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Certain Class of Analytic Functions Involving Salagean Type q-Difference Operator

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Abstract

In this paper, we define a new subclass of analytic functions with negative coefficients involving Salagean type $q-$ difference operator and discuss certain characteristic properties and inclusion relations involving $N_\delta(e)$ of this generalized function class. Further, we determine partial sums results for the function class. The usefulness of the main result not only provide the unification of the results discussed in the literature but also generate certain new results.

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1. Introduction and Preliminaries

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{U}=\{z\in\mathbb{C}:\ |z|<1\}$. We also denote $\mathscr{T}$ a subclass of $\mathscr{A}$ introduced and studied by Silverman [17], consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\ a_n>0;z\in\mathbb{U}. \tag{1.2}$$

For functions $f\in\mathscr{A}$ given by (1.1) and $g\in\mathscr{A}$ given by $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$, we define the Hadamard product (or Convolution) of f and g by $(f*g)(z)=z+\sum\limits_{n=2}^{\infty}a_nb_nz^n,\ z\in\mathbb{U}.$

We briefly recall here the notion of *q-operators* i.e. *q-difference operator* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [8] and Kanas and Răducanu [9] have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$. For details on *q-calculus* one can refer [1, 2, 3, 4, 8, 9, 10, 11, 13, 21, 22] and also the reference cited therein. For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0<q<1$.

For $0<q<1$ the Jackson's *q-derivative* of a function $f\in\mathscr{A}$ is, by definition, given as follows [8]

$$D_qf(z)=\begin{cases}\dfrac{f(z)-f(qz)}{(1-q)z} & for\quad z\neq 0,\\ f'(0) & for\quad z=0,\end{cases} \tag{1.3}$$

and

$$D_q^2f(z)=D_q(D_qf(z)).$$

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From (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n] a_n z^{n-1} \tag{1.4}$$

where

$$[n] = [n]_q = \frac{1-q^n}{1-q}, \tag{1.5}$$

is sometimes called *the basic number n*. If $q \to 1^-$, $[n] \to n$.
For a function $h(z) = z^n$, we obtain $D_q h(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n] z^{n-1}$, and $\lim_{q\to 1^-} D_q h(z) = \lim_{q\to 1^-} \left([n] z^{n-1}\right) = n z^{n-1} = h'(z)$, where h' is the ordinary derivative. Recently for $f \in \mathscr{A}$, Govindaraj and Sivasubramanian [11] defined and discussed the Salagean *q-differential operator* as given below:

$$\begin{aligned} \mathscr{D}_q^0 f(z) &= f(z) \\ \mathscr{D}_q^1 f(z) &= z\mathscr{D}_q f(z) \\ \mathscr{D}_q^m f(z) &= z\mathscr{D}_q(\mathscr{D}_q^{m-1} f(z)) \\ \mathscr{D}_q^m f(z) &= z + \sum_{n=2}^{\infty} [n]^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U}). \end{aligned} \tag{1.6}$$

We note that $\lim_{q\to 1^-}$

$$D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U}) \tag{1.7}$$

the familiar Salagean derivative[16].
In this paper, we define a new subclass of analytic functions with negative coefficients involving Salagean type $q-$ difference operator and discuss certain characteristic properties of this generalized function class and inclusion relations involving $N_\delta(e)$ for the function class. Further by letting $f_k(z) = z + \sum\limits_{n=2}^{k} a_n z^n$ be the sequence of partial sums of the analytic function $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ we determine new sharp lower bounds $\Re\left(\frac{f(z)}{f_k(z)}\right)$, $\Re\left(\frac{f_k(z)}{f(z)}\right)$, $\Re\left(\frac{f'(z)}{f'_k(z)}\right)$, $\Re\left(\frac{f'_k(z)}{f'(z)}\right)$. The usefulness of the main result not only provide the unification of the results discussed in the literature but also generate certain new results.
For $0 \le \mu \le 1$, $0 \le \alpha < 1$, $\beta \ge 0$ and $m \in \mathbb{N}_0$, we let $\mathscr{J}_q^m(\mu,\alpha,\beta)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left(\frac{\mathscr{D}_q^{m+1} f(z)}{(1-\mu)z + \mu\mathscr{D}_q^m f(z)} - \alpha\right) > \beta\left|\frac{\mathscr{D}_q^{m+1} f(z)}{(1-\mu)z + \mu\mathscr{D}_q^m f(z)} - 1\right|, \; z \in \mathbb{U}, \tag{1.8}$$

where $\mathscr{D}_q^m f(z)$ is given by (1.6). We further let $\mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta) = \mathscr{J}_q^m(\mu,\alpha,\beta) \cap \mathscr{T}$.

Remark 1.1. *By taking $\mu = 1$, $0 \le \alpha < 1$, $\beta \ge 0$ and $m \in \mathbb{N}_0$, we let $\mathscr{T}\mathscr{J}_q^m(1,\alpha,\beta) \equiv \mathscr{T}\mathscr{S}\mathscr{P}_q^m(\alpha,\beta)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion*

$$\Re\left(\frac{\mathscr{D}_q^{m+1} f(z)}{\mathscr{D}_q^m f(z)} - \alpha\right) > \beta\left|\frac{\mathscr{D}_q^{m+1} f(z)}{\mathscr{D}_q^m f(z)} - 1\right|, \; z \in \mathbb{U}, \tag{1.9}$$

studied by Govindaraj and Sivasubramanian [9] for subordination results in conic domain.

Remark 1.2. *By taking $\mu = 0$, $m \in \mathbb{N}_0$, $0 \le \alpha < 1$ and $\beta \ge 0$, let $\mathscr{T}\mathscr{J}_q^m(0,\alpha,\beta) \equiv \mathscr{U}\mathscr{S}\mathscr{D}_q^m(\alpha,\beta)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion*

$$\Re\left(\frac{\mathscr{D}_q^{m+1} f(z)}{z} - \alpha\right) > \beta\left|\frac{\mathscr{D}_q^{m+1} f(z)}{z} - 1\right|, \quad z \in \mathbb{U},$$

where $\mathscr{D}_q^m f(z)$ is given by (1.6).

Remark 1.3. *For $\mu = 0$, $\beta = 0$, $0 \le \alpha < 1$ and $m \in \mathbb{N}_0$, let $\mathscr{T}\mathscr{J}_q^m(0,\alpha,0) \equiv \mathscr{R}_q^m(\alpha)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion*

$$\Re\left(\frac{\mathscr{D}_q^{m+1} f(z)}{z}\right) > \alpha \quad z \in \mathbb{U},$$

where $\mathscr{D}_q^m f(z)$ is given by (1.6).

Remark 1.4. *As $\lim_{q\to 1^-}$ and suitably specializing the parameter (as mentioned in the above remarks) we can deduce an interesting subclasses of $\mathscr{A}$ denoted by $\mathscr{S}^m(\mu,\alpha,\beta)$, satisfying the analytic criterion*

$$\Re\left(\frac{\mathscr{D}^{m+1} f(z)}{(1-\mu)z + \mu\mathscr{D}^m f(z)} - \alpha\right) > \beta\left|\frac{\mathscr{D}^{m+1} f(z)}{(1-\mu)z + \mu\mathscr{D}^m f(z)} - 1\right|, \; z \in \mathbb{U}, \tag{1.10}$$

where $\mathscr{D}^m f(z)$ is given by (1.7).

2. Basic Properties

In this section we obtain the characterization properties for the classes $\mathcal{T}\mathcal{J}_q^m(\mu,\alpha,\beta)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathcal{J}_q^m(\mu,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n]^m([n](1+\beta)-\mu(\alpha+\beta))\,|a_n|\leq 1-\alpha, \tag{2.1}$$

where $0\leq\mu\leq 1$, $0\leq\alpha<1,\beta\geq 0$ and $m\in\mathbb{N}_0$. The result is sharp for the function

$$f_n(z)=z-\frac{1-\alpha}{[n]^m([n](1+\beta)-\mu(\alpha+\beta))}z^n.$$

Proof. It suffices to show that

$$\beta\left|\frac{\mathcal{D}_q^{m+1}f(z)}{(1-\mu)z+\mu\mathcal{D}_q^m f(z)}-1\right|-\Re\left(\frac{\mathcal{D}_q^{m+1}f(z)}{(1-\mu)z+\mu\mathcal{D}_q^m f(z)}-1\right)$$
$$\leq 1-\alpha.$$

We have

$$\beta\left|\frac{\mathcal{D}_q^{m+1}f(z)}{(1-\mu)z+\mu\mathcal{D}_q^m f(z)}-1\right|-\Re\left(\frac{\mathcal{D}_q^{m+1}f(z)}{(1-\mu)z+\mu\mathcal{D}_q^m f(z)}-1\right)$$
$$\leq(1+\beta)\left|\frac{\mathcal{D}_q^{m+1}f(z)}{(1-\mu)z+\mu\mathcal{D}_q^m f(z)}-1\right|$$
$$\leq\frac{(1+\beta)\sum\limits_{n=2}^{\infty}[n]^m([n]-\mu)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]^m\mu|a_n||z|^{n-1}}.$$

As $|z|\to 1^-$, the last expression is bounded above by $1-\alpha$ if (2.1) holds. It is obvious that the function f_n satisfies the inequality (2.1),and thus $1-\alpha$ cannot be replaced by a larger number. Therefore we need only to prove that $f\in\mathcal{T}\mathcal{J}_q^m(\mu,\alpha,\beta)$. Since

$$\Re\left(\frac{1-\sum\limits_{n=2}^{\infty}[n]^{m+1}\,a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]^m\mu\,a_nz^{n-1}}-\alpha\right)>\beta\left|\frac{\sum\limits_{n=2}^{\infty}[n]^m([n]-\mu)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]^m\mu a_nz^{n-1}}\right|.$$

Letting $z\to 1^-$ along the real axis, we obtain the desired inequality given in (2.1). □

Corollary 2.2. *If $f\in\mathcal{T}\mathcal{J}_q^m(\mu,\alpha,\beta)$, then*

$$|a_n|\leq\frac{1-\alpha}{[n]^m([n](1+\beta)-\mu(\alpha+\beta))}, \tag{2.2}$$

Equality holds for the function $f(z)=z-\frac{1-\alpha}{[n]^m([n](1+\beta)-\mu(\alpha+\beta))}z^n$.

Corollary 2.3. *A function $f(z)$ of the form (1.1) is in $\mathcal{T}\mathcal{S}\mathcal{P}_q^m(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n]^m([n](1+\beta)-(\alpha+\beta))\,|a_n|\leq 1-\alpha, \tag{2.3}$$

where $0\leq\alpha<1$, $\beta\geq 0$ and $m\in\mathbb{N}_0$. The result is sharp for the function

$$f_n(z)=z-\frac{1-\alpha}{[n]^m([n](1+\beta)-(\alpha+\beta))}z^n.$$

Corollary 2.4. *A function $f(z)$ of the form (1.1) is in $\mathcal{U}\mathcal{S}\mathcal{D}_q^m(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n]^{m+1}(1+\beta)\,|a_n|\leq 1-\alpha, \tag{2.4}$$

where $0\leq\alpha<1$, $\beta\geq 0$ and $m\in\mathbb{N}_0$. The result is sharp for the function

$$f_n(z)=z-\frac{1-\alpha}{[n]^{m+1}(1+\beta)}z^n.$$

.

Corollary 2.5. *A function $f(z)$ of the form (1.1) is in $\mathscr{R}_q^m(\alpha)$ if*

$$\sum_{n=2}^{\infty} [n]^{m+1} \, |a_n| \le 1-\alpha, \tag{2.5}$$

where $0 \le \alpha < 1$ and $m \in \mathbb{N}_0$. The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{[n]^{m+1}} z^n.$$

.

Corollary 2.6. *A function $f(z)$ of the form (1.1) is in $\mathscr{S}^m(\mu,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty} n^m (n(1+\beta) - \mu(\alpha+\beta)) \, |a_n| \le 1-\alpha, \tag{2.6}$$

where $0 \le \mu \le 1$, $0 \le \alpha < 1, \beta \ge 0$ and $m \in \mathbb{N}_0$. The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{n^m(n(1+\beta) - \mu(\alpha+\beta))} z^n.$$

Adopting the techniques of Silvia [20] one can prove the results on distortion bounds, extreme points, radii of convexity and starlikeness for functions $f(z) \in \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$ we skip the details.

3. Inclusion relations involving $N_\delta(e)$.

In this section following [7, 12, 15], we define the n, δ neighborhood of function $f(z) \in \mathscr{T}$ and discuss the inclusion relations involving $N_\delta(e)$.

$$N_\delta(f) = \left\{ g \in \mathscr{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \le \delta \right\}. \tag{3.1}$$

Particularly for the identity function $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in \mathscr{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \le \delta \right\}. \tag{3.2}$$

Theorem 3.1. *Let*

$$\delta = \frac{1-\alpha}{[2]^m([2](1+\beta) - \mu(\alpha+\beta))}. \tag{3.3}$$

Then $\mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta) \subset N_\delta(e)$.

Proof. For $f \in \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$, Theorem 2.1, yields

$$[2]^m([2](1+\beta) - \mu(\alpha+\beta)) \sum_{n=2}^{\infty} a_n \le 1-\alpha$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\alpha}{[2]^m([2](1+\beta) - \mu(\alpha+\beta))}. \tag{3.4}$$

On the other hand, from (2.1) and (3.4) we have

$$\begin{aligned}
[2]^m(1+\beta) \sum_{n=2}^{\infty} [n] a_n &\le 1-\alpha + [2]^m \mu(\alpha+\beta) \sum_{n=2}^{\infty} a_n \\
&\le 1-\alpha + \frac{[2]^m \mu(\alpha+\beta)(1-\alpha)}{[2]^m([2](1+\beta) - \mu(\alpha+\beta))} \\
&\le \frac{[2]^m(1+\beta)(1-\alpha)}{[2](1+\beta) - \mu(\alpha+\beta)}, \\
\sum_{n=2}^{\infty} [n] a_n &\le \frac{1-\alpha}{[2](1+\beta) - \mu(\alpha+\beta)}. \qquad (3.5)
\end{aligned}$$

□

Now we determine the neighborhood for each of the class $\mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$ which we define as follows.
A function $f \in \mathscr{T}$ is said to be in the class $\mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$ if there exists a function $g \in \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta,\eta)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\eta, \ (z\in\mathbb{U}, \ \ 0\le\eta<1). \tag{3.6}$$

Theorem 3.2. *If $g \in \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$ and*

$$\eta = 1-\frac{\delta[2]^m([2](1+\beta)-\mu(\alpha+\beta))}{2\left[[2]^m([2](1+\beta)-\mu(\alpha+\beta))-(1-\alpha)\right]}. \tag{3.7}$$

Then $N_\delta(g)\subset \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta,\eta)$.

Proof. Suppose that $f\in N_\delta(g)$ then we find from (3.5) that

$$\sum_{n=2}^{\infty} n|a_n-b_n| \le \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n-b_n| \le \frac{\delta}{2}.$$

Next, since $g \in \mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta)$, we have

$$\sum_{n=2}^{\infty} b_n \le \frac{1-\alpha}{[2]^m([2](1+\beta)-\mu(\alpha+\beta))}.$$

So that

$$\begin{aligned}
\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n}\\
&\le \frac{\delta}{2}\times\frac{[2]^m([2](1+\beta)-\mu(\alpha+\beta))}{[2]^m([2](1+\beta)-\mu(\alpha+\beta))-(1-\alpha)}\\
&\le 1-\eta.
\end{aligned}$$

provided that η is given precisely by (3.7). Thus by definition, $f\in\mathscr{T}\mathscr{J}_q^m(\mu,\alpha,\beta,\eta)$ for η given by (3.7), which completes the proof. □

4. Partial Sums

Silverman [18] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. In this section following the earlier work by Silverman [18] and also the works cited in [5, 6, 14, 19] on partial sums of analytic functions, we study the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_k(z) = z+\sum_{n=2}^{k} a_n z^n$$

when the coefficients of $f(z)$ are satisfy the condition (2.1).
Throughout this section for our convenience, unless otherwise stated, we let

$$\Phi_n^m = \Phi^m(\mu,\alpha,\beta) = [n]^m([n](1+\beta)-\mu(\alpha+\beta)). \tag{4.1}$$

where $0\le\mu\le1$, $0\le\alpha<1, \beta\ge0$ and $m\in\mathbb{N}_0$.

Theorem 4.1. *If $f\in\mathscr{A}$ of the form (1.1) satisfies the condition (2.1), then*

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \ge \frac{\Phi_{k+1}^m-1+\alpha}{\Phi_{k+1}^m} \qquad (z\in\mathbb{U}) \tag{4.2}$$

where

$$\Phi_k^m = \Phi^m(\mu,\alpha,\beta) \ge \begin{cases} 1-\alpha, & if \ \ n=2,3,\ldots,k \\ \Phi_{k+1}^m, & if \ \ n=k+1,k+2,\ldots. \end{cases} \tag{4.3}$$

The result (4.2) is sharp with the function given by

$$f(z) = z+\frac{1-\alpha}{\Phi_{k+1}^m}z^{k+1}. \tag{4.4}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{\Phi_{k+1}^{m}}{1-\alpha}\left[\frac{f(z)}{f_k(z)}-\frac{\Phi_{k+1}^{m}-1+\alpha}{\Phi_{k+1}^{m}}\right]$$

$$=\frac{1+\sum\limits_{n=2}^{k}a_nz^{n-1}+\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum\limits_{n=2}^{k}a_nz^{n-1}}. \tag{4.5}$$

It suffices to show that $|w(z)|\le 1$. Now, from (4.5) we can write

$$w(z)=\frac{\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}{2+2\sum\limits_{n=2}^{k}a_nz^{n-1}+\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}.$$

Hence we obtain

$$|w(z)|\le\frac{\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n|-\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}.$$

Now $|w(z)|\le 1$ if and only if

$$2\left(\frac{\Phi_{k+1}^{m}}{1-\alpha}\right)\sum_{n=k+1}^{\infty}|a_n|\le 2-2\sum_{n=2}^{k}|a_n|$$

or, equivalently,

$$\sum_{n=2}^{k}|a_n|+\sum_{n=k+1}^{\infty}\frac{\Phi_{k+1}^{m}}{1-\alpha}|a_n|\le 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=2}^{k}|a_n|+\sum_{n=k+1}^{\infty}\frac{\Phi_{k+1}^{m}}{1-\alpha}|a_n|\le\sum_{n=2}^{\infty}\frac{\Phi_{n}^{m}}{1-\alpha}|a_n|$$

which is equivalent to

$$\sum_{n=2}^{k}\left(\frac{\Phi_{n}^{m}-1+\alpha}{1-\alpha}\right)|a_n|+\sum_{n=k+1}^{\infty}\left(\frac{\Phi_{n}^{m}-\Phi_{k+1}^{m}}{1-\alpha}\right)|a_n|\ge 0. \tag{4.6}$$

To see that the function given by (4.4) gives the sharp result, we observe that for $z=re^{i\pi/n}$

$$\frac{f(z)}{f_k(z)}\;=\;1+\frac{1-\alpha}{\Phi_{k+1}^{m}}z^n\to 1-\frac{1-\alpha}{\Phi_{k+1}^{m}}=\frac{\Phi_{k+1}^{m}-1+\alpha}{\Phi_{k+1}^{m}}\quad\text{when } r\to 1^-.$$

□

We next determine bounds for $f_k(z)/f(z)$.

Theorem 4.2. *If $f\in\mathscr{A}$ of the form (1.1) satisfies the condition (2.1), then*

$$\Re\left(\frac{f_k(z)}{f(z)}\right)\ge\frac{\Phi_{k+1}^{m}}{\Phi_{k+1}^{m}+1-\alpha}\qquad(z\in\mathbb{U}), \tag{4.7}$$

where $\Phi_{k+1}^{m}\ge 1-\alpha$ and

$$\Phi_{n}^{m}\ge\begin{cases}1-\alpha, & if\;\; n=2,3,\dots,k\\ \Phi_{k+1}^{m}, & if\;\; n=k+1,k+2,\dots.\end{cases} \tag{4.8}$$

The result (4.7) is sharp with the function given by (4.4).

Proof. The proof follows by defining

$$\frac{1+w(z)}{1-w(z)}=\frac{\Phi_{k+1}^m+1-\alpha}{1-\alpha}\left[\frac{f_k(z)}{f(z)}-\frac{\Phi_{k+1}^m}{\Phi_{k+1}^m+1-\alpha}\right]$$

and much akin to similar arguments in Theorem 4.1. □

We next turns to ratios involving derivatives.

Theorem 4.3. *If $f\in\mathscr{A}$ of the form (1.1) satisfies the condition (2.1), then*

$$\Re\left(\frac{f'(z)}{f_k'(z)}\right)\geq\frac{\Phi_{k+1}^m-(k+1)(1-\alpha)}{\Phi_{k+1}^m}\qquad(z\in\mathbb{U})\tag{4.9}$$

and

$$\Re\left(\frac{f_k'(z)}{f'(z)}\right)\geq\frac{\Phi_{k+1}^m}{\Phi_{k+1}^m+(k+1)(1-\alpha)}\qquad(z\in\mathbb{U})\tag{4.10}$$

where $\Phi_{k+1}^m\geq(k+1)(1-\alpha)$ and

$$\Phi_n^m\geq\begin{cases}n(1-\alpha), & if\ \ n=2,3,\ldots,k\\ n\left(\frac{\Phi_{k+1}^m}{k+1}\right), & if\ \ n=k+1,k+2,\ldots.\end{cases}\tag{4.11}$$

The results are sharp with the function given by (4.4).

Proof. We write

$$\frac{1+w(z)}{1-w(z)}=\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)}\left[\frac{f'(z)}{f_k'(z)}-\left(\frac{\Phi_{k+1}^m-(k+1)(1-\alpha)}{\Phi_{k+1}^m}\right)\right]$$

where

$$w(z)=\frac{\left(\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}{2+2\sum\limits_{n=2}^{k}na_nz^{n-1}+\left(\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)}\right)\sum\limits_{n=k+1}^{\infty}na_nz^{n-1}}.$$

Now $|w(z)|\leq 1$ if and only if

$$\sum_{n=2}^{k}n|a_n|+\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)}\sum_{n=k+1}^{\infty}n|a_n|\leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=2}^{k}n|a_n|+\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)}\sum_{n=k+1}^{\infty}n|a_n|\leq\sum_{n=2}^{\infty}\frac{\Phi_n^m}{1-\alpha}|a_n|$$

which is equivalent to

$$\sum_{n=2}^{k}\left(\frac{\Phi_n^m-n(1-\alpha)}{1-\alpha}\right)|a_n|+\sum_{n=k+1}^{\infty}\frac{(k+1)\Phi_n^m-n\Phi_{k+1}^m}{(k+1)(1-\alpha)}|a_n|\geq 0.$$

To prove the result (4.10), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{(k+1)(1-\alpha)+\Phi_{k+1}^m}{(1-\alpha)(k+1)}\left[\frac{f_k'(z)}{f'(z)}-\frac{\Phi_{k+1}^m}{(k+1)(1-\alpha)+\Phi_{k+1}^m}\right]$$

and by similar arguments in first part we get desired result. □

Concluding Remark: As a special cases of the above theorems on partial sums , we can determine new sharp lower bounds for $\Re\left(\frac{f(z)}{f_k(z)}\right)$, $\Re\left(\frac{f_k(z)}{f(z)}\right)$, $\Re\left(\frac{f'(z)}{f_k'(z)}\right)$, $\Re\left(\frac{f_k'(z)}{f'(z)}\right)$ and also the inclusion relations involving $N_\delta(e)$ for various function classes stated in Remark 1.1-1.4.

References

[1] Ş. Altınkaya, and S. Yalçın, Faber polynomial coefficient estimates for a class of bi-univalent functions based on the symmetric q-derivative operator, Journal of Fractional Calculus and Applications, Vol:8, No:2 (2017), 79-87.
[2] Ş. Altınkaya, and S. Yalçın, On the Fekete-Szegö problem for analytic functions defined by using symmetric q-derivative operator, Konuralp Journal of Mathematics, Vol:5, No:1 (2017), 176-186.
[3] S. Araci, U. Duran, M. Acikgoz and H. M. Srivastava, A certain (p, q)-derivative operator and associated divided differences, J. Inequal. Appl., (2016) 2016:301.
[4] Aral, A., Gupta, V. and Agarwal, R. P., *Applications of q-calculus in operator theory*, Springer, New York, 2013.
[5] B.A. Frasin, Partial sums of certain analytic and univalent functions, Acta Math. Acad. Paed. Nyir. Vol:21 (2005), 135-145.
[6] B.A. Frasin and G.Murugusundaramoorthy Partial sums of certain analytic functions, Mathematica,Tome 53, Vol:76, No:2,(2011), 131-142.
[7] A.W.Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., Vol:8 (1957), 598-601.
[8] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, Vol:46 (1908), 253-281.
[9] S. Kanas and D. Răducanu, Some subclass of analytic functions related to conic domains, Math. Slovaca, Vol:64, No:5 (2014), 1183-1196.
[10] Z. Karahuseyin, Ş. Altınkaya and S. Yalçın, On $H_3(1)$ Hankel determinant for univalent functions defined by using $q-derivative$ operator, Transylv. J. Math. Mech., Vol:9, No:1 (2017), 25-33.
[11] M.Govindaraj, and S. Sivasubramanian, On a class of analytic function related to conic domains involving $q-$calculus, Analysis Math., Vol:43, No:5 (2017), 475-487.
[12] G.Murugusundaramoorthy and H. M. Srivastava, Neighborhoods of certain classes of analytic functions of complex order, Journal of Ineql. In Pure and Appl. Maths., Vol:5, No:2, Art.24 (2004) 1-8.
[13] S. D. Purohit, and R. K. Raina, Fractional q-calculus and certain subclasses of univalent analytic functions, Mathematica Vol:55(78), No:1 (2013), 62-74.
[14] T. Rosy, K.G. Subramanian and G. Murugusundaramoorthy, Neighborhoods and partial sums of starlike based on Ruscheweyeh derivatives, J. Ineq. Pure Appl. Math., Vol:4, No:4, Art. 64 (2003), 1-8.
[15] S. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc., Vol:81 (1981), 521-527.
[16] Salagean, G. S., *Subclasses of univalent functions,* Lecture Notes in Math., Springer-Verlag, 1013 (1983), 362-372.
[17] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., Vol:51 (1975), 109-116.
[18] H. Silverman, Partial sums of starlike and convex functions, J.Math Anal.&.Appl., Vol:209 (1997), 221–227.
[19] T. Sheill-Small, A note on partial sums of convex schlicht functions, Bull. London Math. Soc., Vol:2 (1970), 165-168.
[20] E.M. Silvia., Partial sums of convex functions of order α, Houston.J.Math., Vol:11, No:3 (1985), 397-404.
[21] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inf. Sci. Vol:5, No:3 (2011), 390-444.
[22] K.Vijaya, Certain class of analytic functions based on q-difference operator (communicated)