Sums of the Fibonacci and Lucas Numbers over the Binary Digital Sums

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Abstract

Let $s(n)$ denote the binary digital sum of the positive integer n. Using elementary combinatorial method, we present some known identities as a new form in term of $s(n)$. The sums of the Fibonacci, Lucas and harmonic numbers over the binary digital sums are considered. Moreover, we also give the sums over the binary digital sums, which derived from the binomial theorem.

Keywords: Binary digital sum, Fibonacci number, harmonic number, Lucas number
2010 Mathematics Subject Classification: 11B13, 11B39, 11B65

1. Introduction and statement of the main result

For $n \in \mathbb{N}$ can be written uniquely in base 2 as

$$n = \sum_{m=0}^{\infty} a_m(n) 2^m, \quad a_m(n) \in \{0, 1\}.$$

The binary digital sum of the positive integer n is defined by

$$s(n) := \sum_{m=0}^{\infty} a_m(n).$$

The function $s(n)$ occurs in many branches of mathematics. We refer the reader to the monograph [5] (see Chapter IV, part 4.3) for more details.

The research on an exact expression for the sum

$$\sum_{n=0}^{N} f(s(n)),$$

for a given function $f : \mathbb{N} \to \mathbb{R}$ is a classical problem in number theory, which usually called the sum of binary digits problem. The sum of digits problems was studied by many authors. In 1968, Trollope showed an explicit expression for binary digital sum in case of $f(y) = y$, see in [6]. In 2009, function $f(y) = \binom{y}{m}$ with $m > 0$, $f(y) = y^m$ and $f(y,t) = \exp(ty)$, for some t were considered, see in [3].

In this paper we shall present sums of a given real-valued function over the binary digital sums. By using the elementary combinatorial method, we obtain a new form related to binomial coefficients. Our main result is:

Theorem 1.1. *Let $k \geq 1$. For a given real-valued function $f : \mathbb{N} \to \mathbb{R}$*

$$\sum_{n=1}^{2^k-1} f(s(n)) = \sum_{i=1}^{k} \binom{k}{i} f(i). \tag{1.1}$$

Theorem 1.1 shows that the sum $\sum_{n=1}^{2^k-1} f(s(n))$ is written as an binomial term. Note that there are many identities for sum of some special numbers, for example Fibonacci, Lucas and harmonic numbers, related to binomial coefficients, see in [4]. We could provide sums of these numbers over the binary digital sums.

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2. Certain sums of Fibonacci and Lucas numbers over the binary digital sums

The Fibonacci $\{F_n\}_{n\geq 0}$ and Lucas sequences $\{L_n\}_{n\geq 0}$ are defined by

$$F_{n+2} = F_{n+1} + F_n, \qquad L_{n+2} = L_{n+1} + L_n$$

with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. Sums involving the Fibonacci and Lucas sequences play important roles in various branches of mathematics. In [4] appears the summation of the Fibonacci and Lucas sequences in various types of formulas. In particular, the discovery by E. Lucas in 1876 showed several summations of Fibonacci numbers related to the binomial coefficient in following lemmas.

Lemma 2.1. *For a given integer $n \geq 0$, we have*

$$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}, \qquad [4, see\ Eq.\ 12.2\,],$$

$$\sum_{i=1}^{n} \binom{n}{i} L_i = L_{2n} - 2, \qquad [4, see\ Eq.\ 12.3\,],$$

$$\sum_{i=1}^{n} \binom{n}{i} (-1)^i F_i = (-1)^{n-1} F_n, \qquad [4, see\ Eq.\ 12.4\,],$$

$$\sum_{i=1}^{n} \binom{n}{i} (-1)^i L_i = (-1)^n L_n - 2 \qquad [4, see\ Eq.\ 12.5\,].$$

Applying Theorem 1.1 and Lemma 2.1 by setting $f(n) = F_{s(n)}, L_{s(n)}, (-1)^{s(n)} F_{s(n)}$ and $(-1)^{s(n)} L_{s(n)}$ respectively, we obtain following corollary.

Corollary 2.2. *For a given integer $k \geq 1$, we have*

$$\sum_{n=1}^{2^k-1} F_{s(n)} = F_{2k},$$

$$\sum_{n=1}^{2^k-1} L_{s(n)} = L_{2k} - 2,$$

$$\sum_{n=1}^{2^k-1} (-1)^{s(n)} F_{s(n)} = (-1)^{k-1} F_k,$$

$$\sum_{n=1}^{2^k-1} (-1)^{s(n)} L_{s(n)} = (-1)^k L_k - 2.$$

There are more identities of the sum of the Fibonacci and Lucas numbers related to binomial coefficient, see [4, Exercise 12]. We could state many sums of the Fibonacci and Lucas numbers over the binary digital sums.

3. Harmonic sums over the binary digital sums

In [2], Dilcher showed the following identity for the combinatorial sum

$$\sum_{m=1}^{k} \binom{k}{m} \frac{(-1)^{m-1}}{m^n} = \sum_{k \geq r_1 \geq \dots \geq r_n \geq 1} \frac{1}{r_1 \cdots r_n}, \qquad \alpha \in \mathbb{N}.$$

Using this fact with Theorem 1.1, we obtain

Corollary 3.1. *For any integers $k, l \geq 1$, we have*

$$\sum_{n=1}^{2^k-1} \frac{(-1)^{s(n)-1}}{s(n)^l} = \sum_{k \geq r_1 \geq \dots \geq r_l \geq 1} \frac{1}{r_1 \cdots r_l}. \tag{3.1}$$

A special case of (3.1) is

Corollary 3.2. *For any integers $k \geq 1$, we have*

$$\sum_{n=1}^{2^k-1} \frac{(-1)^{s(n)-1}}{s(n)} = H_k,$$

where H_k are the harmonic numbers.

There is an interesting formula is known as Boole's formula [1], which is stated that

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^n = n!.$$

Combine with Theorem 1.1, we have

Corollary 3.3. *For any integer $k \geq 1$,*

$$\sum_{n=1}^{2^k-1} (-1)^{s(n)} s(n)^k = (-1)^k k!.$$

There are also many identities of the sum of the harmonic numbers related to binomial coefficient, therefore we similary can formulate the harmonic sums over the binary digital sums.

4. Sums over the binary digital sums, which derived form the binomial theorem

In the previous sections, the main idea for our sums over the binary digital sums follows from the sum involving the binomial coefficients. Here we present the sums of other numbers over the binary digital sum, which derived from the binomial theorem. We begin with the simplest form of the binomial theorem. Namely, for any integer $n \geq 0$, $x, y \in \mathbb{R}$,

$$\sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = (x+y)^n.$$

Using the binomial theorem and Theorem 1.1, we have

Corollary 4.1. *For any integers $k \geq 1$, $x \neq 0, y \in \mathbb{R}$, we have*

$$\sum_{n=1}^{2^k-1} \left(\frac{y}{x}\right)^{s(n)} = \left(1 + \frac{y}{x}\right)^k - 1. \tag{4.1}$$

A special case of (4.1) is

Corollary 4.2. *For any integers $k \geq 1$ and for any real number a, we have*

$$\sum_{n=1}^{2^k-1} a^{s(n)} = (a+1)^k - 1.$$

By using the elementary calculus method, there are the following interesting identities. For all integers $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \binom{n}{i} i = n2^{n-1},$$

$$\sum_{i=1}^{n} \binom{n}{i} i^2 = n(n+1)2^{n-2},$$

$$\sum_{i=1}^{n} \binom{n}{i} i^3 = (n+3)n^2 2^{n-3}.$$

In view of these identities and Theorem 1.1, we have

Corollary 4.3. *For any integers $k \geq 1$, we have*

$$\sum_{n=1}^{2^k-1} s(n) = 2^{k-1}k,$$

$$\sum_{n=1}^{2^k-1} s(n)^2 = 2^{k-2}k(k+1),$$

$$\sum_{n=1}^{2^k-1} s(n)^3 = 2^{k-3}k^2(k+3).$$

Furthermore, the well known Vandermonde's identity states that, for all $m, n, r \in \mathbb{N}$

$$\sum_{i=0}^{n} \binom{m}{i} \binom{n}{r-i} = \binom{m+n}{r}. \tag{4.2}$$

If we put $m = n = r$ in (4.2) and apply again Theorem 1.1, then we obtain

Corollary 4.4. *For any integers $k \geq 1$, we have*

$$\sum_{n=1}^{2^k-1} \binom{k}{s(n)} = \binom{2k}{k} - 1.$$

5. Proof of Theorem 1.1

Proof. Let $k \geq 1$. It is obvious that, for $1 \leq n < 2^k$, $1 \leq s(n) \leq k$.

Then, for a given real-valued function $f : \mathbb{N} \to \mathbb{R}$,

$$\sum_{n=1}^{2^k-1} f(s(n)) = \sum_{i=1}^{k} B(i,k) f(i),$$

where $B(i,k) = \#\{1 \leq n < 2^k \mid s(n) = i\}$. Every element of this set can be written as a binary sequences of length k. Namely, for $n \in \{1 \leq n < 2^k \mid s(n) = i\}$, we have

$$n = \sum_{j=0}^{k-1} a_j(n) 2^j, \qquad a_j(n) \in \{0,1\}, j = 0,1,2,...,k-1,$$

and

$$\sum_{j=0}^{k-1} a_j(n) = i, \qquad a_j(n) \in \{0,1\}, j = 0,1,2,...,k-1. \tag{5.1}$$

For a fixed integers $k \geq 1$ and any integer $0 \leq i \leq k$, we know that $\binom{k}{i}$ is the number of compositions of i such that,

$$n_0 + n_1 + \cdots + n_{k-1} = i,$$

where $n_j \in \{0,1\}$, for $j = 0,1,\ldots,k-1$. Thus, (5.1) shows that $B(i,k) = \binom{k}{i}$. This completes the proof.

□

Acknowledgement

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