

Research Article

Quantum arithmetic of Drinfeld modules

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ABSTRACT. We study the quantum invariants of projective varieties over the number fields. Namely, explicit formulas for a functor \mathcal{Q} on such varieties are proved. The case of abelian varieties with complex multiplication is treated in detail.

Keywords: Drinfeld modules, noncommutative tori, projective variety.

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1. INTRODUCTION

Quantum arithmetic deals with a functor \mathcal{Q} on the projective varieties $V(k)$ over a number field k ; we refer the reader to Section 2.3 or [11, Theorem 1.3] for the details. Such a functor takes values in the triples $(\Lambda, [I], K)$ consisting of a real number field K , an ideal class $[I]$ and an order $\Lambda \subset K$, i.e. a subring of the ring of integers of K [4]. The invariant $(\Lambda, [I], K)$ comes from the K -theory of operator algebras related to the quantum mechanics [1]; hence the name. The existence of functor \mathcal{Q} was proved by contradiction [11, Remark 1.4]. Such a proof entails no efficient formula for the number field K in terms of the field of definition of variety $V(k)$, except for the special case of complex multiplication [11, Theorem 4.1]. The aim of our note is to fill the gap by establishing such a formula (Theorem 1.1). We shall use the following notation and facts.

Let $\mathfrak{k} := \mathbf{F}_{p^n}$ be a finite field and $\tau(x) = x^p$. Consider a ring $\mathfrak{k}\langle\tau\rangle$ of the non-commutative polynomials given by the commutation relation $\tau a = a^p \tau$ for all $a \in A$, where $A := \mathfrak{k}[T]$ is the ring of polynomials in variable T over \mathfrak{k} . The Drinfeld module $\text{Drin}_A^r(\mathfrak{k})$ of rank $r \geq 1$ is a homomorphism

$$(1.1) \quad \rho : A \xrightarrow{r} \mathfrak{k}\langle\tau\rangle$$

given by a polynomial $\rho_a = a + c_1 \tau + \cdots + c_r \tau^r$, where $a \in A$ and $c_i \in \mathfrak{k}$ [13, Section 12]. An isogeny between Drinfeld modules $\text{Drin}_A^r(\mathfrak{k})$ and $\widetilde{\text{Drin}}_A^r(\mathfrak{k})$ is a surjective morphism $f : \text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k})$ with a finite kernel, *ibid*. Consider a torsion submodule $\Lambda_\rho[a] := \{\lambda \in \bar{\mathfrak{k}} \mid \rho_a(\lambda) = 0\}$ of the A -module $\bar{\mathfrak{k}}$. Drinfeld modules $\text{Drin}_A^r(\mathfrak{k})$ and associated torsion submodules $\Lambda_\rho[a]$ define generators of a non-abelian class field theory for the function fields. Namely, for each non-zero $a \in A$ the function field $\mathfrak{k}(T)(\Lambda_\rho[a])$ is a Galois extension of the field $\mathfrak{k}(T)$ of rational functions in variable T over \mathfrak{k} , such that its Galois group is isomorphic to a subgroup of the matrix group $GL_r(A/aA)$ [13, Proposition 12.5].

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On the other hand, the norm closure of a representation of the multiplicative semi-group of the ring $\mathfrak{k}(\tau)$ [6] by bounded linear operators on a Hilbert space gives rise to the noncommutative torus \mathcal{A}_{RM}^{2r} having real multiplication (RM) [10]. The latter is a C^* -algebra generated by the unitary operators u_1, \dots, u_{2r} satisfying the commutation relations $\{u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j \mid 1 \leq i, j \leq 2r\}$, where θ_{ij} are algebraic numbers and $\Theta = (\theta_{ij}) \in M_{2r}(\mathbf{R})$ is a skew-symmetric matrix [12]. The K -theory of the C^* -algebra \mathcal{A}_{RM}^{2r} is well known [1, Chapter III] and [12, Section 3]. Namely, the Grothendieck semi-group is given by the formula $K_0^+(\mathcal{A}_{RM}^{2r}) \cong \mathbf{Z} + \alpha_1 \mathbf{Z} + \dots + \alpha_r \mathbf{Z} \subset \mathbf{R}$, where α_i are algebraic integers of degree $2r$ over \mathbf{Q} [10, Section 2.2.2].

The following is true [10, Theorem 3.3]:

- (i) there exists a functor $F : \text{Drin}_A^r(\mathfrak{k}) \mapsto \mathcal{A}_{RM}^{2r}$ from the category of Drinfeld modules \mathfrak{D} to a category of the noncommutative tori \mathfrak{A} , which maps any pair of isogenous (isomorphic, resp.) modules $\text{Drin}_A^r(\mathfrak{k}), \widetilde{\text{Drin}}_A^r(\mathfrak{k}) \in \mathfrak{D}$ to a pair of the homomorphic (isomorphic, resp.) tori $\mathcal{A}_{RM}^{2r}, \widetilde{\mathcal{A}}_{RM}^{2r} \in \mathfrak{A}$;
- (ii) $F(\Lambda_\rho[a]) = \{e^{2\pi i \alpha_i + \log \log \varepsilon} \mid 1 \leq i \leq r\}$, where $\mathcal{A}_{RM}^{2r} = F(\text{Drin}_A^r(\mathfrak{k}))$ and $\log \varepsilon$ is a scaling factor;
- (iii) the number field $k = \mathbf{Q}(F(\Lambda_\rho[a]))$ is the extension of its subfield with the Galois group $G \subseteq GL_r(A/aA)$.

An isogeny $f : \text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k})$ defines a Grothendieck semi-group $K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r}) = \{\mathbf{Z} + \frac{\alpha_1}{m_1} \mathbf{Z} + \dots + \frac{\alpha_r}{m_r} \mathbf{Z} \mid m_i \in \mathbf{N}\}$ (Lemma 3.1 and Corollary 3.2) and an extension of the number field $k \cong \mathbf{Q}(F(\Lambda_\rho[a]))$ (Lemma 3.2). We use these facts to construct an (étale) branched covering $V(k) \rightarrow \mathbf{C}P^n$ [7, Theorem 5] of the n -dimensional projective space $\mathbf{C}P^n$ (Corollary 3.3). Denote by $\log k$ ($\arccos k$, resp.) a number field generated by α_i , such that $k \cong \mathbf{Q}(e^{2\pi i \alpha_i + \log \log \varepsilon})$ ($k \cong \mathbf{Q}(\cos 2\pi \alpha_i \times \log \varepsilon)$, resp.) (Corollary 2.1). Our main result can be formulated as follows.

Theorem 1.1.

$$(1.2) \quad \mathcal{Q}(V(k)) = \begin{cases} (\Lambda, [I], \log k), & \text{if } k \subset \mathbf{C} - \mathbf{R}, \\ (\Lambda, [I], \arccos k), & \text{if } k \subset \mathbf{R}. \end{cases}$$

We start with a brief review of the preliminary facts in Section 2. Theorem 1.1 is proved in Section 3. The case of abelian varieties with complex multiplication is treated in Section 4.

2. PRELIMINARIES

We briefly review noncommutative tori, Drinfeld modules and quantum arithmetic. We refer the reader to [1], [12], [13, Chapters 12 & 13] and [11] for a detailed exposition.

2.1. Noncommutative geometry.

2.1.1. C^* -algebras. The C^* -algebra is an algebra \mathcal{A} over \mathbf{C} with a norm $a \mapsto \|a\|$ and an involution $\{a \mapsto a^* \mid a \in \mathcal{A}\}$ such that \mathcal{A} is complete with respect to the norm, and such that $\|ab\| \leq \|a\| \|b\|$ and $\|a^* a\| = \|a\|^2$ for every $a, b \in \mathcal{A}$. Each commutative C^* -algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space X . Any other algebra \mathcal{A} can be thought of as a noncommutative topological space.

2.1.2. K -theory of C^* -algebras. By $M_\infty(\mathcal{A})$ one understands the algebraic direct limit of the C^* -algebras $M_n(\mathcal{A})$ under the embeddings $a \mapsto \mathbf{diag}(a, 0)$. The direct limit $M_\infty(\mathcal{A})$ can be thought of as the C^* -algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the C^* -algebra \mathcal{A} . Two projections $p, q \in M_\infty(\mathcal{A})$ are equivalent, if there exists an element $v \in M_\infty(\mathcal{A})$, such that $p = v^* v$ and

$q = vv^*$. The equivalence class of projection p is denoted by $[p]$. We write $V(\mathcal{A})$ to denote all equivalence classes of projections in the C^* -algebra $M_\infty(\mathcal{A})$, i.e. $V(\mathcal{A}) := \{[p] : p = p^* = p^2 \in M_\infty(\mathcal{A})\}$. The set $V(\mathcal{A})$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \mathbf{diag}(p, q) = [p' \oplus q']$, where $p' \sim p$, $q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(\mathcal{A})$ is given by $[0]$, where 0 is the zero projection. By the K_0 -group $K_0(\mathcal{A})$ of the unital C^* -algebra \mathcal{A} one understands the Grothendieck group of the abelian semi-group $V(\mathcal{A})$, i.e. a completion of $V(\mathcal{A})$ by the formal elements $[p] - [q]$. The image of $V(\mathcal{A})$ in $K_0(\mathcal{A})$ is a positive cone $K_0^+(\mathcal{A})$ defining the order structure \leq on the abelian group $K_0(\mathcal{A})$. The pair $(K_0(\mathcal{A}), K_0^+(\mathcal{A}))$ is known as a dimension group of the C^* -algebra \mathcal{A} .

2.1.3. Noncommutative tori. The m -dimensional noncommutative torus \mathcal{A}_Θ^m is the universal C^* -algebra generated by unitary operators u_1, \dots, u_m satisfying the commutation relations

$$(2.3) \quad u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j, \quad 1 \leq i, j \leq m$$

for a skew-symmetric matrix $\Theta = (\theta_{ij}) \in M_m(\mathbf{R})$ [12]. It is known that $K_0(\mathcal{A}_\Theta^m) \cong K_1(\mathcal{A}_\Theta^m) \cong \mathbf{Z}^{2m-1}$. The canonical trace τ on the C^* -algebra \mathcal{A}_Θ^m defines a homomorphism from $K_0(\mathcal{A}_\Theta^m)$ to the real line \mathbf{R} ; under the homomorphism, the image of $K_0(\mathcal{A}_\Theta^m)$ is a \mathbf{Z} -module, whose generators $\tau = (\tau_i)$ are polynomials in θ_{ij} . The noncommutative tori \mathcal{A}_Θ^m and $\mathcal{A}_{\Theta'}^m$ are Morita equivalent, if the matrices Θ and Θ' belong to the same orbit of a subgroup $SO(m, m | \mathbf{Z})$ of the group $GL_{2m}(\mathbf{Z})$, which acts on Θ by the formula $\Theta' = (A\Theta + B) / (C\Theta + D)$, where $(A, B, C, D) \in GL_{2m}(\mathbf{Z})$ and the matrices $A, B, C, D \in GL_m(\mathbf{Z})$ satisfy the conditions $A^t D + C^t B = I$, $A^t C + C^t A = 0 = B^t D + D^t B$, where I is the unit matrix and t at the upper right of a matrix means a transpose of the matrix.) The group $SO(m, m | \mathbf{Z})$ can be equivalently defined as a subgroup of the group $SO(m, m | \mathbf{R})$ consisting of linear transformations of the space \mathbf{R}^{2m} , which preserve the quadratic form $x_1 x_{m+1} + x_2 x_{k+2} + \dots + x_k x_{2m}$.

2.2. Non-abelian class field theory. Let $\mathfrak{k} := \mathbf{F}_q(T)$ ($A := \mathbf{F}_q[T]$, resp.) be the field of rational functions (the ring of polynomial functions, resp.) in one variable T over a finite field \mathbf{F}_q , where $q = p^n$ and let $\tau_p(x) = x^p$. Recall that the Drinfeld module $Drin_A^r(\mathfrak{k})$ of rank $r \geq 1$ is a homomorphism

$$(2.4) \quad \rho : A \xrightarrow{r} \mathfrak{k}\langle \tau_p \rangle$$

given by a polynomial $\rho_a = a + c_1 \tau_p + c_2 \tau_p^2 + \dots + c_r \tau_p^r$ with $c_i \in \mathfrak{k}$ and $c_r \neq 0$, such that for all $a \in A$ the constant term of ρ_a is a and $\rho_a \notin \mathfrak{k}$ for at least one $a \in A$ [13, p. 200]. For each non-zero $a \in A$ the function field $\mathfrak{k}(\Lambda_\rho[a])$ is a Galois extension of \mathfrak{k} , such that its Galois group is isomorphic to a subgroup G of the matrix group $GL_r(A/aA)$, where $\Lambda_\rho[a] = \{\lambda \in \bar{\mathfrak{k}} \mid \rho_a(\lambda) = 0\}$ is a torsion submodule of the non-trivial Drinfeld module $Drin_A^r(\mathfrak{k})$ [13, Proposition 12.5]. Clearly, the abelian extensions correspond to the case $r = 1$.

Let G be a left cancellative semigroup generated by τ_p and all $a_i \in \mathfrak{k}$ subject to the commutation relations $\tau_p a_i = a_i^p \tau_p$. Let $C^*(G)$ be the semigroup C^* -algebra [6]. For a Drinfeld module $Drin_A^r(\mathfrak{k})$ defined by (2.4) we consider a homomorphism of the semigroup C^* -algebras:

$$(2.5) \quad C^*(A) \xrightarrow{r} C^*(\mathfrak{k}\langle \tau_p \rangle).$$

It is proved that (2.5) defines a map $F : Drin_A^r(\mathfrak{k}) \mapsto \mathcal{A}_{RM}^{2r}$ [10, Definition 3.1].

Theorem 2.2 ([10]). *The following is true:*

In other words, we omit the additive structure and consider a multiplicative semigroup of the ring $\mathfrak{k}\langle \tau_p \rangle$.

- (i) the map $F : \text{Drin}_A^r(\mathfrak{k}) \mapsto \mathcal{A}_{RM}^{2r}$ is a functor from the category of Drinfeld modules \mathfrak{D} to a category of the noncommutative tori \mathfrak{A} , which maps any pair of isogenous (isomorphic, resp.) modules $\text{Drin}_A^r(\mathfrak{k})$, $\widetilde{\text{Drin}}_A^r(\mathfrak{k}) \in \mathfrak{D}$ to a pair of the homomorphic (isomorphic, resp.) tori \mathcal{A}_{RM}^{2r} , $\widetilde{\mathcal{A}}_{RM}^{2r} \in \mathfrak{A}$;
- (ii) $F(\Lambda_\rho[a]) = \{e^{2\pi i \alpha_i + \log \log \varepsilon} \mid 1 \leq i \leq r\}$, where $\mathcal{A}_{RM}^{2r} = F(\text{Drin}_A^r(\mathfrak{k}))$, α_i are generators of the Grothendieck semi-group $K_0^+(\mathcal{A}_{RM}^{2r})$, $\log \varepsilon$ is a scaling factor and $\Lambda_\rho(a)$ is the torsion submodule of the A -module $\overline{\mathfrak{k}}_\rho$;
- (iii) the Galois group $\text{Gal}(k_0(e^{2\pi i \alpha_i + \log \log \varepsilon}) \mid k_0) \subseteq \text{GL}_r(A/aA)$, where k_0 is a subfield of the number field $\mathbf{Q}(e^{2\pi i \alpha_i + \log \log \varepsilon})$.

Theorem 2.2 implies a non-abelian class field theory as follows. Fix a non-zero $a \in A$ and let $G := \text{Gal}(\mathfrak{k}(\Lambda_\rho[a]) \mid \mathfrak{k}) \subseteq \text{GL}_r(A/aA)$, where $\Lambda_\rho[a]$ is the torsion submodule of the A -module $\overline{\mathfrak{k}}_\rho$. Consider the number field $k = \mathbf{Q}(F(\Lambda_\rho[a]))$. Denote by k_0 the maximal subfield of k which is fixed by the action of all elements of the group G .

Corollary 2.1. (Non-abelian class field theory) *The number field*

$$(2.6) \quad k \cong \begin{cases} k_0(e^{2\pi i \alpha_i + \log \log \varepsilon}), & \text{if } k_0 \subset \mathbf{C} - \mathbf{R}, \\ k_0(\cos 2\pi \alpha_i \times \log \varepsilon), & \text{if } k_0 \subset \mathbf{R}, \end{cases}$$

is a Galois extension of k_0 , such that $\text{Gal}(k \mid k_0) \cong G$.

2.3. Quantum arithmetic. Let V be an n -dimensional projective variety over the field of complex numbers \mathbf{C} . Recall [9, Section 5.3.1] that the Serre C^* -algebra \mathcal{A}_V is the norm closure of a self-adjoint representation of the twisted homogeneous coordinate ring of V by the bounded linear operators acting on a Hilbert space; we refer the reader to [14] or Section 2.1.1 for the details. Let $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V))$ be a dimension group of the C^* -algebra \mathcal{A}_V [1, Section 6.1]. The triple $(\Lambda, [I], K)$ stays for a dimension group generated by the ideal class $[I]$ of an order $\Lambda \subseteq O_K$ in the ring of integers of a number field K [3, Chapter 6], [4] or [9, Theorem 3.5.4].

Definition 2.1. *The Serre C^* -algebra \mathcal{A}_V is said to have real multiplication by the triple $(\Lambda, [I], K)$, if there exists an isomorphism of the dimension group $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (\Lambda, [I], K)$, where $\Lambda \subseteq O_K$ is an order, $[I] \subset \Lambda$ is an ideal class and K is a number field.*

Example 2.1 ([11]). *Let $V \cong \mathcal{E}_{CM}$ be an elliptic curve with complex multiplication by the triple $(L, [I], k)$, where $L = \mathbf{Z} + fO_k$ is an order of conductor $f \geq 1$ in the ring O_k of an imaginary quadratic field $k \cong \mathbf{Q}(\sqrt{-d})$ and $[I]$ an ideal class in L . Then $\mathcal{A}_V \cong \mathcal{A}_{RM}$ is a noncommutative torus with real multiplication by the triple $(\Lambda, [I], K)$, where $[I]$ an ideal class in the order $L = \mathbf{Z} + f'O_k$ of conductor $f' \geq 1$ in the ring O_k of an imaginary quadratic field $K \cong \mathbf{Q}(\sqrt{d})$, such that f' is the least integer satisfying a group isomorphism $\text{Cl}(\mathbf{Z} + f'O_k) \cong \text{Cl}(\mathbf{Z} + fO_k)$, where $\text{Cl}(R)$ is the class group of the ring R .*

Let \mathcal{M}_V be the deformation moduli space of variety V and $m = \dim_{\mathbf{C}} \mathcal{M}_V$. Denote by O_K the ring of integers of a number field K of degree $\deg(K \mid \mathbf{Q}) = 2m$.

Theorem 2.3 ([11]). *The Serre C^* -algebra \mathcal{A}_V has real multiplication by a triple $(\Lambda, [I], K)$, if and only if, the projective variety V is defined over a number field k .*

3. PROOF OF THEOREM 1.1

The idea of proof was outlined in Section 1. For the sake of clarity, let us review the main steps. Each isogeny of the Drinfeld module $\text{Drin}_A^r(\mathfrak{k})$ generates an endomorphism of the Grothendieck semi-group $K_0^+(\mathcal{A}_{RM}^{2r}) \cong \mathbf{Z} + \alpha_1 \mathbf{Z} + \cdots + \alpha_r \mathbf{Z} \subset \mathbf{R}$, where $\mathcal{A}_{RM}^{2r} = F(\text{Drin}_A^r(\mathfrak{k}))$

(Lemma 3.1). These endomorphisms are classified by the r -tuples of integers $m_i \geq 1$ (Corollary 3.2). In particular, each isogeny defines an extension of the field $k = \mathbf{Q}(e^{2\pi i \alpha_i + \log \log \varepsilon})$ by the r roots of degree m_i (Lemma 3.2). Using results of [7, Theorem 5] we construct a branched covering $V(k) \rightarrow \mathbf{C}P^n$ of the n -dimensional projective space $\mathbf{C}P^n$ (Corollary 3.3). The $V(k)$ is proved to satisfy equation (1.2) (Lemma 3.3). We pass to a detailed argument.

Lemma 3.1. *Drinfeld modules $\text{Drin}_A^r(\mathfrak{k})$ and $\widetilde{\text{Drin}}_A^r(\mathfrak{k})$ are isogenous, if and only if, $\tilde{\Lambda} \subseteq \Lambda$, $[I] \subseteq [\tilde{I}]$ and $K \cong \tilde{K}$, where Λ ($\tilde{\Lambda}$, resp.) is the endomorphism ring of $K_0^+(F(\text{Drin}_A^r(\mathfrak{k}))) \cong \mathbf{Z} + \sum_{k=1}^r \alpha_k \mathbf{Z}$ ($K_0^+(F(\widetilde{\text{Drin}}_A^r(\mathfrak{k})))$, resp.), $[I] \subseteq [\tilde{I}]$ are the ideal classes of orders Λ and $\tilde{\Lambda}$ belonging to the same coset in the inclusion of the corresponding ideal class groups and $K = \mathbf{Q}(\alpha_i)$.*

Proof. (i) In view of item (i) of Theorem 2.2, any pair of isogenous Drinfeld modules $\text{Drin}_A^r(\mathfrak{k})$ and $\widetilde{\text{Drin}}_A^r(\mathfrak{k})$ defines a homomorphism of the C^* -algebras $h : \mathcal{A}_{RM}^{2r} \rightarrow \widetilde{\mathcal{A}}_{RM}^{2r}$, where $\mathcal{A}_{RM}^{2r} = F(\text{Drin}_A^r(\mathfrak{k}))$ and $\widetilde{\mathcal{A}}_{RM}^{2r} = F(\widetilde{\text{Drin}}_A^r(\mathfrak{k}))$.

(ii) Consider an induced order-homomorphism $h_* : K_0^+(\mathcal{A}_{RM}^{2r}) \rightarrow K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r})$ of the Grothendieck semi-groups of \mathcal{A}_{RM}^{2r} and $\widetilde{\mathcal{A}}_{RM}^{2r}$ [1, Section 6].

(iii) Recall that $K_0^+(\mathcal{A}_{RM}^{2r}) \cong \mathbf{Z} + \alpha_1 \mathbf{Z} + \cdots + \alpha_r \mathbf{Z} \subset \mathbf{R}$ and $K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r}) \cong \mathbf{Z} + \tilde{\alpha}_1 \mathbf{Z} + \cdots + \tilde{\alpha}_r \mathbf{Z} \subset \mathbf{R}$, where α_i and $\tilde{\alpha}_k$ are algebraic numbers of degree $2r$ over \mathbf{Q} . In view of (ii), one gets $K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r}) \subseteq K_0^+(\mathcal{A}_{RM}^{2r})$ and, therefore, $K \cong \tilde{K}$, where $K = \mathbf{Q}(\alpha_i)$ and $\tilde{K} = \mathbf{Q}(\tilde{\alpha}_k)$.

(iv) On the other hand, we have $\Lambda := \text{End}(K_0^+(\mathcal{A}_{RM}^{2r}))$ and $\tilde{\Lambda} := \text{End}(K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r}))$, where End is the ring of endomorphisms of the respective Grothendieck semi-groups. Since $K_0^+(\widetilde{\mathcal{A}}_{RM}^{2r}) \subseteq K_0^+(\mathcal{A}_{RM}^{2r})$, one gets an inclusion of the rings $\tilde{\Lambda} \subseteq \Lambda$.

(v) Recall that Λ and $\tilde{\Lambda}$ are orders in the ring of integers O_K of the number field $K = \mathbf{Q}(\alpha_i)$. Denote by $Cl(\Lambda)$ and $Cl(\tilde{\Lambda})$ the respective ideal class groups. Since $\tilde{\Lambda} \subseteq \Lambda$, one gets a reverse inclusion of the finite abelian groups $Cl(\Lambda) \subseteq Cl(\tilde{\Lambda})$. If $[I] \in Cl(\Lambda)$, then $[I] \in Cl(\tilde{\Lambda})$ and the corresponding element $[\tilde{I}] \in Cl(\tilde{\Lambda})$ must be in the same coset with $[I]$ defined by the subgroup $Cl(\Lambda)$ of the ideal class group $Cl(\tilde{\Lambda})$.

(vi) The necessary conditions of Lemma 3.1 are established likewise and the proof is left to the reader.

Lemma 3.1 is proved. \square

Corollary 3.2. *The set of all isogenies of the Drinfeld modules $\text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k})$ is bijective with the integer tuples $\{(m_1, \dots, m_r) \mid m_i \geq 1\}$. Moreover, the composition of isogenies correspond to a component-wise multiplication of the tuples.*

Proof. Each $\tilde{\Lambda} \subseteq \Lambda$ has the form $\tilde{\Lambda} = \mathbf{Z} + \tilde{\alpha}_1 \mathbf{Z} + \cdots + \tilde{\alpha}_r \mathbf{Z}$, where $\tilde{\alpha}_i = \frac{\alpha_i}{m_i}$ for some integers $m_i \geq 1$ [2, p. 88]. It is easy to see, that a composition of isogenies $\text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k}) \rightarrow \widetilde{\widetilde{\text{Drin}}}_A^r(\mathfrak{k})$ corresponds to the inclusions $\tilde{\tilde{\Lambda}} \subseteq \tilde{\Lambda} \subseteq \Lambda$. The latter gives rise to a component-wise multiplication of the integer vectors (m_1, \dots, m_r) . Corollary 3.2 follows. \square

Lemma 3.2. *An isogeny $\text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k})$ acts on the image $F(\Lambda_\rho[a]) = \{e^{2\pi i \alpha_i + \log \log \varepsilon} \mid 1 \leq i \leq r\}$ of the torsion submodule $\Lambda_\rho[a]$ via an extension $\tilde{k} = k(\sqrt[m_1]{x}, \dots, \sqrt[m_r]{x})$ ($\tilde{k} = k(\Re \sqrt[m_1]{x}, \dots, \Re \sqrt[m_r]{x})$, resp.) of the imaginary (real, resp.) number field $k = \mathbf{Q}(e^{2\pi i \alpha_i + \log \log \varepsilon})$ ($k = \mathbf{Q}(\cos 2\pi \alpha_i \times \log \varepsilon)$, resp.) by the roots of degree m_i of the elements $x \in k$.*

Proof. (i) Using Corollary 3.2, one obtains a formula for the image of torsion submodule $\tilde{\Lambda}_\rho[a]$:

$$(3.7) \quad F(\tilde{\Lambda}_\rho[a]) = \{e^{2\pi \frac{\alpha_i}{m_i} + \log \log \tilde{\varepsilon}} \mid 1 \leq i \leq r\},$$

where $m_i \geq 1$ are integer numbers. We let $m := LCM(m_1, \dots, m_r)$ be the least common multiple and $\log \tilde{\varepsilon} := (\log \varepsilon)^m$.

(ii) The substitution $\log \tilde{\varepsilon}_i = (\log \varepsilon)^{\frac{1}{m_i}}$ brings (3.7) to the form:

$$(3.8) \quad F(\tilde{\Lambda}_\rho[a]) = \{(e^{2\pi i \alpha_i + \log \log \varepsilon})^{\frac{1}{m_i}} \mid 1 \leq i \leq r\}.$$

It follows from (3.8) that the field k is an extension of degree m of the field \tilde{k} by the roots $\{\sqrt[m_i]{x} \mid 1 \leq i \leq r\}$, where $x \in k$.

(iii) If $k = \mathbf{Q}(\cos 2\pi \alpha_i \times \log \varepsilon)$ is a real number field, we take the real part \Re of the complex numbers in formulas (3.7) and (3.8) and repeat the argument of steps (i) and (ii). Lemma 3.2 is proved. \square

Corollary 3.3. *For every $n \geq 1$ and every isogeny $\text{Drin}_A^r(\mathfrak{k}) \rightarrow \widetilde{\text{Drin}}_A^r(\mathfrak{k})$ there exists a Galois covering $V(k) \rightarrow kP^n$ with the branch locus of index (m_1, \dots, m_r) .*

Proof. The result follows from [7]. Indeed, given an integer $n \geq 1$ and the index of multiplicities (m_1, \dots, m_r) of the branch locus consisting of the union of r hyper-surfaces, one can construct a Galois covering:

$$(3.9) \quad V \longrightarrow \mathbf{C}P^n,$$

where $\mathbf{C}P^n$ is the n -dimensional complex projective space [7, Theorem 5]. Since $k \subset \mathbf{C}$, one can always assume that projective variety V is defined over the number field k , for otherwise one takes a deformation of V in its moduli space. Therefore we get a covering:

$$(3.10) \quad V(k) \longrightarrow \mathbf{C}P^n.$$

\square

Lemma 3.3. *The projective variety $V(k)$ satisfies equation (1.2).*

Proof. (i) Recall that $\log k := \mathbf{Q}(\alpha_i)$ ($\arccos k := \mathbf{Q}(\alpha_i)$, resp.) is a number field, such that $k \cong \mathbf{Q}(e^{2\pi i \alpha_i + \log \log \varepsilon})$ ($k \cong \mathbf{Q}(\cos 2\pi \alpha_i \times \log \varepsilon)$, resp.) Lemmas 3.1, 3.2 and Corollary 3.3 give us a map

$$(3.11) \quad \mathcal{P} : (\Lambda, [I], \log k) \longrightarrow V(k)$$

sending order-isomorphic Handelman triples $(\Lambda, [I], \log k)$ to the k -isomorphic projective varieties $V(k)$.

(ii) On the other hand, there exists a map

$$(3.12) \quad \mathcal{Q} : V(k) \longrightarrow (\Lambda', [I]', K)$$

which sends k -isomorphic projective varieties $V(k)$ to the order-isomorphic Handelman triples (Theorem 2.3).

(iii) One gets from (3.11) and (3.12) a commutative diagram in Figure 1. It follows from the diagram that $(\Lambda, [I], \log k) \cong (\Lambda', [I]', K)$ are order-isomorphic Handelman triples. In particular, the number fields $K \cong \log k$ must be isomorphic. Therefore, the projective variety $V(k)$ satisfies equation (1.2).

Lemma 3.3 is proved. \square

Theorem 1.1 follows from Lemma 3.3.

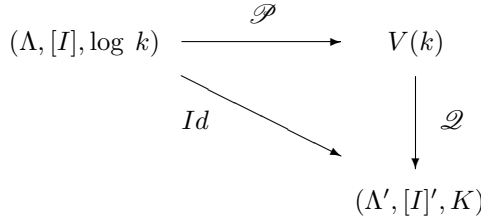


FIGURE 1. Handelman’s triples

4. COMPLEX MULTIPLICATION

We shall illustrate Theorem 1.1 in the simplest case of an n -dimensional abelian variety with complex multiplication $A_{CM}^n(k)$ [5]. Roughly speaking, the following result says that $n = r$, where r is the rank of the Drinfeld module.

Corollary 4.4.

$$(4.13) \quad \mathcal{Q}(A_{CM}^n(k)) = (\Lambda, [I], \log k),$$

where

$$\begin{cases} k & \cong \mathbf{Q}(e^{2\pi i\alpha_1 + \log \log \varepsilon}, \dots, e^{2\pi i\alpha_n + \log \log \varepsilon}), \\ \log k & \cong \mathbf{Q}(\alpha_1, \dots, \alpha_n). \end{cases}$$

Proof. It is known that $\mathcal{Q}(A_{CM}^n) = (\Lambda, [I], K)$, where $\Lambda = \mathbf{Z} + \alpha_1\mathbf{Z} + \dots + \alpha_n\mathbf{Z}$ [9, Remark 6.6.2]. It remains to compare this result with the formula $\Lambda = \mathbf{Z} + \alpha_1\mathbf{Z} + \dots + \alpha_r\mathbf{Z}$ obtained in Lemma 3.1 for the Drinfeld modules. Thus one gets $n = r$ in Theorem 1.1. Corollary 4.4 follows. \square

Example 4.2. Let $n = 1$, i.e. $V(k) \cong \mathcal{E}_{CM}$ is an elliptic curve with complex multiplication (Example 2.1). In this case $k \cong \mathbf{Q}(e^{2\pi i\alpha + \log \log \varepsilon})$ and $\log k \cong \mathbf{Q}(\alpha)$, where α and $\varepsilon \in \log k$ are irrational quadratic numbers. On the other hand, it is well known that $k \cong \mathbf{Q}(j(\mathcal{E}_{CM}))$ is the Hilbert class field $\mathcal{H}(\mathbf{Q}(\sqrt{-d}))$ of an imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$ corresponding to complex multiplication with $j(\mathcal{E}_{CM})$ being the j -invariant of \mathcal{E}_{CM} . In other words, the algebraic number $e^{2\pi i\alpha + \log \log \varepsilon}$ is a generator of the field $\mathcal{H}(\mathbf{Q}(\sqrt{-d}))$ distinct from the Weierstrass generator $j(\mathcal{E}_{CM})$. This fact was first observed in [8].

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