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Coefficient estimates for a subclass of analytic bi-pseudo-starlike functions of Ma-Minda type

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Abstract

In this paper, we introduce a new subclass $\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to this class, we obtain initial coefficient bounds. Our results generalize and improve some earlier results in the literature.

Keywords: *Analytic functions; univalent functions; bi-univalent functions; coefficient bounds; subordination; pseudo-starlike functions.*
2010 Mathematics Subject Classification: *30C45*

1. Introduction

Let $\mathbb{R}=(-\infty,\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N}:=\{1,2,3,\ldots\}=\mathbb{N}_0\setminus\{0\}$$

be the set of positive integers.

Let $\mathscr{A}$ denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C}\quad\text{and}\quad|z|<1\}.$$

We also denote by $\mathscr{S}$ the class of all functions in the normalized analytic function class $\mathscr{A}$ which are univalent in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z)\prec g(z)\qquad(z\in\mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0)=0\qquad\text{and}\qquad|\omega(z)|<1\quad(z\in\mathbb{U})$$

such that

$$f(z)=g(\omega(z))\quad(z\in\mathbb{U}).$$

Indeed, it is known that

$$f(z)\prec g(z)\quad(z\in\mathbb{U})\Rightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z)\prec g(z)\quad(z\in\mathbb{U})\Leftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

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Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [7] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathscr{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathscr{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\, r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class Σ, see [13] (see also [2]). In fact, the aforecited work of Srivastava et al. [13] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Xu et al. [14, 15], and others (see, for example, [4, 5, 6, 8, 9, 12, 17, 18]).

Let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{1.3}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function φ satisfies the above conditions.

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(z)|\} < 1.$$

We suppose also that

$$u(z) = p_1 z + p_2 z^2 + p_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{1.4}$$

and

$$v(z) = q_1 z + q_2 z^2 + q_3 z^3 + \cdots \qquad (z \in \mathbb{U}). \tag{1.5}$$

We observe that

$$|p_1| \leq 1, \quad |p_2| \leq 1 - |p_1|^2, \quad |q_1| \leq 1, \quad |q_2| \leq 1 - |q_1|^2. \tag{1.6}$$

By simple computations, we have

$$\varphi(u(z)) = 1 + B_1 p_1 z + \left(B_1 p_2 + B_2 p_1^2\right) z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{1.7}$$

and

$$\varphi(v(w)) = 1 + B_1 q_1 w + \left(B_1 q_2 + B_2 q_1^2\right) w^2 + \cdots \qquad (w \in \mathbb{U}). \tag{1.8}$$

Recently, Babalola [3] defined the class $\mathscr{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as follows:

Suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. A function $f \in \mathscr{A}$ given by (1.1) belongs to the class $\mathscr{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk $\mathbb{U}$ if and only if

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}).$$

Babalola [3] proved that all pseudo-starlike functions are Bazilevič of type $1 - 1/\lambda$, order $\beta^{1/\lambda}$ and univalent in $\mathbb{U}$.

Motivated by the abovementioned works, we define the following subclass of function class Σ.

Definition 1.1. *For $\lambda \geq 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathscr{LB}_\Sigma^\lambda(\varphi)$ if the following conditions are satisfied:*

$$\frac{z(f'(z))^\lambda}{f(z)} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} \prec \varphi(w) \qquad (w \in \mathbb{U}),$$

where the function $g = f^{-1}$ is defined by (1.2).

Remark 1.2. *In the following special cases of Definition 1.1, we show how the class of analytic bi-univalent functions $\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ for suitable choices of λ and φ lead to certain known classes of analytic bi-univalent functions studied earlier in the literature.*
(**i**) *For $\lambda=1$, we get the class $\mathscr{LB}_{\Sigma}^{1}(\varphi)=\mathscr{ST}_{\Sigma}(\varphi)$ of Ma-Minda bi-starlike functions introduced and studied by Ali et al. [1].*
(**ii**) *If we let*

$$\varphi(z):=\varphi_{\alpha}(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^{2}z^{2}+\cdots \qquad (0<\alpha\leq 1, z\in\mathbb{U}),$$

then the class $\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ reduces to the class denoted by $\mathscr{LB}_{\Sigma}^{\lambda}(\alpha)$ which is the subclass of the functions $f\in\Sigma$ satisfying

$$\left|\arg\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right)\right|<\frac{\alpha\pi}{2} \qquad \text{and} \qquad \left|\arg\left(\frac{w(g'(w))^{\lambda}}{g(w)}\right)\right|<\frac{\alpha\pi}{2},$$

where the function $g=f^{-1}$ is defined by (1.2).
(**iii**) *If we let*

$$\varphi(z):=\varphi_{\beta}(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^{2}+\cdots \qquad (0\leq\beta<1, z\in\mathbb{U}),$$

then the class $\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ reduces to the class denoted by $\mathscr{LB}_{\Sigma}(\lambda,\beta)$ which is the subclass of the functions $f\in\Sigma$ satisfying

$$\Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right)>\beta \qquad \text{and} \qquad \Re\left(\frac{w(g'(w))^{\lambda}}{g(w)}\right)>\beta,$$

where the function $g=f^{-1}$ is defined by (1.2).

The classes $\mathscr{LB}_{\Sigma}^{\lambda}(\alpha)$ and $\mathscr{LB}_{\Sigma}(\lambda,\beta)$ are introduced and studied by Joshi et al. [10]. In the special case $\lambda=1$, we get the classes $\mathscr{LB}_{\Sigma}^{1}(\alpha)=\mathscr{S}_{\Sigma}^{*}[\alpha]$ and $\mathscr{LB}_{\Sigma}(1,\beta)=\mathscr{S}_{\Sigma}^{*}(\beta)$ introduced and studied by Brannan and Taha [2].

In order to derive our main results, we need the following lemma.

Lemma 1.3. *[16] Let $k,l\in\mathbb{R}$ and $z_1,z_2\in\mathbb{C}$. If $|z_1|<R$ and $|z_2|<R$, then*

$$|(k+l)z_1+(k-l)z_2|\leq\begin{cases}2R|k| & , \quad |k|\geq|l|\\ 2R|l| & \quad |k|\leq|l|\end{cases}.$$

2. Main Results

Theorem 2.1. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ and $\lambda\geq 1$. Then*

$$|a_2|\leq\sqrt{\frac{B_1^3}{(2\lambda-1)\left[(2\lambda-1)B_1+\left|\lambda B_1^2-(2\lambda-1)B_2\right|\right]}} \tag{2.1}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1^2}{(2\lambda-1)^2} & , \quad B_1\leq\frac{(2\lambda-1)^2}{3\lambda-1}\\ \left(1-\frac{(2\lambda-1)^2}{(3\lambda-1)B_1}\right)\frac{B_1^3}{(2\lambda-1)\left[(2\lambda-1)B_1+\left|\lambda B_1^2-(2\lambda-1)B_2\right|\right]}+\frac{B_1}{3\lambda-1} & , \quad B_1\geq\frac{(2\lambda-1)^2}{3\lambda-1}\end{cases}. \tag{2.2}$$

Proof. Let $f\in\mathscr{LB}_{\Sigma}^{\lambda}(\varphi)$ and $g=f^{-1}$ be defined by (1.2). Then there are analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, such that

$$\frac{z(f'(z))^{\lambda}}{f(z)}=\varphi(u(z)) \tag{2.3}$$

and

$$\frac{w(g'(w))^{\lambda}}{g(w)}=\varphi(v(w)). \tag{2.4}$$

It follows from (1.7), (1.8), (2.3) and (2.4) that

$$(2\lambda-1)a_2 = B_1p_1 \tag{2.5}$$

$$\left(2\lambda^2-4\lambda+1\right)a_2^2+(3\lambda-1)a_3 = B_1p_2+B_2p_1^2 \tag{2.6}$$

$$-(2\lambda-1)a_2 = B_1q_1 \tag{2.7}$$

$$\left(2\lambda^2+2\lambda-1\right)a_2^2-(3\lambda-1)a_3 = B_1q_2+B_2q_1^2. \tag{2.8}$$

From (2.5) and (2.7), we find that

$$p_1 = -q_1 \tag{2.9}$$

and

$$2(2\lambda-1)^2 a_2^2 = B_1^2\left(p_1^2+q_1^2\right). \tag{2.10}$$

Also from (2.6), (2.8) and (2.10), we have

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{2(2\lambda-1)\left[\lambda B_1^2-(2\lambda-1)B_2\right]}. \tag{2.11}$$

In view of (2.9) and (2.11), together with (1.6), we get

$$|a_2|^2 \le \frac{B_1^3\left(1-|p_1|^2\right)}{(2\lambda-1)\left|\lambda B_1^2-(2\lambda-1)B_2\right|}. \tag{2.12}$$

Substituting (2.5) in (2.12) we obtain

$$|a_2| \le \sqrt{\frac{B_1^3}{(2\lambda-1)\left[(2\lambda-1)B_1+\left|\lambda B_1^2-(2\lambda-1)B_2\right|\right]}}, \tag{2.13}$$

which is desired inequality (2.1).
On the other hand, by subtracting (2.8) from (2.6) and a computation using (2.9) finally lead to

$$a_3 = a_2^2 + \frac{B_1(p_2-q_2)}{2(3\lambda-1)}. \tag{2.14}$$

From (1.6), (2.5), (2.9) and (2.14), it follows that

$$\begin{aligned} |a_3| &\le |a_2|^2 + \frac{B_1}{2(3\lambda-1)}(|p_2|+|q_2|) \\ &\le |a_2|^2 + \frac{B_1}{3\lambda-1}\left(1-|p_1|^2\right) \\ &= \left(1-\frac{(2\lambda-1)^2}{(3\lambda-1)B_1}\right)|a_2|^2 + \frac{B_1}{3\lambda-1}. \end{aligned} \tag{2.15}$$

Substituting (2.5) and (2.13) in (2.15) we obtain the desired inequality (2.2). □

Remark 2.2. *Theorem 2.1 is an improvement of the estimates obtained by Mazi and Altınkaya [11, Corollary 5].*

If we take $\lambda = 1$ in Theorem 2.1, then we have the following Corollary 1.

Corollary 1. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{S}\mathscr{T}_\Sigma(\varphi)$. Then

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{B_1+\left|B_1^2-B_2\right|}}$$

and

$$|a_3| \le \begin{cases} B_1^2 & , \quad B_1 \le \frac{1}{2} \\ \left(1-\frac{1}{2B_1}\right)\frac{B_1^3}{B_1+\left|B_1^2-B_2\right|}+\frac{B_1}{2} & , \quad B_1 \ge \frac{1}{2} \end{cases}.$$

Remark 2.3. *Corollary* 1 *is an improvement of the estimates obtained by Mazi and Altınkaya [11, Corollary 4].*

If we consider the function φ_α, defined in Remark 1.2 (ii), in Theorem 2.1, then we get the following consequence.

Corollary 2. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{L}\mathscr{B}_\Sigma^\lambda(\alpha)$ and $\lambda \ge 1$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{(2\lambda-1)(2\lambda-1+\alpha)}}$$

and

$$|a_3| \le \begin{cases} \frac{4\alpha^2}{(2\lambda-1)^2} & , \quad 0<\alpha\le\frac{(2\lambda-1)^2}{2(3\lambda-1)} \\ \left(1-\frac{(2\lambda-1)^2}{2(3\lambda-1)\alpha}\right)\frac{4\alpha^2}{(2\lambda-1)(2\lambda-1+\alpha)}+\frac{2\alpha}{3\lambda-1} & , \quad \frac{(2\lambda-1)^2}{2(3\lambda-1)}\le\alpha\le 1 \end{cases}.$$

Remark 2.4. *Note that the coefficient estimates on $|a_3|$ in Corollary 2 is an improvement of the estimate obtained by Joshi et al. [10, Theorem 1].*

If we take $\lambda = 1$ in Corollary 2, then we get the following consequence.

Corollary 3. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{S}^*_\Sigma[\alpha]$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}$$

and

$$|a_3| \leq \begin{cases} 4\alpha^2 & , \quad 0 < \alpha \leq \frac{1}{4} \\ \frac{5\alpha^2}{1+\alpha} & , \quad \frac{1}{4} \leq \alpha < 1 \end{cases}.$$

If we consider the function φ_β, defined in Remark 1.2 (iii), in Theorem 2.1, then we get the following consequence.

Corollary 4. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{LB}_\Sigma(\lambda,\beta)$ and $\lambda \geq 1$. Then

$$|a_2| \leq \frac{2(1-\beta)}{\sqrt{(2\lambda-1)(2\lambda-1+|2\lambda\beta-1|)}}$$

and

$$|a_3| \leq \begin{cases} \left(1-\frac{(2\lambda-1)^2}{2(3\lambda-1)(1-\beta)}\right)\frac{4(1-\beta)^2}{(2\lambda-1)(2\lambda-1+|2\lambda\beta-1|)}+\frac{2(1-\beta)}{3\lambda-1} & , \quad 0 \leq \beta \leq 1-\frac{(2\lambda-1)^2}{2(3\lambda-1)} \\ \frac{4(1-\beta)^2}{(2\lambda-1)^2} & , \quad 1-\frac{(2\lambda-1)^2}{2(3\lambda-1)} \leq \beta < 1 \end{cases}.$$

Remark 2.5. *Note that Corollary 4 is an improvement of the estimates obtained by Joshi et al. [10, Theorem 2].*

If we take $\lambda = 1$ in Corollary 4, then we get the following consequence.

Corollary 5. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathscr{S}^*_\Sigma(\beta)$. Then

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)} & , \quad 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{\sqrt{2\beta}} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{5-6\beta}{2} & , \quad 0 \leq \beta < \frac{1}{2} \\ \frac{(1-\beta)(3-2\beta)}{2\beta} & , \quad \frac{1}{2} \leq \beta \leq \frac{3}{4} \\ 4(1-\beta)^2 & , \quad \frac{3}{4} \leq \beta < 1 \end{cases}.$$

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