is an easier method than the finite element method in the terms of ease of implementation for the numerical solution of the RLW equation.

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