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An Efficient Step Size Strategy for Numerical Approximation of Dynamical Systems with Hurwitz Stable

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Abstract

In this study, we have given an algorithm and a step size strategy for numerical solution of Hurwitz stable differential equation systems. The algorithm is suited for implementation using computer algebra systems. So we also have given numerical examples from various field using this algorithm and a Maple procedure for the algorithm.

Keywords: *Variable step size; step size strategy; Hurwitz stable matrix; differential equation systems; numerical integration.*
2010 Mathematics Subject Classification: *34A30; 65D30; 65L05.*

1. Introduction

Most of real-life problems are modeled by differential equations. The calculation of the analytical solution of these problems is either impossible or impractical. Therefore, there is a need to use numerical methods. Usually, it is important to use variable step size for the accuracy and efficiency of solution in numerical integration of initial value problems ([11]). Also,the authors of this article investigated how variable step size determined for numerical integrations of initial value problems with different aspects in [5, 6, 7, 8].
Consider the following Cauchy problem on $D = \{(t,X) : |t-t_0| \le T, |x_j - x_{j0}| \le b_j\}$

$$X' = F(t,X), X(t_0) = X_0 \tag{1.1}$$

where $X(t) = (x_j(t))$, $X_0 = (x_{j0})$, $x_{j0} = x_j(t_0)$, $F(t,X) = (f_j)$, $f_j = f_j(t,x_1,x_2,...,x_N)$, $F(t,X) \in C^1([t_0-T,t_0+T] \times R^N)$, $X(t)$, X_0 and $b = (b_j) \in R^N$.
In [4, 7] a step size strategy for $F(t,X) = AX$ is proposed by

$$h_i \le \frac{1}{\alpha \sqrt[4]{N^5}} \left(\frac{2\delta_L}{\beta_{i-1}}\right)^{\frac{1}{2}} \tag{1.2}$$

such that the local error $||LE_i|| \le \delta_L$. Here, the user determines error level δ_L. For general case of $F(t,X)$ the step size strategies are also proposed in [8].
In this paper, the step size strategy in [4, 7] is developed for the system

$$X'(t) = AX(t), A \in R^{N\times N},$$

where the matrix A is a Hurwitz stable matrix. In Section 2; some basic concepts are given and the concept of Hurwitz stability is remained. The step size strategy for linear differential equation systems is reviewed. In Section 3; the step size strategy for the Hurwitz systems and the algorithm which calculates the step sizes according to the given strategy and numerical solutions are given. The numerical solutions obtained with the new strategy and algorithm are compared with the results in [4, 7]. Finally, the given strategy and algorithm are applied to some industrial problems.

2. Preliminaries

Firstly, we should be noted that we use the Frobenius norm for every $A = (a_{ij}) \in R^{N\times N}$ and as a norm in R^N we use Euclidean norm, in this study.

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Let we consider the following Cauchy problem on D.

$$X'(t) = AX(t), X(t_0) = X_0 \tag{2.1}$$

Here, $D = \{(t,X) : |t-t_0| \le T, |x_j - x_{j0}| \le b_j\}$, $X(t) = (x_j(t))$, $X_0 = (x_{j0})$, $x_{j0} = x_j(t_0)$, $A = (a_{ij}) \in R^{N\times N}$, $X(t)$, X_0 and $b = (b_j) \in R^N$.
Let in $i-th$ step numerical solution be $Y_i = (y_{ij}) \in R^N$. We let construct the Cauchy problem

$$Z' = AZ(t), Z(t_{i-1}) = Y_{i-1}, Y_0 = X_0, t \in [t_{i-1}, t_i). \tag{2.2}$$

The vector of local error LE_i is defined by $LE_i = Y_i - Z(t_i)$.
We use Euler method which is defined in [14] by

$$Y_{i+1} = Y_i + h_{i+1}F_i, \tag{2.3}$$

where $F_i = (f_{ij}) \in R^N$, $Y_i = (y_{ij}) \in R^N$ and $h_{i+1} = t_{i+1} - t_i$. The local error vector for the Euler's method is given by

$$LE_i = -\frac{h_i^2}{2}A^2 Z(\tau_{ij}), \tau_{ij} \in [t_{i-1}, t_i) \tag{2.4}$$

and the upper bound of the local error $||LE_i||$ is given by

$$||LE_i|| \le (\tfrac{1}{2}\alpha^2\beta_{i-1})\sqrt{N^5}h_i^2,$$

where $\alpha = \max_{1\le i,j\le N}|a_{ij}|$ and $\max_{1\le j\le N}(\sup_{t_{i-1}\le\tau_i<t_i}|z_j(\tau_i)|) \le \beta_{i-1}$.

2.1. Concept of Hurwitz Stability

The following theorem explains the concept of Hurwitz stability.

Theorem 2.1.(Lyapunov) Let $A = (a_{ij}) \in R^{N\times N}$. The matrix A is asymptotically stable if and only if there is a solution $H = H^* > 0$ of the Lyapunov matrix equation $A^*H + HA = -I$. Here, $H = H^* > 0$ indicates that the symmetric matrix H is a positive definite matrix and A^* is the transposition of the matrix A (see, [1]).
Lyapunov theorem states if such H does exist then all the eigenvalues of matrix A lie strictly in the left-hand half-plane ([1, 10]).
Also, define the region $C_H = \{z \in C : Rez < 0\}$ in the complex plane C. If $\sigma(A) \subset C_H$ then $A \in R^{N\times N}$ is said to be Hurwitz stable ([15]).
Here $\sigma(A)$ is the spectrum of A.
According to these statements, Hurwitz stability and asymptotically stability are the equivalent concepts. Because of this equivalence, we chose to use the concept of "Hurwitz stability" in this paper.
The stability parameter of the system (2.1) which shows the quality of Hurwitz stability and Hurwitz stability is given in [1, 10] as follows:

$$\kappa(A) = 2||A||||H||. \tag{2.5}$$

If there is a solution $H = H^* > 0$ of the Lyapunov equation $A^*H + HA = -I$, then the stability parameter is shown as $\kappa(A) < \infty$. Otherwise it is $\kappa(A) = \infty$. For the solution of the system (2.1) the following holds

$$||X(t)|| \le \sqrt{\kappa(A)}e^{-\frac{t||A||}{\kappa(A)}}||X(0)|| \tag{2.6}$$

see ([1, 2]).

2.2. The step size strategy (SSSLS)for linear systems

According to equation (2.4), the upper bound of local error for the system (2.1) is given by

$$||LE_i|| \le (\frac{1}{2}\alpha^2\beta_{i-1})\sqrt{N^5}h_i^2, \tag{2.7}$$

where

$$\alpha = \max_{1\le i,j\le N}|a_{ij}|, \max_{1\le j\le N}(\sup_{t_{i-1}\le\tau_i<t_i}|z_j(\tau_i)|) \le \beta_{i-1}. \tag{2.8}$$

From the inequality (2.7) in step i, the step size is calculated by

$$h_i \le \frac{1}{\alpha\sqrt[4]{N^5}}(\frac{2\delta_L}{\beta_{i-1}})^{\frac{1}{2}} \tag{2.9}$$

such that $||LE_i|| \le \delta_L$, where δ_L is the error level that is determined by user ([4, 7]).

3. Step Size Strategy and Algorithm for the Hurwitz Stable Systems

In this section, we will give a step size strategy for the differential equation systems with Hurwitz stable. This strategy is development of the strategy given in [4, 7]. Then we will give an algorithm to be used in computer calculations of this strategy.

3.1. New Strategy (SSSHS)

We let assume that (2.1) is a Hurwitz stable system and consider the following theorem before giving a step-size strategy.

Theorem 3.1 The upper bound of local error for the Cauchy problem (2.1) is

$$||LE_i|| \leq \frac{1}{2}||A||^2\beta_{i-1}\sqrt{\kappa(A)}e^{-\frac{t_{i-1}||A||}{\kappa(A)}}h_i^2, \tag{3.1}$$

where $\kappa(A)$ and β_{i-1} are given by (2.5) and (2.8), respectively.

Proof The local error of the Cauchy problem (2.1) is known to be

$$||LE_i|| \leq \frac{1}{2}h_i^2||A||^2||Z(\tau_i)||^2 \tag{3.2}$$

in the interval $[t_{i-1},t_i)$ from [4, 7]. Here, Y_i is the solution vector obtained from the numerical method in step i and $Z(t)$ is the same as defined in (2.2). If we write

$$||Z(\tau_i)|| \leq \sqrt{\kappa(A)}e^{-\frac{\tau_i||A||}{\kappa(A)}}||Z(t_{i-1})||, \tau_i \in (t_{i-1},t_i) \tag{3.3}$$

in the inequality (2.6), we obtain

$$||LE_i|| \leq \frac{1}{2}h_i^2||A||^2\sqrt{\kappa(A)}e^{-\frac{t_{i-1}||A||}{\kappa(A)}}||Z(t_{i-1})|| \tag{3.4}$$

by inequalities (3.2) and (3.3).

Strategy 1 According to the inequality (3.1), the step-size is computed with

$$h_i \leq (\frac{2\delta_L}{\beta_{i-1}})^{\frac{1}{2}}\frac{1}{||A||}\kappa(A)^{-\frac{1}{4}}e^{t_{i-1}\frac{||A||}{2\kappa(A)}} \tag{3.5}$$

in the ith step of the numerical integration of Cauchy problem (2.1) such as the local error is smaller than error level δ_L .

Note 1 In accordance with our goal, ith step size can be chosen from the inequality (3.5). Theoretically, if the step sizes are computed by the inequality (3.5), the error level δ_L are not exceeded. However, sometimes it may be observed that $||LE_i|| > \delta_L$ in a few steps, in practice. This situation can be caused by several reasons. Especially, the effects of floating point arithmetic can be one of the reasons (see [4, 9]). It is well known that another reason can be the instability of the numerical method ([12]). Actually; if the value T is selected according to the nature of the real-life problems then the local error is usually expected to be than error level δ_L .

3.2. New Algorithm (SSAHS)

This algorithm is development of the algorithm in [4, 7] for the Hurwitz stable systems. SSAHS calculates the step sizes given by the inequality (3.5) and the numerical solution of the Cauchy problem (2.1) with Hurwitz stable.
SSAHS is as follows.

Algorithm 1

Step 0 (Inputs) t_0, T, $\kappa(A)$, b, h^*, δ_L, X_0, A.
Step 1 Calculate β_0 and $||A||$.
Step 2 Calculate β_{k-1} and $\hat{h}_k$; $\hat{h}_k = (\frac{2\delta_L}{\beta_{k-1}})^{\frac{1}{2}}\kappa(A)^{-\frac{1}{4}}\frac{1}{||A||}e^{t_{k-1}\frac{||A||}{2\kappa(A)}}$.
Step 3 Control step size $\hat{h}_k$ with algorithm **K**.
Step 4 Calculate $t_k = t_{k-1}+h_k$ and $Y_k = (I+h_kA)Y_{k-1}$. Replace k by $k+1$ and go to step 2.

Here, **K** is the step size control algorithm given in [3].

Example 1 Let us calculate the step sizes and numerical solution of Cauchy problem

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -7 & -1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.6}$$

on $D = \{(t,X) : t \in [0,T], |x_j - x_{j0}| \leq 5|\}$ using SSALS and SSAHS. The stability parameter of the system (3.6) is $\kappa(A) = 1,41271$. For $T = 10$, $h^* = 10^{-12}$ and $\delta_L = 10^{-1}$, outputs obtained from each algorithm are seen in Table 1.

Both SSSLS and SSSHS can be used for the numerical integration of the Cauchy problem

$$X'(t) = AX(t), X(t_0) = X_0,$$

where A is the Hurwitz stable matrix.
Because of the matrix $A = \begin{pmatrix} -7 & -1 \\ 1 & -5 \end{pmatrix}$ is a Hurwitz stable matrix, it is more advantageous to be preferred SSSHS. Because, larger step sizes are obtained by SSSHS. Therefore, the processing time is shorter. The process takes only 19 steps by SSAHS while ending with step 834 by SSALS as it can be seen from the above tables.
Table 2 and Figure 3.1 show us the number of steps and CPU times for $T = 1$, $T = 3$, $T = 5$, $T = 7$ and $T = 10$ with SSAHS and SSALS.

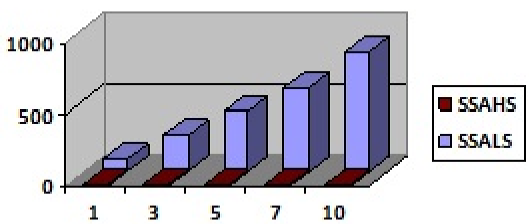
Table 1: The values of h_i and $||LE||_i$ obtained from SSALS and SSAHS for Example 1.

SSALS		SSAHS	
i	h_i	i	h_i
1	0.1096614132e-1	**1**	0.2364413537e-1
2	0.1102676853e-1	**2**	0.2538987050e-1
3	0.1108464547e-1	**3**	0.2736546418e-1
4	0.1113981458e-1	**4**	0.2961906983e-1
⋮	⋮	⋮	⋮
18	0.1166621209e-1	**18**	0.9368440981
19	0.1168950242e-1	**19**	7.631745801
⋮	⋮		
833	0.120128059e-1		
834	0.9595176e-2		

SSALS		SSAHS	
i	$\|\| LE_i \|\|$	i	$\|\| LE_i \|\|$
1	0.299517487164503889e-2	**1**	0.135812138725487745e-1
2	0.282878713234895572e-2	**2**	0.133933394693476360e-1
3	0.266913067468298312e-2	**3**	0.131378540255884385e-1
4	0.251620149825517890e-2	**4**	0.128075308770921152e-1
⋮	⋮	⋮	⋮
18	0.102062881396393574e-2	**18**	0.115630790914989846e-2
19	0.952937390818749163e-3	**19**	0.517510313204872091e-1
⋮	⋮		
833	0.371781669457522694e-31		
834	0.221153743684013559e-31		

Table 2: The number of steps and Cpu times for $T = 1$, $T = 3$, $T = 5$, $T = 7$ and $T = 10$ with SSALS and SSAHS for Example 1.

T	Number of steps SSALS	SSAHS	CPU time SSALS	SSAHS
1	85	16	0.17s	0.15s
3	252	19	0.92s	0.17s
5	418	19	1.71s	0.17s
7	585	19	2.71s	0.17s
10	834	19	4.60s	0.18s



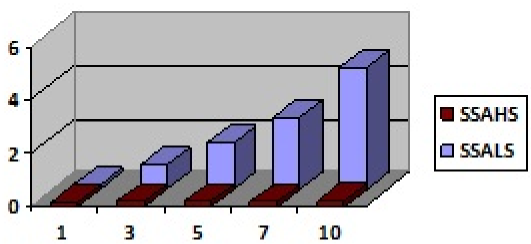


Figure 3.1: The number of steps and CPU times for values T with SSAHS and SSALS for Example 1.

4. Applications

We have considered only real-life problems and applied SSAHS to these problems, in this section. We have used Maple procedures for Cauchy problems given by equation (2.1) in all calculations.

4.1. Mixing problem (Brine Tank Cascade)

Let brine tanks A, B, C be given as in Figure 4.1 and V_1, V_2, V_3 be volumes of each tank, respectively.

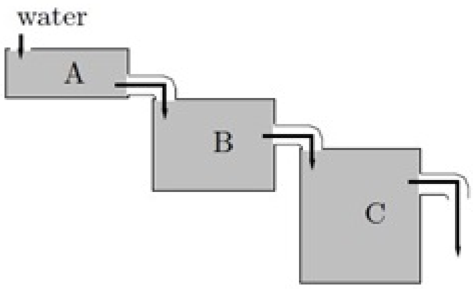


Figure 4.1: The brine tanks in cascade

Suppose that the water enters the tank A at rate r, drains from A to B, from B to C and from C in the same rate. Let assume that there is a uniform salt concentration in each tank and $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t.
We suppose added to tank A water containing no salt. Therefore, the amount of salt in each tank is modeled by the following differential system

$$X'(t) = \begin{pmatrix} -\frac{V_1}{r} & 0 & 0 \\ \frac{V_1}{r} & -\frac{V_2}{r} & 0 \\ 0 & \frac{V_2}{r} & -\frac{V_3}{r} \end{pmatrix} X(t)$$

where $X(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}^T$ (for example, [16]).
To apply SSAHS we let take $V_1 = 20$ (gal), $V_2 = 40$ (gal), $V_3 = 50$ (gal), $r = 10$ gal/min and the initial value $X_0 = \begin{pmatrix} 10 & 0 & 0 \end{pmatrix}^T$. In this case the differential system can written as

$$X'(t) = \begin{pmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{pmatrix} X(t), X(0) = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

Let us examine the behavior the amount of salt in each tank for three hours. Consider Cauchy problem (4.1) on $D = \{(t,X) : t \in [0,180], |x_j - x_j(0)| \leq 1\}$ and use the values of $h^* = 10^{-12}$ and $\delta_L = 10^{-1}$. The stability parameter of the system (4.1) is $\kappa(A) = 9.89988$. The behavior the amount of salt in each tank have been illustrated in Figure 4.2.

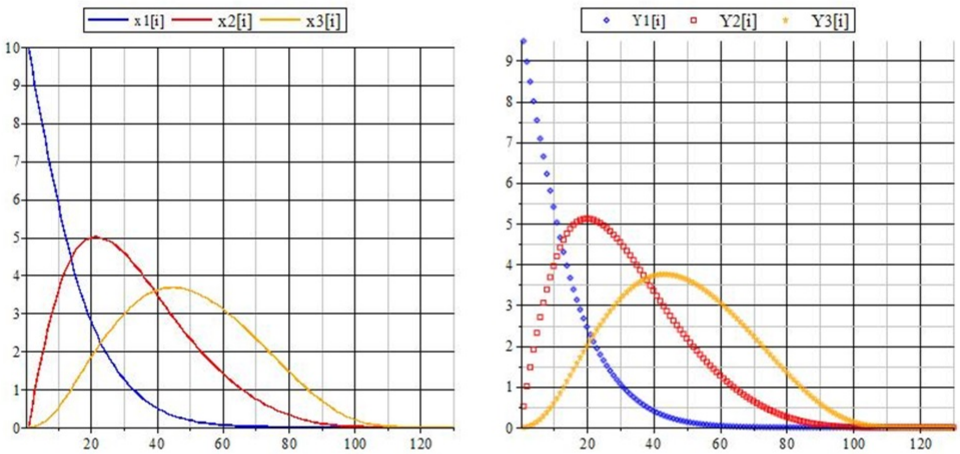


Figure 4.2: Exact and numerical solutions of Cauchy problem (4.1) calculated with SSAHS for the mixing problem.

The results have been summarized in Table 3 and Figure 4.3 illustrates the values of h_i and $||LE_i||$ calculated by SSAHS.

Table 3: The values of h_i and $||LE_i||$ calculated by SSAHS for mixing problem.

i	h_i	$\| LE_i \|$
1	0.1036119930	0.246045857464538519e-1
2	0.1065498685	0.244271245651936017e-1
3	0.1096460506	0.242348555069961911e-1
4	0.1129125257	0.240266839847081588e-1
⋮	⋮	⋮
128	21.19927411	0.133354507708755610e-9
129	46.50202739	0.108793598420230885e-8
130	1.0539370	0.218504620931324559e-10

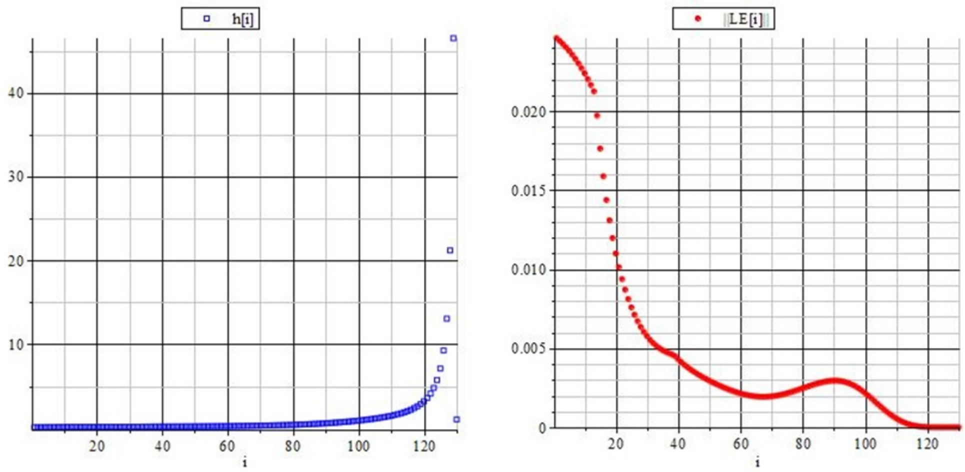


Figure 4.3: h_i and $||LE_i||$ calculated by SSAHS for the mixing problem (4.1).

4.2. Irregular Heartbeats and Lidocaine

The drug lidocaine is used to treate the illness of ventricular arrhythmia or irregular heartbeat. Let's give the dynamic model for this treatment, $x_1(t)$ is amount of lidocaine in the bloodstream, $x_2(t)$ is amount of lidocaine in body tissue and $X(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$.
On $D = \{(t,X) : t \in [0,120], |x_j - x_{j0}| \leq 5\}$, consider following system which valids for a special body weight only(for example, [16]);

$$X'(t) = \begin{pmatrix} -0.09 & 0.038 \\ 0.066 & -0.038 \end{pmatrix} X(t), X(0) = \begin{pmatrix} 0 \\ 200 \end{pmatrix}. \tag{4.2}$$

Let us apply SSAHS to Cauchy problem (4.2) and examine its behavior for two hours. We let use the values of $h^* = 10^{-12}$ and $\delta_L = 10^{-1}$. The stability parameter of the system (4.2) is $\kappa(A) = 17.1613$. The behavior the amount of lidocaine in the bloodstream and the amount of lidocaine in body tissue have been illustrated in Figure 4.4.

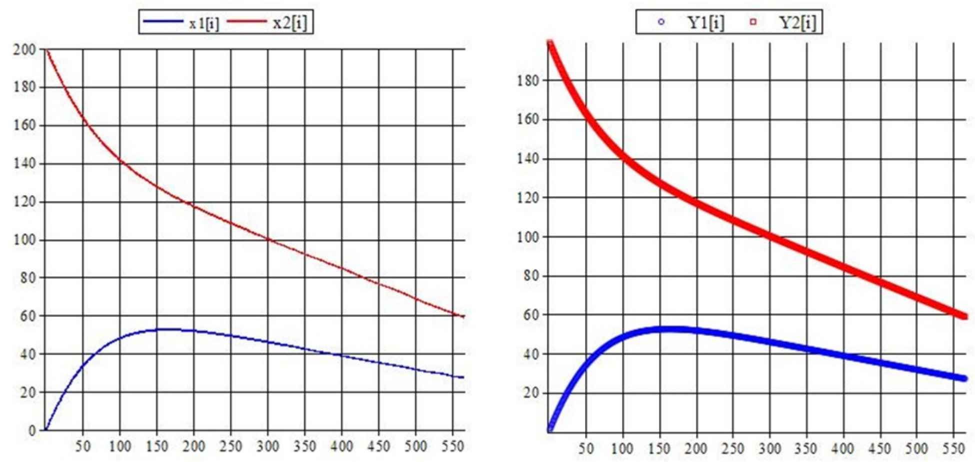


Figure 4.4: Exact and numerical solutions of Cauchy problem (4.2) calculated with SSAHS for irregular heartbeats and Lidocaine problem.

The results have been summarized in Table 4 and Figure 4.5 illustrates the values of h_i and $||LE_i||$ calculated by SSAHS.

Table 4: The values of h_i and $||LE_i||$ calculated by SSAHS for irregular heartbeats and lidocaine problem.

i	h_i	$\|\| LE_i \|\|$
1	0.1241087639	0.960540502266953207e-2
2	0.1244509033	0.951420019572411270e-2
3	0.1247925215	0.942325051852235286e-2
4	0.1251336039	0.933256518586074793e-2
⋮	⋮	⋮
564	0.3407702670	0.216837352597045227e-3
565	0.3415955568	0.217325842328989908e-3
566	0.1190267	0.263326300541874501e-4

4.3. Biomass Transfer

Consider a forest having several varieties of trees. Let variables x_1 , x_2, x_3, t be defined as biomass decayed into humus, biomass of dead trees, biomass of living trees and time in decades, respectively. A typical biological model is

$$X'(t) = \begin{pmatrix} -1 & 3 & 0 \\ 0 & -3 & 5 \\ 0 & 0 & -5 \end{pmatrix} X(t), X(0) = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}, \tag{4.3}$$

where $X(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}^T$ (for example, [16]). Let us apply SSAHS to Cauchy problem (4.3) to examine the behavior biomass decayed into humus, biomass of dead trees and biomass of living trees for five decade. Consider Cauchy problem (4.3) on $D = \{(t,X) : t \in [0,5], |x_j - x_{j0}| \leq 1\}$ and use the values of $h^* = 10^{-12}$ and $\delta_L = 10^{-1}$. The stability parameter of the system (4.3) is $\kappa(A) = 19.4504$. The behavior of x_1 , x_2, x_3 have been illustrated in Figure 4.6.
The results have been summarized in Table 5 and Figure 4.7 illustrates the values of h_i and $||LE_i||$ calculated by SSAHS.

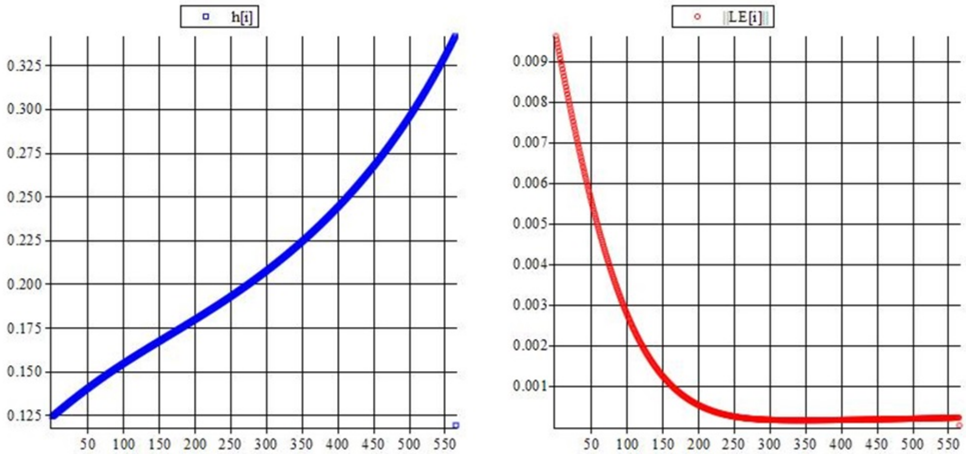


Figure 4.5: h_i and $||LE_i||$ calculated by SSAHS for the irregular heartbeats and Lidocaine problem (4.2).

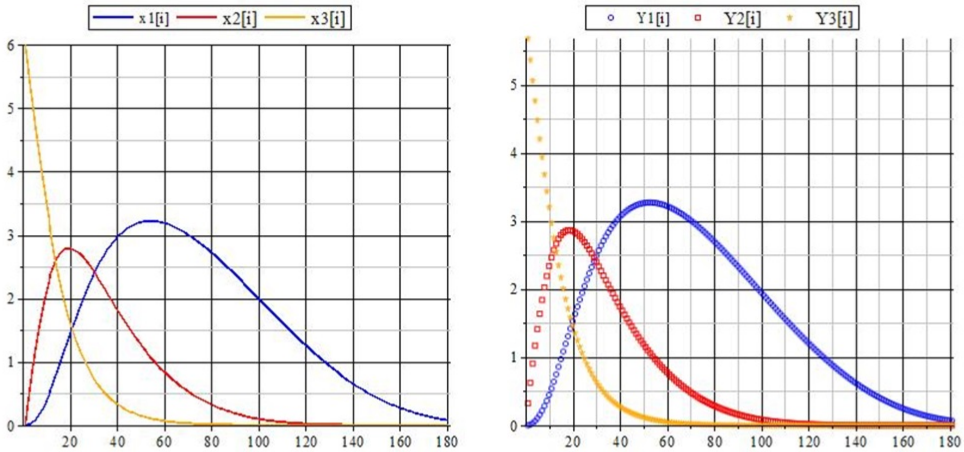


Figure 4.6: Exact and numerical solutions of Cauchy problem (4.3) calculated with SSAHS for the biomass transfer problem.

Table 5: The values of h_i and $||LE_i||$ calculated by SSAHS for biomass transfer problem.

i	h_i	$\|\| LE_i \|\|$
1	0.0107601553438753	0.168219886603259219e-1
2	0.0110400008426164	0.165359783843472205e-1
3	0.0113325947672574	0.162355854639435263e-1
4	0.0116386423120766	0.159202215777695076e-1
⋮	⋮	⋮
64	0.0167724395009129	0.883449648401477602e-3
65	0.0168648895806415	0.848078463039397888e-3
66	0.00453738405232285	0.589744791508788806e-4

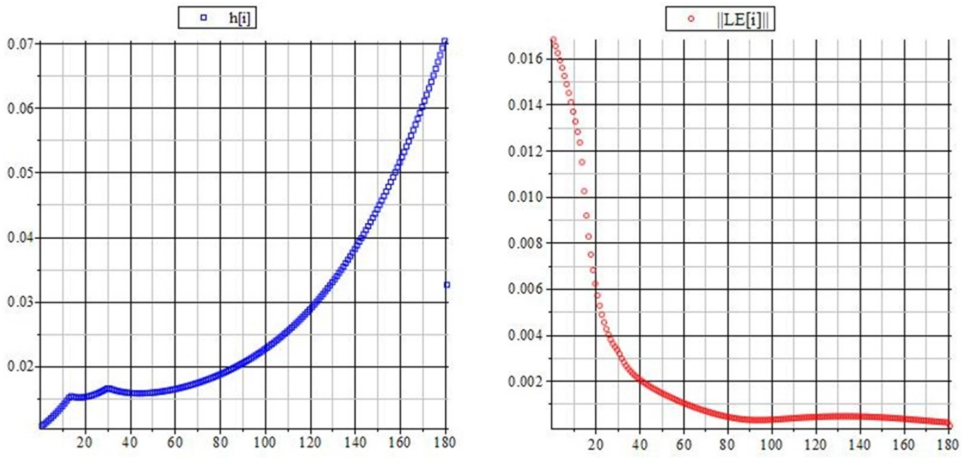


Figure 4.7: h_i and $||LE_i||$ calculated by SSAHS for the biomass transfer problem (4.3).

5. Conclusions

In this work, a step size strategy named as SSSHS and an algorithm named as SSAHS were given for the system

$$X'(t) = AX(t), X(t_0) = X_0,$$

where A is the Hurwitz stable matrix. SSAHS calculates the step sizes based on SSSHS and the numerical solutions. SSAHS is suitable to write computer procedure. To calculate the step sizes and numerical solutions, the Maple procedures have been used. A numerical example was given to compare with SSSLS (SSALS) and SSSHS (SSAHS). Both SSSLS (SSALS) and SSSHS (SSAHS) can be used for the numerical integration of the Cauchy problem

$$X'(t) = AX(t), X(t_0) = X_0,$$

where A is the Hurwitz stable matrix. However, it was seen that it is more advantageous to be preferred SSSHS (SSAHS). Because, the larger step sizes are obtained by SSSHS (SSAHS). Therefore, the processing time is shorter. SSSHS and SSAHS were applied to real life problems which named as mixing problem, irregular heartbeats and Lidocaine problem and biomass transfer problem.
In this study, the Euclidean norm was used as the vector norm. By using different norms, different step sizes instead of the step sizes given by inequality (3.5) can also selected. It should be noted that an expected situation.

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