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On Conharmonically Flatness of Lorentzian α-Sasakian Manifolds

Nesrin Calışkan[1*] and Ayşe Funda Sağlamer[2]

[1]*Department of Mathematics and Science Education, Faculty of Education, Usak University, 64200, Uşak-Turkey*
[2]*Department of Mathematics, Faculty of Science and Arts, Dumlupinar University, 43100, Kutahya-Turkey*
[*]*Corresponding author E-mail: caliskan.nesrin35@gmail.com*

Abstract

Conharmonically flatness of Lorentzian α-Sasakian manifolds is characterized and some structure theorems are discussed. In this manner, conharmonically flat, φ-conharmonically flat, ξ-conharmonically flat and quasi-conharmonically flat Lorentzian α-Sasakian manifolds are investigated.

Keywords: *Conharmonic curvature tensor, conharmonically flatness, φ-conharmonically flatness, ξ-conharmonically flatness, quasi-conharmonically flatness.*
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1. Introduction

Lorentzian α-Sasakian manifolds take part as a special type of trans-Sasakian structures in [14] by Yildiz and Murathan including their characteristics of conformally and quasiconformally flatness. Then, Lorentzian α-Sasakian manifolds' Weyl pseudosymmetry and partially Ricci-Pseudo symmetry properties are studied by Prakasha et al. in [8]. Lokesh et al., searched W_2 curvature tensor related to concircular, conharmonic, quasi-conformal and pseudo projective curvature tensors for these kind of structures in [7]. Taleshian & Asghari searched Ricci-semisymmetry, ϕ-Ricci semisymmetry and ϕ-symmetry of such manifolds in [12]. Barman studied both locally ϕ-symmetry and ξ-projectively flatness of the aforementioned structures via semi-symmetric metric connection in [1]. Ravikumar et al., discussed projectively and concircularly pseudo symmetric besides generalized projective and concircular ϕ-recurrent types of these manifolds in [10]. Dey and Bhattacharyya searched ϕ-recurrency and generalized ϕ-recurrency of projective curvature tensor and quasi-projectively flatness of such spaces in [4]. Ravikumar et al., studied T-Curvature tensor for such kind of structures in [11]. Prakasha & Chavan discussed on M-projective curvature tensor of the same spaces in [9].
Conharmonic curvature tensor is introduced by Ishii [6] and it has been studying for several structures like K-contact and Sasakian manifolds, generalized Sasakian-space-forms, LP-Sasakian manifolds [5], [3], [13]. Also, Bhattacharyya discussed conharmonically flat LP-Sasakian manifolds [2]. In this paper, Lorentzian α-Sasakian manifolds' conharmonic curvature tensor's extended flatness characteristics are discussed.

2. Preliminaries

Assume, $M^{m=(2n+1)}$ be smooth and so the tangent space to M of any point of M contructs the Lie algebra stucture $\mathfrak{L}$. Hence, M is called Lorentzian α-Sasakian manifold with the quadruple (φ,ξ,η,g) if below properties hold:

$$\varphi^2 = I+\eta o\xi, i_\xi g = \eta \tag{2.1}$$

$$\eta(\xi) = -1, \eta o\varphi = 0, \varphi(\xi) = 0, \tag{2.2}$$

where ξ and η denote the Reeb vector and covector fields and additionally, ϕ and g denote the endomorphism and Lorentzian metric structure. Also the below relations hold for the Lorentzian α-Sasakian manifolds:

$$\nabla_T \xi = -\alpha\varphi(T) \tag{2.3}$$

Email addresses: **caliskan.nesrin35@gmail.com (Nesrin Çalışkan) , afundasaglamer@gmail.com (Ayşe Funda Sağlamer)**

$$g(T,U)=g(\varphi T,\varphi U)-\eta(T)\eta(U) \tag{2.4}$$

$$\eta(R(T,U)V)=\alpha^2(g(U,V)\eta(T)-g(T,V)\eta(U)) \tag{2.5}$$

$$R(T,U)\xi=\alpha^2(\eta(U)T-\eta(T)U) \tag{2.6}$$

$$R(\xi,T)U=\alpha^2(g(T,U)\xi-\eta(U)T) \tag{2.7}$$

$$R(\xi,T)\xi=\alpha^2(T+\eta(T)\xi) \tag{2.8}$$

$$S(\varphi T,\varphi U)=S(T,U)+2n\alpha^2\eta(T)\eta(U) \tag{2.9}$$

$$S(T,\xi)=S(\xi,T)=2n\alpha^2\eta(T) \tag{2.10}$$

$$S(\xi,\xi)=-2n\alpha^2 \tag{2.11}$$

$$Q\xi=2n\alpha^2\xi \tag{2.12}$$

for $T,U,V\in\mathfrak{L}$. In the aforementioned relations, S is the Ricci curvature tensor satisfying $S(T,U)=g(QT,U)$ with the notation Q of the Ricci operator. Besides the associated Lorentzian metric's covariant differentiation is notated by ∇ .

Definition 2.1. *The conharmonic curvature tensor of the Lorentzian α-Sasakian manifold M^{2n+1} is described in the sense Ishi introduced in [6]:*

$$K(T,U)V=R(T,U)V-\frac{1}{2n-1}[S(U,V)T-S(T,V)U+g(U,V)QT-g(T,V)QU] \tag{2.13}$$

for $T,U,V\in\mathfrak{L}$.

3. Conharmonically Flat Lorentzian α-Sasakian manifolds

Definition 3.1. *The Lorentzian α-Sasakian manifold M is called conharmonically flat if the following relation holds:*

$$K(T,U)V=0 \tag{3.1}$$

for $T,U,V\in\mathfrak{L}$.

Theorem 3.2. *If the conharmonically flat Lorentzian α-Sasakian manifold M^{2n+1} is Einstein, then it is locally isometric to $S^{2n+1}(c)$ with the constant c.*

Proof. If M^{2n+1} is conharmonically flat, then the aforementioned equation (3.1) holds. By using (2.13) in (3.1),

$$R(T,U)V=\frac{1}{2n-1}[S(U,V)T-S(T,V)U+g(U,V)QT-g(T,V)QU] \tag{3.2}$$

is obtained. Due to manifold is Einstein, λ is constant, $S(T,U)=\lambda g(T,U)$ holds and enables to

$$g(R(T,U)V,W)=\frac{2\lambda}{2n-1}[g(U,V)g(T,W)-g(T,V)g(U,W)] \tag{3.3}$$

for $T,U,V,W\in\mathfrak{L}$.

$$[\frac{2\lambda}{2n-1}-\alpha^2][g(U,V)+\eta(U)\eta(V)]=0 \tag{3.4}$$

for $U,V\in\mathfrak{L}$. This means that $\frac{2\lambda}{2n-1}-\alpha^2=0$ or $g(U,V)+\eta(U)\eta(V)=0$. Due to (2.4) holds for the Lorentzian α-Sasakian manifold and also $g(\varphi U,\varphi V)\neq 0$, it is possible to get the below relation

$$\lambda=\frac{(2n-1)\alpha^2}{2} \tag{3.5}$$

In consequence of (3.3) and (3.5), the following relation is obtained:

$$R(T,U)V=\alpha^2[g(U,V)T-g(T,V)U] \tag{3.6}$$

Thereby M^{2n+1} is a space of constant curvature with the $c=\alpha^2$ constant.
By virtue of (3.6), we get hereinbelow

$$(\nabla_W R)(T,U)V=0 \tag{3.7}$$

for $T,U,V,W\in\mathfrak{L}$. So, the conharmonically flat Einstein Lorentzian α-Sasakian manifold M is a space of constant curvature and hence locally isometric to $S^{2n+1}(\alpha^2)$.

□

4. Quasi-Conharmonically Flat Lorentzian α-Sasakian Manifolds

Definition 4.1. *The Lorentzian α-Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called quasi-conharmonically flat if the below relation is verified:*

$$g(K(T,U)V, \varphi W) = 0 \tag{4.1}$$

for $T,U,V,W \in \mathfrak{L}$.

Theorem 4.2. *Assume that Lorentzian α-Sasakian manifold M^{2n+1} be quasi-conharmonically flat. So, we get the following relation:*

$$S(U,V) = (r - 2n\alpha^2 - 2n + 1)g(U,V) \tag{4.2}$$

for $U,V \in \mathfrak{L}$.

Proof. By using (2.1) and (2.13),

$$\begin{aligned} g(K(T,U)V, \varphi W) = g(R(T,U)V, \varphi W) - \frac{1}{2n-1}[S(U,V)g(T,\varphi W) \\ -S(T,V)g(U,\varphi W) + g(U,V)S(T,\varphi W) - g(T,V)S(U,\varphi W)] \end{aligned} \tag{4.3}$$

is obtained for $T,U,V,W \in \mathfrak{L}$. Since $\{e_i\}$ the orthonormal basis , if we take summation on $i = 1,2,\ldots,2n+1$ and putting $X = \varphi e_i$ and $W = e_i$ in (2.4), we get

$$\begin{aligned} g(K(\varphi e_i,U)V, \varphi e_i) = g(R(\varphi e_i,U)V, \varphi e_i) - \frac{1}{2n-1}[S(U,V)g(\varphi e_i,\varphi e_i) \\ -S(\varphi e_i,V)g(U,\varphi e_i) + g(U,V)S(\varphi e_i,\varphi e_i) - g(\varphi e_i,V)S(U,\varphi e_i)] \end{aligned} \tag{4.4}$$

for $U,V \in \mathfrak{L}$. and after necessary calculations, we obtain the following relation:

$$\begin{aligned} g(K(\varphi e_i,U)V, \varphi e_i) = S(U,V) + g(U,V) - \frac{1}{2n-1}[2nS(U,V) \\ -S(U,V) + \eta(U)S(\xi,V) + (r-2n\alpha^2)g(U,V) - S(U,V)] \end{aligned} \tag{4.5}$$

Owing to M^{2n+1} is quasi-conharmonically flat with the help of (2.13), equation (4.2) is verified. □

Lemma 4.3. *The quasi-conharmonically flat Lorentzian α-Sasakian manifold M^{2n+1} is an Einstein manifold.*

Theorem 4.4. *The Lorentzian α-Sasakian manifold M^{2n+1} is called quasi-conharmonically flat iff the below relation holds:*

$$\begin{aligned} \eta(R(T,U)V) = \alpha^2[\eta(T)\eta(V)U - \eta(U)\eta(V)T] + \frac{1}{(2n-1)}\{2n\alpha^2[\eta(U)g(\varphi T,\varphi V) \\ -\eta(T)g(\varphi U,\varphi V)] + [\eta(U)S(\varphi T,\varphi V) - \eta(T)S(\varphi U,\varphi V)]\} \end{aligned} \tag{4.6}$$

for $T,U,V \in \mathfrak{L}$.

Proof. Due to the Lorentzian α-Sasakian manifold M^{2n+1} is quasi-conharmonically flat, by the use of relations (2.1), (2.13) and taking $W = \varphi W$, we obtain

$$\begin{aligned} g(R(T,U)V, \varphi^2 W) = \frac{1}{2n-1}[S(U,V)g(\varphi T,\varphi W) \\ -S(T,V)G(\varphi U,\varphi W) + g(U,V)S(\varphi T,\varphi W) - g(T,V)S(\varphi U,\varphi W)] \end{aligned} \tag{4.7}$$

for $T,U,V,W \in \mathfrak{L}$.
Finally, we get (4.6) via the use of (2.13) and (2.1) in (2.3) and vice versa. □

5. ξ-Conharmonically Flat Lorentzian α-Sasakian Manifold

Definition 5.1. *Suppose the Lorentzian α-Sasakian manifold M^{2n+1} be ξ-conharmonically flat, then the following relation holds:*

$$K(T,U)\xi = 0 \tag{5.1}$$

for $T,U \in \mathfrak{L}$.

Theorem 5.2. *Assume M^{2n+1} be a ξ-conharmonically flat Lorentzian α-Sasakian manifold, then*

$$R(T,\xi)\xi = -\frac{1}{2n-1}\{S(U,\xi)T - S(T,\xi)U + \eta(U)QT - \eta(T)QU\} \tag{5.2}$$

for $T,U \in \mathfrak{L}$.

Proof. If M^{2n+1} is ξ-conharmonically flat then (5.1) holds. Taking $V=\xi$ in (2.13) and considering (2.1),

$$g(K(T,U)\xi,W)=g(R(T,U)\xi,W)-\frac{1}{2n-1}\{S(U,\xi)g(T,W)$$
$$-S(T,\xi)g(U,W)+\eta(U)S(T,W)-\eta(T)S(U,W)\} \tag{5.3}$$

for $T,U,W\in\mathfrak{L}$.
Owing to the fact that $\{e_i\}$ is orthonormal basis, $\{\varphi e_i\}$ is orthonormal basis either. Summarizing $i=1,2,\ldots,(2n+1)$ in (5.3) and changing $T=W=e_i$, it takes the following form:

$$g(K(e_i,U)\xi,e_i)=g(R(e_i,U)\xi,e_i)-\frac{1}{2n-1}\{S(U,\xi)g(e_i,e_i)$$
$$-S(e_i,\xi)g(U,e_i)+\eta(U)S(e_i,e_i)\} \tag{5.4}$$

for $U\in\mathfrak{L}$. By using (2.1), (2.2) we have (5.3).

□

Theorem 5.3. *For the Lorentzian α-Sasakian manifold M^{2n+1} necessary and sufficient condition to be ξ-conharmonically flat is: M^{2n+1} is an η-Einstein manifold.*

Proof. For the Lorentzian α-Sasakian manifold M^{2n+1} by using $U=\xi$ in (5.3), it is possible to obtain

$$S(T,W)=(-\alpha^2)g(T,W)+(2n-1)\alpha^2\eta(T)\eta(W) \tag{5.5}$$

for $T,W\in\mathfrak{L}$. Namely, M^{2n+1} is η-Einstein and vice versa. □

6. φ-Conharmonically Flat Lorentzian α-Sasakian Manifold

Definition 6.1. *Assume, M^{2n+1} be the Lorentzian α-Sasakian manifold, then M^{2n+1} is called φ-conharmonically flat if*

$$g(K(\varphi T,\varphi U)\varphi V,\varphi W)=0 \tag{6.1}$$

for $T,U,V,W\in\mathfrak{L}$.

Theorem 6.2. *The Lorentzian α-Sasakian manifold M^{2n+1} is called φ-conharmonically flat iff the equation (6.2) holds.*

$$g(K(\varphi T,\varphi U)\varphi V,\varphi W)=-\frac{1}{2(2n-1)}\{g(\varphi U,\varphi V)g(\varphi T,\varphi W)$$
$$-g(\varphi T,\varphi V)g(\varphi U,\varphi W)\} \tag{6.2}$$

for $T,U,V,W\in\mathfrak{L}$.

Proof. Considering (2.13) for M^{2n+1}, it is possible to have the below relation:

$$S(\varphi T,\varphi W)=g(Q(\varphi T),\varphi W). \tag{6.3}$$

In consequence of this, the equation herein below can be written:

$$g(K(\varphi T,\varphi U)\varphi V,\varphi W)=g(R(\varphi T,\varphi U)\varphi V,\varphi W)-\frac{1}{2n-1}\{S(\varphi Y,\varphi V)g(\varphi T,\varphi W)$$
$$-S(\varphi T,\varphi V)g(\varphi U,\varphi W)+g(\varphi U,\varphi V)S(\varphi T,\varphi W)-g(\varphi T,\varphi V)S(\varphi U,\varphi W)\} \tag{6.4}$$

Owing to the fact that $\{e_i\}$ is orthonormal basis, $\{\varphi e_i\}$ is orthonormal basis either. If we summarize $i=1,2,\ldots,(2n+1)$ in (4.3) and take $T=W=e_i$, we have the following relation:

$$g(K(\varphi e_i,\varphi U)\varphi V,\varphi e_i)=g(R(\varphi e_i,\varphi U)\varphi V,\varphi e_i)-\frac{1}{2n-1}\{S(\varphi U,\varphi V)g(\varphi e_i,\varphi e_i)$$
$$-S(\varphi e_i,\varphi V)g(\varphi U,\varphi e_i)+g(\varphi U,\varphi V)S(\varphi e_i,\varphi e_i)-g(\varphi e_i,\varphi V)S(\varphi U,\varphi e_i)\} \tag{6.5}$$

for $U,V\in\mathfrak{L}$. Due to M is φ-conharmonically flat, (6.1) holds and by virtue of (6.5),

$$S(\varphi U,\varphi V)=(r-2n\alpha^2-2n+1)g(\varphi U,\varphi V) \tag{6.6}$$

holds for $U,V\in\mathfrak{L}$. Then, using (2.4) and (6.1) in (6.4), we get (6.2). □

References

[1] A. Barman, On Lorentzian α-Sasakian Manifolds Admitting a Type of Semi-symmetric Metric Connection, Novi Sad J. Math Vol:44, No.2 (2014), 77-88.
[2] A. Bhattacharyya, On a Type of Conharmonically Flat LP-Sasakian Manifold, Anal. Stiin, Ale. Univ." AL. I. CUZA" IASI SI a Mathematica Vol: 47, (2001), 183-188.
[3] U.C. De and R.N. Singh and S.K. Pandey, On the Conharmonic Curvature Tensor of Generalized Sasakian-space-forms, International Scholarly Research Network ISRN Geometry Vol:2012, (2012), 1-14.
[4] S. Dey and A. Bhattacharyya, Some curvature properties of Lorentzian α -Sasakian manifolds, Journal of Dynamical Systems and Geometric Theories Vol:14, No.1 (2016), 85-98.
[5] M.K. Dwivedi and J.S. Kim, On Conharmonic Curvature Tensor in K-contact and Sasakian Manifolds, Bull. Malays. Math. Sci. Soc. (2) Vol:34, No.1 (2011), 171-180.
[6] SY. Ishi, On Conharmonic Transformation, Tensor NS. Vol:7 (1957), 73-80.
[7] S. Lokesh and C.S.B. Venkatesha and K.T.P. Kumar, On Lorentzian α-Sasakian Manifolds, Bulletin of the Malaysian Mathematical Sciences Society. Vol:2, No.3 (2012), 177-182.
[8] D.G. Prakasha and C.S. Bagewadi and N.S. Basavarajappa, On Pseudosymmetric Lorentzian α-Sasakian manifolds, IJPAM. Vol:48, No.1 (2008), 57-65.
[9] D.G. Prakasha and V. Chavan, On M-Projective Curvature Tensor of Lorentzian α-Sasakian manifolds, International Journal of Pure Mathematical Sciences. Vol:18, (2017), 22-31.
[10] N.S. Ravikumar and K. Nagana Gouda, Some Results On Lorentzian α-Sasakian manifolds, Mathematica Aertia. Vol:6, No.6 (2016), 867-875.
[11] N.S. Ravikumar and K. Nagana Gouda and N. Srikantha, T Curvature Tensor On Lorentzian α-Sasakian manifolds, IJPAM. Vol:112, No.1 (2017), 81-91.
[12] A. Taleshian and N. Asghari, On Lorentzian α-Sasakian manifolds, The Journal of Mathematics and Computer Science. Vol:4, No.3 (2012), 295-300.
[13] A. Taleshian and D.G. Prakasha and K. Vikas and N. Asghari, On the Conharmonic Curvature Tensor of LP-Sasakian Manifolds, Palestine Journal of Mathematics. Vol:5, No.1 (2016), 177-184.
[14] A. Yildiz and C. Murathan, On Lorentzian α-Sasakian Manifolds, Kyungpook Mathematical Journal. Vol:45, No.1 (2005), 95-103.