

Research Article

## Several integral inequalities for strongly $(\alpha, m)$ -convex functions with modulus

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**ABSTRACT.** In the paper, the authors introduce the concept of strongly  $(\alpha, m)$ -convex functions with modulus, establish several new integral identities, and, by virtue of these newly-established identities and Hölder's inequality, present some new integral inequalities of the Hermite–Hadamard type for strongly  $(\alpha, m)$ -convex functions with modulus.

**Keywords:** Hermite–Hadamard's inequality, strongly  $(\alpha, m)$ -convex function, Hölder inequality.

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### 1. INTRODUCTION

The following definitions are well known in the literature:

**Definition 1.1.** A function  $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$  is said to be convex on the interval  $I$  if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ .

**Definition 1.2 ([10]).** A function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $m$ -convex if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for all  $x, y \in [0, b]$ ,  $t \in [0, 1]$ , and  $m \in (0, 1]$ , where  $b > 0$ .

**Definition 1.3 ([7]).** A function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $(\alpha, m)$ -convex if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

is valid for all  $x, y \in [0, b]$ ,  $t \in [0, 1]$ , and  $(\alpha, m) \in (0, 1]^2$ , where  $b > 0$ .

**Definition 1.4 ([9]).** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be strongly convex with modulus  $c > 0$ , if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

is valid for all  $x, y \in [a, b]$  and  $t \in [0, 1]$ , where  $b > a$ .

In recent decades, numerous inequalities of the Hermite–Hadamard type for various kinds of convex functions have been established. Some of them may be recited as follows:

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**Theorem 1.1** ([3, Theorem 2.2]). Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $a, b \in I^\circ$  with  $a < b$ , where  $I^\circ$  denotes the interior of the interval  $I$ . If  $|f'|$  is convex on  $[a, b]$ , then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

**Theorem 1.2** ([2, p. 48, Theorem 2] and [5]). Let  $f : \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}$  be  $m$ -convex and  $m \in (0, 1]$ . If  $f \in L_1([a, b])$  for  $0 \leq a < b < \infty$ , then

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2} \right\}.$$

**Theorem 1.3** ([1, Theorem 3.1]). Let  $I \supseteq \mathbb{R}_0 = [0, \infty)$  be an open real interval and let  $f : I \rightarrow \mathbb{R}$  be a differentiable function on  $I$  such that  $f' \in L_1([a, b])$  for  $0 \leq a < b < \infty$ . If  $|f'|^q$  is  $(\alpha, m)$ -convex on  $[a, b]$  for some given numbers  $m, \alpha \in (0, 1]$  and  $q \geq 1$ , then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left( \frac{1}{2} \right)^{1-1/q} \min \left\{ \left[ v_1 |f'(a)|^q + v_2 m \left| f' \left( \frac{b}{m} \right) \right|^q \right]^{1/q}, \left[ v_2 m \left| f' \left( \frac{a}{m} \right) \right|^q + v_1 |f'(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)} \left( \alpha + \frac{1}{2^\alpha} \right)$$

and

$$v_2 = \frac{1}{(\alpha+1)(\alpha+2)} \left( \frac{\alpha^2 + \alpha + 2}{2} - \frac{1}{2^\alpha} \right).$$

**Theorem 1.4** ([6, Theorems 2.2 and 2.4]). Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}_0$  be differentiable on  $I^\circ$  and  $g : [a, b] \rightarrow \mathbb{R}_0$  be continuous and symmetric with respect to  $\frac{a+b}{2}$  for  $a, b \in I$  with  $a < b$ . Denote

$$L(t) = \frac{1+t}{2}a + \frac{1-t}{2}b \quad \text{and} \quad U(t) = \frac{1-t}{2}a + \frac{1+t}{2}b$$

for  $t \in [0, 1]$ .

(1) If  $|f'|$  is convex on  $[a, b]$ , then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x)g(x) dx \right| \\ & \leq \frac{b-a}{4} [|f'(a)| + |f'(b)|] \int_0^1 \int_{L(t)}^{U(t)} g(x) dx dt. \end{aligned}$$

(2) If  $|f'|^q$  is convex on  $[a, b]$  for  $q \geq 1$ , then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x)g(x) dx \right| \\ & \leq \frac{b-a}{2} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q} \int_0^1 \int_{L(t)}^{U(t)} g(x) dx dt. \end{aligned}$$

In [11], the concept of  $(\alpha, m)$ -geometric arithmetic- $F$ -convex function is introduced and several Hermite–Hadamard type integral inequalities about  $(\alpha, m)$ -GA- $F$ -convex function are established.

In recent years, some other kinds of the Hermite–Hadamard type inequalities were created in, for example, [4, 8, 12, 13] and related references therein.

In this paper, we will introduce the concept of strongly  $(\alpha, m)$ -convex functions with modulus, establish several new integral identities, and, by these newly-established identities and Hölder's inequality, present some new integral inequalities of the Hermite–Hadamard type for strongly  $(\alpha, m)$ -convex functions with modulus.

## 2. DEFINITION AND INTEGRAL IDENTITIES

We now introduce the concept of strongly  $(\alpha, m)$ -convex functions.

**Definition 2.5.** For  $(\alpha, m) \in (0, 1]^2$ , a function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be strongly  $(\alpha, m)$ -convex with a modulus  $c > 0$ , if

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y) - ct(1 - t)(x - y)^2$$

is valid for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ .

**Example 2.1.** Let  $f(x) = x^2$  for  $x \in \mathbb{R}_+$  and take  $\alpha = m = \frac{1}{2}$  and  $0 < c \leq \frac{1}{10}$ . Then the function  $f(x) = x^2$  is strongly  $(\frac{1}{2}, \frac{1}{2})$ -convex on  $\mathbb{R}_+$  with the modulus  $0 < c \leq \frac{1}{10}$ .

*Proof.* For all  $x, y \in \mathbb{R}_+$  and  $t \in [0, 1]$ , putting  $u = \frac{x}{y}$  yields

$$\begin{aligned} & t^\alpha f(x) + m(1 - t^\alpha)f(y) - ct(1 - t)(x - y)^2 - f(tx + m(1 - t)y) \\ &= t^\alpha x^2 + m(1 - t^\alpha)y^2 - ct(1 - t)(x - y)^2 - (tx + m(1 - t)y)^2 \\ &\geq y^2 \left[ t^{1/2}u^2 + \frac{1 - t^{1/2}}{2} - \frac{t(1 - t)}{10}(u - 1)^2 - \left( tu + \frac{1 - t}{2} \right)^2 \right] \\ &= \frac{y^2(1 - t^{1/2})}{20} \{ 2(10 + 9t^{1/2} + 9t)t^{1/2}u^2 - 16t(1 + t^{1/2})u + 5(1 - t^{1/2}) + 3t(1 + t^{1/2}) \}. \end{aligned}$$

It is clear that

$$10 + 9t^{1/2} + 9t > 0 \quad \text{and} \quad 5(1 - t^{1/2}) + 3t(1 + t^{1/2}) > 0$$

for all  $t \in [0, 1]$ . The discriminant  $\Delta = B^2 - 4AC$  of the quadratic function  $Au^2 + Bu + C = 0$  with respect to  $u$  in the above braces is

$$\begin{aligned} & [-16t(1 + t^{1/2})]^2 - 4 \times 2(10 + 9t^{1/2} + 9t)t^{1/2}[5(1 - t^{1/2}) + 3t(1 + t^{1/2})] \\ &= 8w(32w^5 + 37w^4 + 23w^3 - 57w^2 - 25w - 50), \quad w = t^{1/2} \\ &\leq 8w[(32 + 37 + 23)w^2 - 57w^2 - 25w - 50] \\ &= 8w(35w^2 - 25w - 50) \\ &\leq 8w(35w - 25w - 50) \\ &= 8w(10w - 50) \\ &\leq -320w \\ &\leq 0 \end{aligned}$$

for  $t \in [0, 1]$ . As a result, making use of the criterion for extreme values of quadratic functions, we derive that

$$t^\alpha f(x) + m(1 - t^\alpha)f(y) - ct(1 - t)(x - y)^2 - f(tx + m(1 - t)y) \geq 0$$

for  $x, y \in \mathbb{R}_+$ ,  $t \in [0, 1]$ ,  $\alpha = m = \frac{1}{2}$ , and  $0 < c \leq \frac{1}{10}$ . Consequently, the function  $f(x) = x^2$  is strongly  $(\frac{1}{2}, \frac{1}{2})$ -convex on  $\mathbb{R}_+$  with the modulus  $0 < c \leq \frac{1}{10}$ .  $\square$

For establishing new integral inequalities of the Hermite–Hadamard type involving strongly  $(\alpha, m)$ -convex functions, we need the following integral identities.

**Lemma 2.1.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function on  $I$  and  $g : [a, b] \rightarrow \mathbb{R}$  be a continuously differentiable function on  $I^\circ$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $f'' \in L_1([a, b])$ , then

$$\begin{aligned}
 \Theta(a, b; g) &\triangleq \frac{f(a) + f(b)}{2(b-a)} \int_a^b g(x) dx + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
 &+ \frac{b-a}{8} \int_0^1 (1-t)[f(u(t)) + f(v(t))][g'(v(t)) - g'(u(t))] dt \\
 &- \frac{1}{2} \int_0^1 [f(u(t)) + f(v(t))][g(v(t)) + g(u(t))] dt \\
 &= \frac{b-a}{8} \int_0^1 (1-t)[f''(u(t)) + f''(v(t))] \left[ \int_{u(t)}^{v(t)} g(x) dx \right] dt,
 \end{aligned}
 \tag{2.1}$$

where

$$u(t) = ta + (1-t)\frac{a+b}{2} \quad \text{and} \quad v(t) = (1-t)\frac{a+b}{2} + tb, \quad t \in [0, 1].
 \tag{2.2}$$

*Proof.* By integration by parts, we have

$$\begin{aligned}
 &\int_0^1 (1-t) \left[ \int_{u(t)}^{v(t)} g(x) dx \right] f''(u(t)) dt \\
 &= \int_0^1 (1-t)[g(v(t)) + g(u(t))] f'(u(t)) dt - \frac{2}{b-a} \int_0^1 \left[ \int_{u(t)}^{v(t)} g(x) dx \right] f'(u(t)) dt \\
 &= \frac{4}{b-a} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_0^1 (1-t)f(u(t))[g'(v(t)) - g'(u(t))] dt \\
 &- \frac{4}{b-a} \int_0^1 f(u(t))[g(v(t)) + g(u(t))] dt + \frac{4}{(b-a)^2} f(a) \left[ \int_a^b g(x) dx \right].
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 &\int_0^1 (1-t) \left( \int_{u(t)}^{v(t)} g(x) dx \right) f''(v(t)) dt \\
 &= \frac{4}{b-a} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_0^1 (1-t)f(v(t))[g'(v(t)) - g'(u(t))] dt \\
 &- \frac{4}{b-a} \int_0^1 f(v(t))[g(v(t)) + g(u(t))] dt + \frac{4}{(b-a)^2} f(b) \left( \int_a^b g(x) dx \right).
 \end{aligned}$$

Lemma 2.1 is thus proved. □

**Lemma 2.2.** Under conditions of Lemma 2.1, if  $g(x) = 1$  for  $x \in [a, b]$ , then

$$\begin{aligned}
 &\frac{1}{4} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{(b-a)^2}{16} \int_0^1 t(1-t)[f''(u(t)) + f''(v(t))] dt,
 \end{aligned}$$

where  $u(t)$  and  $v(t)$  are defined as in (2.2).

### 3. SOME INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

Now we are in position to establish new integral inequalities of the Hermite–Hadamard type involving strongly  $(\alpha, m)$ -convex functions.

**Theorem 3.5.** For  $(\alpha, m) \in (0, 1]^2$ , let  $f : [0, b^*] \rightarrow \mathbb{R}$  be twice differentiable on  $[0, b^*]$  for  $b^* > 0$  and  $g : [a, b] \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}_0$  be continuous on  $[a, b]$  for  $a < b$  with  $\frac{b}{m} \leq b^*$ . If  $|f''|^q$  is strongly  $(\alpha, m)$ -convex on  $[0, \frac{b}{m}]$  for  $q \geq 1$  and  $f'' \in L_1([0, \frac{b}{m}])$ , then

$$(3.3) \quad |\Theta(a, b; g)| \leq \frac{(b-a)^2 \|g\|_\infty}{48} \times \left\{ \left[ \frac{6|f''(a)|^q}{(\alpha+2)(\alpha+3)} + \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{(\alpha+2)(\alpha+3)} - \frac{c[b-(2m-1)a]^2}{20m^2} \right]^{1/q} + \left[ \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{(\alpha+2)(\alpha+3)} + \frac{6|f''(b)|^q}{(\alpha+2)(\alpha+3)} - \frac{c[(2m-1)b-a]^2}{20m^2} \right]^{1/q} \right\},$$

where  $\Theta(a, b; g)$  is defined as in (2.1) and  $\|g\|_\infty = \sup_{t \in [a, b]} g(t)$ .

*Proof.* Using Lemma 2.1 and Hölder’s inequality, we acquire

$$(3.4) \quad \begin{aligned} |\Theta(a, b; g)| &\leq \frac{b-a}{8} \int_0^1 (1-t) [|f''(u(t))| + |f''(v(t))|] \left[ \int_{u(t)}^{v(t)} g(x) dx \right] dt \\ &\leq \frac{(b-a)^2 \|g\|_\infty}{8} \int_0^1 t(1-t) [|f''(u(t))| + |f''(v(t))|] dt \\ &\leq \frac{(b-a)^2 \|g\|_\infty}{8} \left[ \int_0^1 t(1-t) dt \right]^{1-1/q} \left[ \left( \int_0^1 t(1-t) |f''(u(t))|^q dt \right)^{1/q} \right. \\ &\quad \left. + \left( \int_0^1 t(1-t) |f''(v(t))|^q dt \right)^{1/q} \right]. \end{aligned}$$

Utilizing the strongly  $(\alpha, m)$ -convexity of  $|f''|^q$  on  $[0, \frac{b}{m}]$ , we have

$$(3.5) \quad \begin{aligned} &\int_0^1 t(1-t) |f''(u(t))|^q dt \\ &\leq \int_0^1 t(1-t) \left[ t^\alpha |f''(a)|^q + m(1-t^\alpha) \left| f''\left(\frac{a+b}{2m}\right) \right|^q - c \left( \frac{b-(2m-1)a}{2m} \right)^2 t(1-t) \right] dt \\ &= \frac{|f''(a)|^q}{(\alpha+2)(\alpha+3)} + \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{6(\alpha+2)(\alpha+3)} - \frac{c[b-(2m-1)a]^2}{120m^2} \end{aligned}$$

and

$$(3.6) \quad \int_0^1 t(1-t) |f''(v(t))|^q dt \leq \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{6(\alpha+2)(\alpha+3)} + \frac{|f''(b)|^q}{(\alpha+2)(\alpha+3)} - \frac{c[(2m-1)b-a]^2}{120m^2}.$$

Substituting the inequalities (3.5) and (3.6) into the inequality (3.4) leads to the inequality (3.3). Theorem 3.5 is proved.  $\square$

**Corollary 3.1.** Under conditions of Theorem 3.5, if  $q = 1$ , then

$$|\Theta(a, b; g)| \leq \frac{(b-a)^2 \|g\|_\infty}{24} \left[ \frac{3|f''(a)|}{(\alpha+2)(\alpha+3)} + \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|}{(\alpha+2)(\alpha+3)} \right. \\ \left. + \frac{3|f''(b)|^q}{(\alpha+2)(\alpha+3)} - \frac{c[b-(2m-1)a]^2 + c[(2m-1)b-a]^2}{40m^2} \right].$$

In particular, if  $\alpha = m = 1$ , we have

$$|\Theta(a, b; g)| \leq \frac{(b-a)^2 \|g\|_\infty}{480} \left\{ 5|f''(a)| + 10 \left| f''\left(\frac{a+b}{2}\right) \right| + 5|f''(b)| - c(b-a)^2 \right\}.$$

**Theorem 3.6.** For  $(\alpha, m) \in (0, 1]^2$ , let  $f : [0, b^*] \rightarrow \mathbb{R}$  be twice differentiable on  $[0, b^*]$  for  $b^* > 0$  and  $0 \leq a < b$  with  $\frac{b}{m} \leq b^*$ . If  $|f''|^q$  is strongly  $(\alpha, m)$ -convex on  $[0, \frac{b}{m}]$  for  $q \geq 1$  and  $f'' \in L_1([0, \frac{b}{m}])$ , then

$$\left| \frac{1}{4} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)^2}{96} \left\{ \left[ \frac{6|f''(a)|^q}{(\alpha+2)(\alpha+3)} + \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{(\alpha+2)(\alpha+3)} - \frac{c[b-(2m-1)a]^2}{20m^2} \right]^{1/q} \right. \\ \left. + \left[ \frac{\alpha(\alpha+5)m|f''(\frac{a+b}{2m})|^q}{(\alpha+2)(\alpha+3)} + \frac{6|f''(b)|^q}{(\alpha+2)(\alpha+3)} - \frac{c[(2m-1)b-a]^2}{20m^2} \right]^{1/q} \right\}.$$

*Proof.* Similar to the proof of Theorem 3.5, applying Lemma 2.2, the strongly  $(\alpha, m)$ -convexity, and Hölder's inequality results in

$$\left| \frac{1}{4} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)^2}{16} \int_0^1 t(1-t) [|f''(M(t))| + |f''(V(t))|] dt.$$

The rest is the same as in the proof of Theorem 3.5. Theorem 3.6 is thus proved.  $\square$

**Theorem 3.7.** For  $(\alpha, m) \in (0, 1]^2$ , let  $f : [0, b^*] \rightarrow \mathbb{R}$  be twice differentiable on  $[0, b^*]$  for  $b^* > 0$  and  $g : [a, b] \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}_0$  be continuous on  $[a, b]$  for  $a < b$  with  $\frac{b}{m} \leq b^*$ . If  $|f''|^q$  is strongly  $(\alpha, m)$ -convex on  $[0, \frac{b}{m}]$  and  $f'' \in L_1([0, \frac{b}{m}])$  for  $q > 1$  and  $q \geq \ell \geq 0$ , then

$$(3.7) \quad |\Theta(a, b; g)| \leq \frac{(b-a)^2 \|g\|_\infty}{8} \left[ B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ \times \left\{ \left[ \frac{1}{\alpha+1} |f''(a)|^q + \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q - \frac{c[b-(2m-1)a]^2}{24m^2} \right]^{1/q} \right. \\ \left. + \left[ \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q + \frac{1}{\alpha+1} |f''(b)|^q - \frac{c[(2m-1)b-a]^2}{24m^2} \right]^{1/q} \right\},$$

where  $\Theta(a, b; g)$  is defined as in (2.1) and the beta function  $B(\alpha, \beta)$  can be defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \Re(\alpha), \Re(\beta) > 0.$$

*Proof.* Similar to the proof of Theorem 3.5, employing the strongly  $(\alpha, m)$ -convexity, Lemma 2.1, and Hölder's inequality produces

$$\begin{aligned}
 |\Theta(a, b; g)| &\leq \frac{(b-a)^2 \|g\|_\infty}{8} \int_0^1 t(1-t) [|f''(M(t))| + |f''(V(t))|] dt \\
 (3.8) \qquad &\leq \frac{(b-a)^2 \|g\|_\infty}{8} \left( \int_0^1 [t(1-t)]^{q/(q-1)} dt \right)^{1-1/q} \\
 &\quad \times \left[ \left( \int_0^1 |f''(u(t))|^q dt \right)^{1/q} + \left( \int_0^1 |f''(v(t))|^q dt \right)^{1/q} \right].
 \end{aligned}$$

Using the strongly  $(\alpha, m)$ -convexity of  $|f'|^q$  on  $[0, \frac{b}{m}]$ , we obtain

$$\begin{aligned}
 (3.9) \qquad \int_0^1 |f''(u(t))|^q dt &\leq \int_0^1 \left[ t^\alpha |f''(a)|^q + m(1-t^\alpha) \left| f''\left(\frac{a+b}{2m}\right) \right|^q - c \left( \frac{b-(2m-1)a}{2m} \right)^2 t(1-t) \right] dt \\
 &= \frac{1}{\alpha+1} |f''(a)|^q + \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q - \frac{c[b-(2m-1)a]^2}{24m^2}
 \end{aligned}$$

and

$$(3.10) \qquad \int_0^1 |f''(u(t))|^q dt \leq \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q + \frac{1}{\alpha+1} |f''(b)|^q - \frac{c[(2m-1)b-a]^2}{24m^2}.$$

Combining the inequalities (3.8), (3.9), and (3.10) yields the inequality (3.7). The proof of Theorem 3.7 is completed.  $\square$

In Theorem 3.7, letting  $g(x) = 1$  for  $x \in [a, b] \subseteq \mathbb{R}_0$  leads to the following corollary.

**Corollary 3.2.** For  $(\alpha, m) \in (0, 1]^2$ , let  $f : [0, b^*] \rightarrow \mathbb{R}$  be twice differentiable on  $[0, b^*]$  for  $b^* > 0$  and  $0 \leq a < b$  with  $\frac{b}{m} \leq b^*$ . If  $|f''|^q$  is strongly  $(\alpha, m)$ -convex on  $[0, \frac{b}{m}]$  and  $f'' \in L_1([0, \frac{b}{m}])$  for  $q > 1$ , then

$$\begin{aligned}
 &\left| \frac{1}{4} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 &\leq \frac{(b-a)^2}{16} \left[ B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\
 &\quad \times \left\{ \left[ \frac{1}{\alpha+1} |f''(a)|^q + \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q - \frac{c[b-(2m-1)a]^2}{24m^2} \right]^{1/q} \right. \\
 &\quad \left. + \left[ \frac{m\alpha}{\alpha+1} \left| f''\left(\frac{a+b}{2m}\right) \right|^q + \frac{1}{\alpha+1} |f''(b)|^q - \frac{c[(2m-1)b-a]^2}{24m^2} \right]^{1/q} \right\},
 \end{aligned}$$

where  $B(\alpha, \beta)$  is the beta function.

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