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On Subclasses Of Bi-Starlike Functions Defined By Tremblay Fractional Derivative Operator

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Abstract

In this paper, we introduce and investigate new subclasses of strongly bi-starlike and bi-starlike functions defined by Tremblay fractional derivative operator in the open unit disk. Also we obtain upper bounds for the coefficients $|a_2|$ and $|a_3|$ of functions belonging to these classes. Unlike recent studies, we use different technique for obtain the upper bounds on the coefficients $|a_3|$. Theorems proved in this paper generalizes the results given in [3].

Keywords: *bi-starlike functions, Tremblay operator, Bi-univalent functions, Fractional derivative, strongly bi-starlike functions, Coefficient bounds*
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1. Introduction

Let $\mathscr{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{and} |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let we denote by $\mathscr{S}$, the subclass of $\mathscr{A}$ which elements are univalent in $\mathbb{U}$ ([4]).
From The Koebe's One-Quarter Theorem ([4]) says that "the range of every $f(z) \in \mathscr{S}$ contains the disk $\{w \in \mathbb{C} : |w| < 1/4\}$". Therefore every $f \in \mathscr{S}$ has an inverse and the inverse function f^{-1} satisfy the following:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

The function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

$$= w + \sum_{n=2}^{\infty} b_n w^n.$$

Let $f \in \mathscr{A}$. If both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$, then we say that f is *bi-univalent* in $\mathbb{U}$. The class of all bi-univalent functions in $\mathbb{U}$ given by (1.1) is denoted by Σ.
The reader can find a detailed information about the function class Σ in [16] (see also [3],[9],[22]).

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Coefficient estimates for various subclasses of bi-univalent functions have been previously studied by some authors including Ali *et al.* [2], Frasin [6], Kumar *et al.* [8], Sümer Eker [1],[21], Magesh and Yamini [10], Srivastava *et al.* [15],[19],[20].
We need to following definitions of fractional integral and fractional derivative for our results. (For details, see [11],[12], [17], [18]).
Definition 1. For a function f, the fractional integral of order δ is defined, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi\, ;\ (\delta > 0),$$

where f is an analytic function in a simply-connected region of complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring, $log(z-\xi)$ to be real when $z-\xi > 0$.
Definition 2. The fractional derivative of order δ is defined, for a function f , by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi\ (0 \le \delta < 1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.
Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z)\ \ (0 \le \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

By virtue of Definitions 1, 2 and 3, we have

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \qquad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \qquad (n \in \mathbb{N}, 0 \le \delta < 1).$$

In his thesis, Tremblay [23] investigated a fractional calculus operator defined in terms of the Riemann-Liouville fractional differential operator. Recently, Ibrahim and Jahangiri [7] extended the Tremblay Operator in the complex plane.

Definition 4. Let $f \in \mathscr{A}$. The Tremblay fractional derivative operator $\mathrm{T}_z^{\mu,\gamma}$ of a function f is defined, for all $z \in \mathbb{U}$, by

$$\mathrm{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z)$$

$(0 < \gamma \le 1;\ 0 < \mu \le 1,\ 0 \le \mu-\gamma < 1,\ \mu \ge \gamma)$.

Obviously, if we choose $\mu = \gamma = 1$, we obtain

$\mathrm{T}_z^{1,1} f(z) = f(z)$.

In [5], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 5. Let $f(z) \in \mathscr{A}$. The modified Tremblay operator denoted by $\mathfrak{T}^{\mu,\gamma} : \mathscr{A} \to \mathscr{A}$ and defined as:

$$\begin{aligned} \mathfrak{T}_z^{\mu,\gamma} f(z) &= \frac{\gamma}{\mu} \mathrm{T}_z^{\mu,\gamma} f(z) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)} a_n z^n \end{aligned}$$

$(0 < \gamma \le 1;\ 0 < \mu \le 1,\ 0 \le \mu-\gamma < 1,\ \mu \ge \gamma)$.

where $\mathrm{T}_z^{\mu,\gamma}$ denote the Tremblay fractional derivative operator. For more information about Tremblay Operator see [20]).
The object of the present paper is to introduce new subclasses of strongly bi-starlike and bi-starlike functions defined by modified Tremblay operator and find estimates on the modulus of the coefficients $|a_2|$ and $|a_3|$ for functions in this class. In the sequel, it is assumed that $0 < \gamma \leqq 1;\ 0 < \mu \leqq 1,\ 0 < \mu-\gamma < 1,\ \mu \ge \gamma$.

2. Main Results

Definition 6. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma}(\alpha,\mu,\gamma)$ if the following conditions are satisfied:

$$f\in\Sigma \text{ and } \left|arg\frac{z(\mathfrak{T}_z^{\mu,\gamma}f)'(z)}{\mathfrak{T}_z^{\mu,\gamma}f(z)}\right|<\frac{\alpha\pi}{2} \qquad (0<\alpha\leq 1, z\in\mathbb{U}) \tag{2.1}$$

and

$$\left|arg\frac{w(\mathfrak{T}_w^{\mu,\gamma}g)'(w)}{\mathfrak{T}_w^{\mu,\gamma}g(w)}\right|<\frac{\alpha\pi}{2} \qquad (0<\alpha\leq 1, w\in\mathbb{U}) \tag{2.2}$$

where the function $g(w)$ is given by (1.2).
It is clear that for $\gamma=\mu$, this class is reduced to $S_{\Sigma}^{*}(\alpha)$ of class of strongly bi-starlike of order α $(0<\alpha\leq 1)$, which is introduced by Brannan and Taha [3].

Theorem 1. If $f(z)$ given by (1.1) be in the class $\mathcal{N}_{\Sigma}(\alpha,\mu,\gamma)$, then

$$|a_2|\leq 2\alpha(\gamma+1)\sqrt{\frac{(\gamma+2)}{(\mu+1)\left[4\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}} \tag{2.3}$$

and

$$|a_3|\leq\frac{2\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)[\gamma(\mu+3)+2]}. \tag{2.4}$$

Proof. For f given by (1.1), we can write from (2.1) and (2.2)

$$\frac{z(\mathfrak{T}_z^{\mu,\gamma}f)'(z)}{\mathfrak{T}_z^{\mu,\gamma}f(z)}=[p(z)]^{\alpha} \tag{2.5}$$

$$\frac{w(\mathfrak{T}_w^{\mu,\gamma}g)'(w)}{\mathfrak{T}_w^{\mu,\gamma}g(w)}=[q(w)]^{\alpha} \tag{2.6}$$

where $p(z)$ and $q(w)$ are in Caratheódory Class $\mathcal{P}$. So $p(z)$ and $q(w)$ are have the following series expansions:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.7}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{2.8}$$

(see for details [4]). Now, equating the coefficients (2.5) and (2.6), we find that

$$\frac{\mu+1}{\gamma+1}a_2=\alpha p_1, \tag{2.9}$$

$$\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3-\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.10}$$

$$-\frac{\mu+1}{\gamma+1}a_2=\alpha q_1 \tag{2.11}$$

and

$$\left[\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}-\left(\frac{\mu+1}{\gamma+1}\right)^2\right]a_2^2-\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1=-q_1 \tag{2.13}$$

and

$$2\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14), we get

$$a_2^2=\frac{\alpha^2(p_2+q_2)(\gamma+2)(\gamma+1)^2}{(\mu+1)\left[4\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}. \tag{2.15}$$

It is well known that from the Carathe͘ódory Lemma, the coefficients of $|p_n|\le 2$ and $|q_n|\le 2$ for $n\in\mathbb{N}$ (see [4]). If we take absolute value of both side of a_2^2 and if we apply the Carathéodory Lemma to coefficients p_2 and q_2 we obtain

$$|a_2|\le\sqrt{\frac{4\alpha^2(\gamma+2)(\gamma+1)^2}{(\mu+1)\left[4\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}}.$$

This gives desired bound for $|a_2|$ as asserted in (2.3).

Now, in order to find the bound on $|a_3|$, from (2.10), (2.12) and (2.13), we can write

$$4a_3=\alpha\lambda\left\{\frac{4(\mu+2)(\gamma+1)-(\mu+1)(\gamma+2)}{(\mu+2)(\mu+1)}p_2+\frac{(\gamma+2)}{\mu+2}q_2+\frac{2(\alpha-1)(\gamma+1)}{\mu+1}p_1^2\right\} \tag{2.16}$$

where

$$\lambda=\frac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}.$$

If $\alpha=1$, then

$$|a_3|\le\frac{2\lambda(\gamma+1)}{\mu+1}.$$

Now, we consider the case $0<\alpha<1$. From (2.16), we can write

$$4Re(a_3)=\alpha\lambda Re\left\{\frac{4(\mu+2)(\gamma+1)-(\mu+1)(\gamma+2)}{(\mu+2)(\mu+1)}p_2+\frac{(\gamma+2)}{\mu+2}q_2+\frac{2(\alpha-1)(\gamma+1)}{\mu+1}p_1^2\right\} \tag{2.17}$$

From Herglotz's Representation formula (see [13]) for the functions $p(z)$ and $q(w)$, we have

$$p(z)=\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu_1(t),$$

and

$$q(w)=\int_0^{2\pi}\frac{1+we^{-it}}{1-we^{-it}}d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0,2\pi]$ and $\mu_i(2\pi)-\mu_i(0)=1$, $i=1,2$.
We also have

$$p_n=2\int_0^{2\pi}e^{-int}d\mu_1(t),\qquad n=1,2,\ldots,$$

and

$$q_n=2\int_0^{2\pi}e^{-int}d\mu_2(t),\qquad n=1,2,\ldots.$$

Now (2.17) can be written as follows :

$$4Re(a_3)=2\lambda\alpha\left\{\left(\frac{4(\gamma+1)}{\mu+1}-\frac{\gamma+2}{\mu+2}\right)\int_0^{2\pi}cos2td\mu_1(t)+\frac{\gamma+2}{\mu+2}\int_0^{2\pi}cos2td\mu_2(t)\right\}$$

$$-\frac{8\lambda\alpha(1-\alpha)(\gamma+1)}{\mu+1}\left[\left(\int_0^{2\pi}costd\mu_1(t)\right)^2-\left(\int_0^{2\pi}sintd\mu_1(t)\right)^2\right]$$

$$\leq 2\lambda\alpha\left\{\left(\frac{4(\gamma+1)}{\mu+1}-\frac{\gamma+2}{\mu+2}\right)\int_0^{2\pi} cos2td\mu_1(t)+\frac{\gamma+2}{\mu+2}\int_0^{2\pi} cos2td\mu_2(t)+\frac{4(1-\alpha)(\gamma+1)}{\mu+1}\left(\int_0^{2\pi} sintd\mu_1(t)\right)^2\right\}$$

$$=2\lambda\alpha\left\{\left(\frac{4(\gamma+1)}{\mu+1}-\frac{\gamma+2}{\mu+2}\right)\left(\int_0^{2\pi}(1-2sin^2t)d\mu_1(t)\right)+\frac{\gamma+2}{\mu+2}\left(\int_0^{2\pi}(1-2sin^2t)d\mu_2(t)\right)+\frac{4(1-\alpha)(\gamma+1)}{\mu+1}\left(\int_0^{2\pi} sintd\mu_1(t)\right)^2\right\}.$$

By Jensen's inequality (see [14]), we have

$$\left(\int_0^{2\pi}|sint|d\mu(t)\right)^2\leq\int_0^{2\pi} sin^2td\mu(t).$$

Hence

$$4Re(a_3)\leq 2\lambda\alpha\left\{\frac{4(\gamma+1)}{\mu+1}-\left(\frac{4(1+\alpha)(\gamma+1)}{\mu+1}-\frac{2(\gamma+2)}{\mu+2}\right)\int_0^{2\pi} sin^2td\mu_1(t)-\frac{2(\gamma+2)}{\mu+2}\int_0^{2\pi} sin^2td\mu_2(t)\right\}$$

and thus

$$Re(a_3)\leq\frac{2\lambda\alpha(\gamma+1)}{\mu+1}$$

which implies

$$|a_3|\leq\frac{2\lambda\alpha(\gamma+1)}{\mu+1}.$$

This completes the proof of theorem. □

If we choose $\gamma=\mu$, in the Theorem 1, we have the following corollary.

Corollary 1. [3] Let $f(z)$ given by (1.1) belong to $S^*_\Sigma(\alpha)$ $(0<\alpha\leq 1)$. Then

$$|a_2|\leq\frac{2\alpha}{\sqrt{1+\alpha}}\qquad\text{and}\qquad |a_3|\leq 2\alpha.$$

3. The Class $\mathcal{N}_\Sigma(\beta,\mu,\gamma)$ and Coefficient Estimates For The Functions In This Class

Definition 7. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma(\beta,\mu,\gamma)$ if the following conditions are satisfied:

$$f\in\Sigma\qquad and\qquad Re\left\{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f)'(z)}{\mathfrak{T}_z^{\mu,\gamma}f(z)}\right\}>\beta\qquad(0\leq\beta<1, z\in\mathbb{U})\tag{3.1}$$

and

$$Re\left\{\frac{w(\mathfrak{T}_w^{\mu,\gamma}g)'(w)}{\mathfrak{T}_w^{\mu,\gamma}g(w)}\right\}>\beta\qquad(0\leq\beta<1, w\in\mathbb{U})\tag{3.2}$$

where the function g is inverse of the function f given by (1.2),
For $\gamma=\mu$, the class of $\mathcal{N}_\Sigma(\beta,\mu,\gamma)$ is reduced to $S^*_\Sigma(\beta)$ of bi-starlike of order β $(0\leq\beta<1)$, which is introduced by Brannan and Taha [3].

Theorem 2. If $f(z)$ given by (1.1) in the class $\mathcal{N}_\Sigma(\beta,\mu,\gamma)$, then

$$|a_2|\leq\sqrt{\frac{2\lambda(1-\beta)(\gamma+1)}{\mu+1}}\tag{3.3}$$

and

$$|a_3|\leq\frac{2(1-\beta)\lambda(\gamma+1)}{\mu+1}\tag{3.4}$$

where

$$\lambda=\frac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}.$$

Proof. We can write the inequalities in (3.1) and (3.2) as in the following:

$$\frac{z(\mathfrak{T}_z^{\mu,\gamma}f)'(z)}{\mathfrak{T}_z^{\mu,\gamma}f(z)}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$\frac{w(\mathfrak{T}_w^{\mu,\gamma}g)'(w)}{\mathfrak{T}_w^{\mu,\gamma}g(w)}=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8), respectively. Like the proof of Theorem 1, by equating coefficients of (3.5) and (3.6) yields,

$$\frac{\mu+1}{\gamma+1}a_2=(1-\beta)p_1, \tag{3.7}$$

$$\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3-\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)p_2, \tag{3.8}$$

$$-\frac{\mu+1}{\gamma+1}a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$\left[\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}-\left(\frac{\mu+1}{\gamma+1}\right)^2\right]a_2^2-\frac{2(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}a_3=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_1=-q_1 \tag{3.11}$$

and

$$2\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also from (3.8) and (3.10) we obtain

$$\frac{2(\mu+1)}{\lambda(\gamma+1)}a_2^2=(1-\beta)(p_2+q_2), \tag{3.13}$$

where $\lambda=\frac{(\gamma+1)(\gamma+2)}{\gamma(\mu+3)+2}$. Thus, clearly we have

$$|a_2|^2\leq\frac{\lambda(\gamma+1)}{2(\mu+1)}(1-\beta)\left(|p_2|+|q_2|\right). \tag{3.14}$$

If we apply the Carathéodory Lemma to coefficients of p_2, q_2 we find the upper bound on $|a_2|$ as given in (3.3).
For the purpose of to find the bound on $|a_3|$, we multiply $\frac{4(\mu+1)(\mu+2)}{(\gamma+1)(\gamma+2)}-\frac{(\mu+1)^2}{(\gamma+1)^2}$ and $\frac{(\mu+1)^2}{(\gamma+1)^2}$ to the equations (3.8) and (3.10) respectively and on adding them we obtain:

$$\frac{4(\mu+1)(\mu+2)}{\lambda}a_3=(1-\beta)\left\{(3\mu\gamma+2\mu+7\gamma+6)p_2+(\mu+1)(\gamma+2)q_2\right\}. \tag{3.15}$$

Now, let's take absolute value of the both side of (3.15). After then, if we apply the Carathéodory Lemma to coefficients of p_2, q_2 we find

$$|a_3|\leq\frac{2(1-\beta)\lambda(\gamma+1)}{\mu+1},$$

which is asserted in (3.4). □

In the Theorem 2, if we choose $\gamma=\mu$, we obtain:

Corollary 2. [3] If $f(z)$ given by (1.1) belongs to $S_\Sigma^*(\beta)$ $(0\leq\beta<1)$, then

$$|a_2|\leq\sqrt{2(1-\beta)}\qquad\text{and}\qquad|a_3|\leq 2(1-\beta).$$

4. Conclusion

In our present paper, we introduce new subclasses $\mathcal{N}_{\Sigma}(\alpha,\mu,\gamma)$ and $\mathcal{N}_{\Sigma}(\beta,\mu,\gamma)$ of strongly bi-starlike and bi-starlike functions using Tremblay fractional derivative operator. Furthermore we obtained upper bounds for $|a_2|$ and $|a_3|$ for the functions in these classes. Unlike recent studies about bi-univalent functions, we have used Brannan and Taha's technique for obtain the upper bounds on the coefficients $|a_3|$. For $\gamma=\mu$, we can concluded the results which were given by Brannan and Taha [3]. In fact, our Theorems generalizes the results given in [3].

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