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On Some Properties of Incomplete Trivariate Generalized Tribonacci Polynomials

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Abstract

In this paper we define incomplete trivariate generalized Tribonacci polynomials and obtain some properties of them using tables and sum formulas. Especially, we obtain a recurrence relation of these new class of polynomials.

Keywords: *Trivariate generalized Fibonacci polynomials; incomplete trivariate generalized Fibonacci polynomials.*
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1. Introduction

From [6], we know that Tribonacci numbers studied in 1963 by M. Feinberg when he was a 14-years-old. Tribonacci numbers T_n are defined by $T_0 = 0, T_1 = 1, T_2 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for any integer $n > 2$. In [4], Tribonacci polynomials are defined by $t_0(x) = 0, t_1(x) = 1, t_2(x) = x^2$ and $t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x)$ for any integer $n > 2$.
In [5], trivariate Fibonacci polynomials defined by the following recurrence relations

$$H_n(x,y,z) = xH_{n-1}(x,y,z) + yH_{n-2}(x,y,z) + zH_{n-3}(x,y,z)$$

with the initial conditions

$$H_0(x,y,z) = 0, H_1(x,y,z) = 1, H_2(x,y,z) = x.$$

Note that $H_n(1,1,1) = T_n$ and $H_n(x^2,x,1) = t_n(x)$. In [2], it was given incomplete k-Pell, k-Pell-Lucas and modified k-Pell numbers and their some properties. In [3], it was given a proof of the conjecture given in [7] about generating function of the incomplete Tribonacci numbers and given Tribonacci polynomial triangle. In [5], it was obtained some properties of the trivariate Fibonacci and Lucas polynomials such as Binet formulas, generating functions of them (see for more details [5]). In this study we define trivariate generalized Tribonacci polynomials and incomplete trivariate generalized Tribonacci polynomials, then we investigate some properties of these polynomials and construct interesting tables of them.

2. Incomplete Trivariate Generalized Tribonacci Polynomials

At first, we give the definition of trivariate generalized Tribonacci polynomials.

Definition 2.1. *Let $p(x,y,z), q(x,y,z)$ and $r(x,y,z)$ be polynomials with 3 real variables and real coefficients. Trivariate generalized Tribonacci polynomials are defined by the following recurrence relation:*

$$M_n(p,q,r) = p(x,y,z)M_{n-1}(p,q,r) + q(x,y,z)M_{n-2}(p,q,r) + r(x,y,z)M_{n-3}(p,q,r), \quad n \geq 3 \tag{2.1}$$

with the initial conditions

$$M_0(p,q,r) = 0, M_1(p,q,r) = 1 \text{ and } M_2(p,q,r) = p(x,y,z).$$

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n/i	0	1	2	3	$\cdots$
0	1				
1	p	q			
2	p^2	$2pq+r$	q^2		
3	p^3	$3p^2q+2pr$	$3pq^2+2qr$	q^3	
4	p^4	$4p^3q+3p^2r$	$6p^2q^2+6pqr+r^2$	$4pq^3+3q^2r$	
$\vdots$					

Table 1: Trivariate Generalized Tribonacci Polynomial Triangle

The first six trivariate generalized Tribonacci polynomials are showed in the following table:

$$M_1(p,q,r)=1$$
$$M_2(p,q,r)=p(x,y,z)$$
$$M_3(p,q,r)=p^2(x,y,z)+q(x,y,z)$$
$$M_4(p,q,r)=p^3(x,y,z)+2p(x,y,z)q(x,y,z)+r(x,y,z)$$
$$M_5(p,q,r)=p^4(x,y,z)+3p^2(x,y,z)q(x,y,z)+2p(x,y,z)r(x,y,z)+q^2(x,y,z)$$
$$M_6(p,q,r)=p^5(x,y,z)+4p^3(x,y,z)q(x,y,z)+3p^2(x,y,z)r(x,y,z)+3p(x,y,z)q^2(x,y,z)+2q(x,y,z)r(x,y,z).$$

For $p(x,y,z)=x$, $q(x,y,z)=y$ and $r(x,y,z)=z$ we have trivariate Fibonacci polynomial $H_n(x,y,z)$; for $p(x,y,z)=x^2$, $q(x,y,z)=x$ and $r(x,y,z)=1$ we have Tribonacci polynomial $t_n(x)$; for $p(x,y,z)=1$, $q(x,y,z)=1$ and $r(x,y,z)=1$ we have classical Tribonacci numbers T_n (for more details see [5], [6] and [7]). Tribonacci triangle was given in [1], Tribonacci polynomial triangle was given in [7] and trivariate Fibonacci polynomials triangle was given in [5]. In this study we give a table like as aforementioned tables for the polynomials $M_n(p,q,r)$. Now we define a new class of polynomials $S(n,i)(p,q,r)$ related to the polynomials $M_n(p,q,r)$.

Definition 2.2. *For $n\geqslant 0$ and $i\geqslant 0$, we define polynomials $S(n,i)(p,q,r)$ as*

$$S(n,i)(p,q,r)=\sum_{j=0}^{i}\binom{i}{j}\binom{n-j}{i}p^{n-i-j}(x,y,z)q^{i-j}(x,y,z)r^{j}(x,y,z). \tag{2.2}$$

Using the definition of $S(n,i)(p,q,r)$, we give a table named as trivariate generalized Tribonacci polynomial triangle as the sum of elements on the rising diagonal lines in this table is the trivariate generalized Tribonacci polynomials $M_n(p,q,r)$.

Notice that the polynomials $S(n,i)(p,q,r)$ appear on the $n-th$ row and $i-th$ column of this table. Then, we get

$$S(n+1,i)(p,q,r)=pS(n,i)(p,q,r)+qS(n,i-1)(p,q,r)+rS(n-1,i-1)(p,q,r), \tag{2.3}$$

where $S(n,0)(p,q,r)=p^n$ and $S(n,n)(p,q,r)=q^n$. Since the sum of elements on the rising diagonal lines in Table 1 is the trivariate generalized Tribonacci polynomials $M_n(p,q,r)$, we have

$$M_n(p,q,r)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}S(n-1-i,i)(p,q,r). \tag{2.4}$$

Combining the equations (2.2) and (2.4), we have

$$M_n(p,q,r)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}p^{n-2i-j-1}(x,y,z)q^{i-j}(x,y,z)r^{j}(x,y,z). \tag{2.5}$$

Now we define the incomplete trivariate generalized Tribonacci polynomials using the equation (2.4) and (2.5).

Definition 2.3. *For $n\geq 1$, let $p(x,y,z), q(x,y,z)$ and $r(x,y,z)$ be polynomials with 3 real variables and real coefficients. Incomplete trivariate generalized Tribonacci polynomials are defined by the following equation:*

$$\begin{aligned} M_n^{(s)}(p,q,r) &= \sum_{i=0}^{s}S(n-1-i,i)(p,q,r) \\ &= \sum_{i=0}^{s}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}p^{n-2i-j-1}(x,y,z)q^{i-j}(x,y,z)r^{j}(x,y,z),\quad 0\leq s\leq\left\lfloor\frac{n-1}{2}\right\rfloor, \end{aligned} \tag{2.6}$$

If we take $p(x,y,z)=x$, $q(x,y,z)=y$ and $r(x,y,z)=z$, we can also define incomplete trivariate Tribonacci polynomials.

If we take $p(x,y,z)=x^2$, $q(x,y,z)=x$ and $r(x,y,z)=1$ in Definition 2.3, we have incomplete tribonacci polynomials (for more details see [7]).

From now on, we will briefly denote $M_n(p,q,r)$ by M_n, $M_n^{(s)}(p,q,r)$ by $M_n^{(s)}$, $p(x,y,z)$ by p, $q(x,y,z)$ by q, $r(x,y,z)$ by r, $S(n,i)(p,q,r)$ by $S(n,i)$ and incomplete trivariate generalized Tribonacci polynomials by incomplete generalized Tribonacci polynomials.

n/s	0	1	2	3
1	1			
2	p			
3	p^2	p^2+q		
4	p^3	$p^3+2pq+r$		
5	p^4	p^4+3p^2q+2pr	$p^4+3p^2q+2pr+q^2$	
6	p^5	$p^5+4p^3q+3p^2r$	$p^5+4p^3q+3p^2r+3pq^2+2qr$	
7	p^6	$p^6+5p^4q+4p^3r$	$p^6+5p^4q+4p^3r+6p^2q^2+6pqr+r^2$	$p^6+5p^4q+4p^3r+6p^2q^2+6pqr+q^3+r^2$
8	p^7	$p^7+6p^5q+5p^4r$	$p^7+6p^5q+5p^4r+10p^3q^2+12p^2qr+3pr^2$	$p^7+6p^5q+5p^4r+10p^3q^2+12p^2qr+3pr^2+4pq^3+3q^2r$

Table 2: Polynomials of $M_n^{(s)}(p,q,r)$ for $1\leq n\leq 8$ and $0\leq s\leq 3$.

From the equation (2.6), we obtain the followings:

$$M_n^{(0)}=p^{n-1},\ \text{for } n\geq 1,$$
$$M_n^{(1)}=p^{n-1}+(n-2)p^{n-3}q+(n-3)p^{n-4}r,\ \text{for } n\geq 3,$$
$$M_n^{(\lfloor\frac{n-1}{2}\rfloor)}=M_n,\ \text{for } n\geq 1,$$
$$M_n^{(\lfloor\frac{n-1}{2}\rfloor)}=\begin{cases} M_n-\left(\frac{n}{2}pq^{\frac{n-2}{2}}+\frac{n-2}{2}q^{\frac{n-4}{2}}r\right) & \text{for } n\geq 3 \text{ and } n \text{ even;}\\ M_n-q^{\frac{n-1}{2}} & \text{for } n\geq 3 \text{ and } n \text{ odd.}\end{cases}$$

Now, we consider some properties of incomplete generalized Tribonacci polynomials $M_n^{(s)}$.

Lemma 2.4. *The recurrence relation of the incomplete generalized Tribonacci polynomials $M_n^{(s)}$ is*

$$M_{n+3}^{(s+1)}=pM_{n+2}^{(s+1)}+qM_{n+1}^{(s)}+rM_n^{(s)},\ 0\leq s\leq\left\lfloor\frac{n-1}{2}\right\rfloor. \tag{2.7}$$

We can express the equation (2.7), in a different way as

$$M_{n+3}^{(s)}=pM_{n+2}^{(s)}+qM_{n+1}^{(s)}+rM_n^{(s)}-[qS(n-s,s)+rS(n-1-s,s)] \tag{2.8}$$

Proof. From Definition 2.3, we get

$$\begin{aligned} & pM_{n+2}^{(s+1)}+qM_{n+1}^{(s)}+rM_n^{(s)}\\ =\ & p\sum_{i=0}^{s+1}S(n+1-i,i)+q\sum_{i=0}^{s}S(n-i,i)+r\sum_{i=0}^{s}S(n-1-i,i)\\ =\ & p\sum_{i=0}^{s+1}S(n+1-i,i)+q\sum_{i=1}^{s+1}S(n+1-i,i)+r\sum_{i=1}^{s+1}S(n-i,i-1)\\ =\ & \sum_{i=0}^{s+1}[pS(n+1-i,i)+qS(n+1-i,i-1)+rS(n-i,i-1)]-qS(n+1,-1)-rS(n,-1)\\ =\ & \sum_{i=1}^{s+1}S(n+2-i,i)\\ =\ & M_{n+3}^{(s+1)}.\end{aligned}$$

Considering Definition 2.3 and the equation (2.3), we obtain

$$\begin{aligned} M_{n+3}^{(s)} &= \sum_{i=0}^{s}S(n+2-i,i)\\ &= p\sum_{i=0}^{s}S(n+1-i,i)+q\sum_{i=0}^{s}S(n+1-i,i-1)+r\sum_{i=0}^{s}S(n-i,i-1)\\ &= pM_{n+2}^{(s)}+q\sum_{i=0}^{s}S(n-i,i)+r\sum_{i=0}^{s}S(n-1-i,i)-[qS(n-s,s)+rS(n-1-s,s)]\\ &= pM_{n+2}^{(s)}+qM_{n+1}^{(s)}+rM_n^{(s)}-[qS(n-s,s)+rS(n-1-s,s)].\end{aligned}$$

So we find

$$M_{n+3}^{(s)}=pM_{n+2}^{(s)}+qM_{n+1}^{(s)}+rM_n^{(s)}-[qS(n-s,s)+rS(n-1-s,s)].$$

Then the proof is completed. □

If we put $p=x,q=y$ and $r=z$ in Lemma 2.4 , we find the corresponding equations similar to the equations (2.7) and (2.8) for incomplete trivariate Tribonacci polynomials. Also Lemma 2.4 generalizes Proposition 2 on page 6 in [7].

Lemma 2.5. *For $l=\left\lfloor \frac{n-1}{2} \right\rfloor$, we have the following equality*:

$$\sum_{s=0}^{l} M_n^{(s)} = (l+1)M_n - \sum_{i=0}^{l}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j-1}{i} p^{n-2i-j-1}q^{i-j}r^j.$$

Proof. From the definition of incomplete generalized Tribonacci polynomials, we know

$$M_n^{(s)} = \sum_{i=0}^{s} S(n-1-i,i).$$

So, we get

$$\begin{aligned}
\sum_{s=0}^{l} M_n^{(s)} &= S(n-1-0,0)+[S(n-1-0,0)+S(n-1-1,1)]+...+ \\
&\quad [S(n-1-0,0)+S(n-1-1,1)+...+S(n-1-l,l)] \\
&= (l+1)S(n-1-0,0)+lS(n-1-1,1) \\
&\quad +(l-1)S(n-1-2,2)+...+S(n-1-l,l).
\end{aligned}$$

Thus, we find

$$\begin{aligned}
\sum_{s=0}^{l} M_n^{(s)} &= \sum_{i=0}^{l}(l+1-i)S(n-i-1,i) \\
&= \sum_{i=0}^{l}(l+1)S(n-i-1,i) - \sum_{i=0}^{l} iS(n-i-1,i) \\
&= (l+1)\sum_{i=0}^{l}S(n-i-1,i) - \sum_{i=0}^{l} iS(n-i-1,i) \\
&= (l+1)M_n(p,q,r) - \sum_{i=0}^{l} i \sum_{j=0}^{l}\binom{i}{j}\binom{n-i-j-1}{i} p^{n-2i-j-1}q^{i-j}r^j.
\end{aligned}$$

□

Lemma 2.5 generalizes Proposition 5 on page 8 in [7]. If we put $p=x, q=y$ and $r=z$ in Lemma 2.5 ,we get a lemma similar to the Lemma 2.5 for incomplete trivariate Tribonacci polynomials.

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