Konuralp Journal of Mathematics, 6 (2) (2018) 311-320

Konuralp Journal of Mathematics

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*
e-ISSN: 2147-625X

The q-Dunkl wavelet packets

Ahmed Fitouhi[1], Slim Bouaziz[1*] and Kamel Mezlini[3]

[1] *Department of Mathematics, Faculty of Sciences of Tunis, El-Manar University, Tunis, Tunisia*
[2] *Department of Mathematics, Preparatory Institute of Engineer Studies of El-Manar, Tunis, Tunisia*
[3] *Department of Mathematics, Inst. Sup. Sc. Appl. Tech. of Mateur, Tunisia*
[*] *Corresponding author E-mail: bouazizslim1@yahoo.fr*

Abstract

Using the q-harmonic analysis associated with the q-Dunkl operator, we study three types of q-wavelet packets and their corresponding q-wavelet transforms. We give for these wavelet transforms the related Plancherel and inversion formulas as well as their q-scale discrete scaling functions.

Keywords: *q-Harmonic analysis; Packets; Wavelets.*
2010 Mathematics Subject Classification: *33D15; 33D60; 42C40; 42A38; 44A15; 44A20.*

1. Introduction

In 1982, J. Morlet introduced wavelets as a tool to study the analysis of seismic data. Taking account of the success of this method, he, joint with A. Grossmann, reactivated a collaboration between fundamental physics and theoretical signal processing, which led to the formalization of the continuous wavelet transforms, using some elements of classical harmonic analysis (see [6]). Since then, their results were generalized to many fields and many generalized Fourier analysis. The wavelet theory is motivated by the fact that certain algorithms that decompose a signal on the whole family of scales, can be utilized as an effective tool for multiscale analysis. In practical applications involving fast numerical algorithms, the continuous wavelet can be computed at discrete grid points. This theory involves breaking up a complicated function into many simple pieces at different scales and positions. It allows a greatly flexibility with more desirable features such as discretization by wavelet packets, and readiness for better implementation [3, 8, 13]. In general, wavelet packet decomposition divides the frequency space into various parts and allows better frequency localization of the signal.

In another corner of the world, in the time and place, the q-theory, called also in some literature ”quantum calculus”, began to arise. Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to the large number of its application domains. For instance, a lot of work has been carried out while developing some q-analogues of Fourier analysis using elements of quantum calculus (see [1, 9, 10, 11, 12] and references therein).
Since the classical harmonic analysis plays central role in the theory of wavelets and wavelet packets, it is natural to ask if we can apply the q-harmonic analysis to build new wavelets and wavelets packets. Many papers treated the notion of q-wavelets (see [2, 4, 5] and references therein) and gave some applications using q-harmonic analysis. In this paper, we are concerned with the notion of q-wavelet packets. We shall use the harmonic analysis associated with the q-Dunkl operator presented in [1, 2] to study some types of q-wavelet packets. We present a general construction, allowing the development of some types of q-wavelet packets starting from the so-mentioned q-Dunkl continuous wavelet analysis. Furthermore, we study its corresponding q-wavelet packet transform and we prove for this transform a Plancherel formula and an inversion theorem.
This paper is organized as follows: in Section 2, we present some notations and notions needed in the sequel. Section 3 is devoted to present some elements of the q-Dunkl harmonic analysis. We define and study in Sections 4 and 5, the q-Dunkl wavelet packets and the corresponding q-wavelet packets transforms as well as its q-scale discrete scaling functions.

2. Notations and preliminaries

Throughout this paper, we assume $q \in]0,1[$ and we denote

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

Email addresses**: , ahmed.fitouhi@fst.rnu.tn (Ahmed Fitouhi), bouazizslim1@yahoo.fr (Slim Bouaziz), kamel.mezlini@lamsin.rnu.tn (Kamel Mezlini)**

For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad n!_q = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The Rubin's q-differential operator is defined in [10, 11] by

$$\partial_q(f)(z) = \begin{cases} \frac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim_{x\to 0}\partial_q(f)(x) & if \quad z = 0. \end{cases} \tag{2.1}$$

Note that if f is differentiable at z, then $\partial_q(f)(z)$ tend to $f'(z)$ as q tends to 1.

The q-Jackson integrals [7] from 0 to a, from 0 to ∞ and from $-\infty$ to ∞ are defined by (see [7])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad \int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n \tag{2.2}$$

and

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n), \tag{2.3}$$

provided the sums converge absolutely.
The q-Jackson integral [7] in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.4}$$

In the particular case $a = bq^n$, $n \in \mathbb{N}$, the relation (2.4) becomes

$$\int_a^b f(x)d_qx = (1-q)b\sum_{k=0}^{n-1} f(q^k b)q^k. \tag{2.5}$$

In the sequel, we will need the following sets and spaces.
• $\mathscr{C}_{q,0}(\mathbb{R}_q)$ the space of bounded functions on $\mathbb{R}_q$, continued at 0 and vanishing at ∞.
• $\mathscr{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n,m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} | x^m \partial_q^n f(x) | < \infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

• $\mathscr{D}_q(\mathbb{R}_q)$ the subspace of $\mathscr{S}_q(\mathbb{R}_q)$ constituted of functions with compact supports.
• $\mathscr{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n \in \mathbb{N}, \;\; a \geq 0, \quad P_{n,a}(f) = \sup\left\{|\partial_q^k f(x)|; 0 \leq k \leq n; x \in [-a,a]\cap\mathbb{R}_q\right\} < \infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

• $L^p_{\alpha,q}(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_qx\right)^{\frac{1}{p}} < \infty \right\}$, $\quad p > 0$ and $\alpha \in \mathbb{R}$,
• $L^\infty_q(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}$.

3. Elements of q-Dunkl harmonic analysis

In this section, we collect some basic results and properties from the q-Dunkl operator theory, studied in [1]. In particular, we recall some properties of a generalized q-Dunkl translation as well as its related convolution product.

For $\alpha \geq -\frac{1}{2}$, the q-Dunkl transform is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by (see [1])

$$\mathscr{F}_D^{\alpha,q}(f)(\lambda) = c_{\alpha,q}\int_{-\infty}^{\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x).|x|^{2\alpha+1}d_qx, \quad \lambda \in \widetilde{\mathbb{R}}_q, \tag{3.1}$$

where

$$c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \tag{3.2}$$

and $\psi_\lambda^{\alpha,q}$ is the q-Dunkl kernel defined by

$$\psi_\lambda^{\alpha,q}: x \longmapsto j_\alpha(\lambda x;q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x;q^2), \tag{3.3}$$

with $j_\alpha(x;q^2)$ is the normalized third Jackson's q-Bessel function given by:
$j_\alpha(x;q^2) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}$.
It was proved in [1] that for all $\lambda \in \mathbb{C}$, the function: $x \longmapsto \psi_\lambda^{\alpha,q}(x)$ is the unique analytic solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q}(f) & = i\lambda f \\ f(0) & = 1, \end{cases} \tag{3.4}$$

where $\Lambda_{\alpha,q}$ is the q-Dunkl operator defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q \left[f_e + q^{2\alpha+1} f_o \right](x) + [2\alpha+1]_q \frac{f(x)-f(-x)}{2x}, \tag{3.5}$$

with f_e and f_o are respectively the even and the odd parts of f.
We recall that the q-Dunkl operator $\Lambda_{\alpha,q}$ lives the spaces $\mathscr{D}_q(\mathbb{R}_q)$ and $\mathscr{S}_q(\mathbb{R}_q)$ invariant.
It is worthy to claim that letting $q \uparrow 1$ subject to the condition

$$\frac{\ln(1-q)}{\ln(q)} \in 2\mathbb{Z}, \tag{3.6}$$

$\mathscr{F}_D^{\alpha,q}$ tends, at least formally, the classical Dunkl transform. In the remainder of this paper, we assume that the condition (3.6) holds.
Some other properties of the q-Dunkl kernel and the q-Dunkl transform are given in the following results (see [1]).

Theorem 3.1.
i) $\psi_\lambda^{\alpha,q}(x) = \psi_x^{\alpha,q}(\lambda)$, $\psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax)$, $\overline{\psi_\lambda^{\alpha,q}(x)} = \psi_{-\lambda}^{\alpha,q}(x)$, $\forall \lambda, x \in \mathbb{R}$, $a \in \mathbb{C}$.
ii) If $\alpha = -\frac{1}{2}$, *then* $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x;q^2)$.
For $\alpha > -\frac{1}{2}$, $\psi_\lambda^{\alpha,q}$ *has the following q-integral representation of Mehler type*

$$\psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)e(i\lambda xt;q^2)d_qt. \tag{3.7}$$

iii) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q}$ *is bounded on* $\widetilde{\mathbb{R}}_q$ *and we have*

$$\mid \psi_\lambda^{\alpha,q}(x) \mid \leq \frac{4}{(q;q)_\infty}, \quad \forall x \in \widetilde{\mathbb{R}}_q. \tag{3.8}$$

iv) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q} \in \mathscr{S}_q(\mathbb{R}_q)$.
v) The function $\psi_\lambda^{\alpha,q}$ *verifies the following orthogonality relation: For all* $x,y \in \mathbb{R}_q$,

$$\int_{-\infty}^{\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1}dq\lambda = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{x,y}}{(1-q)|xy|^{\alpha+1}}. \tag{3.9}$$

Theorem 3.2. *(1) If* $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ *then* $\mathscr{F}_D^{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_q)$,

$$\|\mathscr{F}_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q}, \tag{3.10}$$

and

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\mathbb{R}_q}} \mathscr{F}_D^{\alpha,q}(f)(\lambda) = 0, \quad \lim_{\substack{|\lambda|\to 0 \\ \lambda\in\widetilde{\mathbb{R}}_q}} \mathscr{F}_D^{\alpha,q}(f)(\lambda) = \mathscr{F}_D^{\alpha,q}(f)(0). \tag{3.11}$$

(2) For $f \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$\mathscr{F}_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda\mathscr{F}_D^{\alpha,q}(f)(\lambda). \tag{3.12}$$

(3) For $f,g \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$\int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q}(f)(\lambda)g(\lambda)|\lambda|^{2\alpha+1}d_q\lambda = \int_{-\infty}^{\infty} f(x)\mathscr{F}_D^{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx. \tag{3.13}$$

Theorem 3.3. *For all* $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\begin{aligned} \forall x \in \mathbb{R}_q, \quad f(x) &= c_{\alpha,q}\int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1}d_q\lambda \\ &= \overline{\mathscr{F}_D^{\alpha,q}(\overline{\mathscr{F}_D^{\alpha,q}(f)})}(x). \end{aligned} \tag{3.14}$$

Theorem 3.4. *i) Plancherel formula*
For $\alpha \geq -1/2$, the q-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $\mathscr{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in \mathscr{S}_q(\mathbb{R}_q)$, we have

$$\|\mathscr{F}_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{3.15}$$

ii) Plancherel theorem
The q-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. Its inverse transform $(\mathscr{F}_D^{\alpha,q})^{-1}$ is given by :

$$(\mathscr{F}_D^{\alpha,q})^{-1}(f)(x) = c_{\alpha,q}\int_{-\infty}^{\infty} f(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1}d_q\lambda = \mathscr{F}_D^{\alpha,q}(f)(-x). \tag{3.16}$$

We are now in a position to define the generalized q-Dunkl translation operator.

Definition 3.5. *The generalized q-Dunkl translation operator is defined for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$ by*

$$T_y^{\alpha,q}(f)(x) = c_{\alpha,q}\int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)|\lambda|^{2\alpha+1}d_q\lambda, \tag{3.17}$$

$$T_0^{\alpha;q}(f) = f.$$

It verifies the following properties.

Theorem 3.6.
1) For all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and all $x \in \mathbb{R}_q$,

$$\lim_{\substack{y\to 0\\ y\in\mathbb{R}_q}} T_y^{\alpha,q}(f)(x) = f(x).$$

2) For all $x, y \in \mathbb{R}_q$, $T_y^{\alpha,q}(f)(x) = T_x^{\alpha,q}(f)(y)$.
3) If $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$) then $T_y^{\alpha,q}(f) \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$) and we have

$$\| T_y^{\alpha,q}(f) \|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q} . \tag{3.18}$$

4) For all $x, y, \lambda \in \mathbb{R}_q$, $T_y^{\alpha,q}\left(\psi_\lambda^{\alpha,q}\right)(x) = \psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)$.
5) For $f \in L^2_{\alpha,q}(\mathbb{R}_q), x, y \in \mathbb{R}_q$, we have

$$\mathscr{F}_D^{\alpha,q}(T_y^{\alpha,q}f)(\lambda) = \psi_\lambda^{\alpha,q}(y)\mathscr{F}_D^{\alpha,q}(f)(\lambda). \tag{3.19}$$

6) For $f \in \mathscr{S}_q(\mathbb{R}_q)$ and $y \in \mathbb{R}_q$, we have

$$\Lambda_{\alpha,q}T_y^{\alpha,q}f = T_y^{\alpha,q}\Lambda_{\alpha,q}f.$$

Definition 3.7. *The q-convolution product is defined for $f, g \in \mathscr{S}_q(\mathbb{R}_q)$ by:*

$$f *_D g(x) = c_{\alpha,q}\int_{-\infty}^{\infty} T_x^{\alpha,q}f(-y)g(y)|y|^{2\alpha+1}d_qy. \tag{3.20}$$

In the following theorems, we present some of its properties.

Theorem 3.8. *For $f, g \in \mathscr{S}_q(\mathbb{R}_q)$, we have*
i) $\mathscr{F}_D^{\alpha,q}(f *_D g) = \mathscr{F}_D^{\alpha,q}(f).\mathscr{F}_D^{\alpha,q}(g)$.
ii) $f *_D g = g *_D f$.
iii) $(f *_D g) *_D h = f *_D (g *_D h)$.

Theorem 3.9. *Let f and g be in $\mathscr{S}_q(\mathbb{R}_q)$. Then*
1) $f *_D g \in \mathscr{S}_q(\mathbb{R}_q)$,
2)

$$\int_{-\infty}^{\infty} | f *_D g(x) |^2 |x|^{2\alpha+1}d_qx = \int_{-\infty}^{\infty} | \mathscr{F}_D^{\alpha,q}(f)(x) |^2| \mathscr{F}_D^{\alpha,q}(g)(x) |^2 |x|^{2\alpha+1}d_qx, \tag{3.21}$$

We finish this section by the following useful result.

Theorem 3.10. *For $a \in \mathbb{R}_q$ the operator H_a defined for $g \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$), by*

$$H_a(g)(x) = \frac{1}{|a|^{2\alpha+2}}g\left(\frac{x}{a}\right)$$

is linear and bijective from $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$) into itself and we have

$$\|H_a(g)\|_{2,\alpha,q} = \frac{1}{|a|^{\alpha+1}}\|g\|_{2,\alpha,q} \tag{3.22}$$

and

$$\mathscr{F}_D^{\alpha,q}(H_a(g))(\lambda) = \mathscr{F}_D^{\alpha,q}(g)(a\lambda), \ \lambda \in \widetilde{\mathbb{R}}_q. \tag{3.23}$$

Proof. The linearity and the bijectivity of H_a are clear. In Particular, $H_a^{-1} = H_{\frac{1}{a}}$. The change of variables $u = \frac{x}{a}$ completes the proof of the result. □

4. q-Dunkl Wavelet Packets

We recall that a Dunkl's q-wavelet is a square q-integrable function g on $\mathbb{R}_q$ satisfying the following admissibility condition:

$$0 < C_g = \int_0^\infty |\mathscr{F}_D^{\alpha,q}(g)(a)|^2 \frac{d_q a}{a} = \int_0^\infty |\mathscr{F}_D^{\alpha,q}(g)(-a)|^2 \frac{d_q a}{a} < \infty. \tag{4.1}$$

We consider a Dunkl's q-wavelet g and a strictly decreasing scale sequence $(r_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}_{q,+}$ satisfying $\lim_{j\to-\infty} r_j = \infty$, $\lim_{j\to\infty} r_j = 0$. We state the following introductory result.

Theorem 4.1. *For all $j \in \mathbb{Z}$, we have :*

1. *the function $\lambda \longmapsto \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathscr{F}_D^{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a}\right)^{\frac{1}{2}}$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$,*
2. *there exists a function $g_j^P \in L^2_{\alpha,q}(\mathbb{R}_q)$ such that for all $\lambda \in \mathbb{R}_q$,*

$$\mathscr{F}_D^{\alpha,q}(g_j^P)(\lambda) = \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathscr{F}_D^{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a}\right)^{\frac{1}{2}}.$$

Proof. Fix $j \in \mathbb{Z}$.
(1) On the one hand, r_j and r_{j+1} are two elements of $\mathbb{R}_{q,+}$ satisfying $r_{j+1} < r_j$, then there exists a positive integer n such that $r_{j+1} = q^n r_j$. So, using the relation (2.5) and Theorem 3.10, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathscr{F}_D^{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a}\right)|\lambda|^{2\alpha+1} d_q\lambda &= \frac{1-q}{C_g}\int_{-\infty}^{\infty}\sum_{k=0}^{n-1} |\mathscr{F}_D^{\alpha,q}(g)(\lambda q^k r_j)|^2 |\lambda|^{2\alpha+1} d_q\lambda \\ &= \frac{1-q}{C_g}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(g)(\lambda q^k r_j)|^2 |\lambda|^{2\alpha+1} d_q\lambda.\end{aligned}$$

On the other hand, the change of variable $u = \lambda q^k r_j$, ($0 \le k \le n-1$), together with Theorem 3.4 lead to

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathscr{F}_D^{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a}\right)|\lambda|^{2\alpha+1} d_q\lambda &= \frac{1-q}{C_g}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} \frac{|\mathscr{F}_D^{\alpha,q}(g)(u)|^2}{(r_j q^k)^{2\alpha+2}} |u|^{2\alpha+1} d_q u \\ &= \frac{1-q}{C_g}\|\mathscr{F}_D^{\alpha,q}(g)(u)\|^2_{2,\alpha,q}\sum_{k=0}^{n-1}\frac{1}{(r_j q^k)^{2\alpha+2}} \\ &= \frac{q^{2\alpha+2}}{C_g[2\alpha+2]_q}\left(\frac{1}{r_{j+1}^{2\alpha+2}} - \frac{1}{r_j^{2\alpha+2}}\right)\|g\|^2_{2,\alpha,q}.\end{aligned}$$

(2) The result follows from Theorem 3.4. □

Definition 4.2. *i) The sequence $(g_j^P)_{j\in\mathbb{Z}}$ is called Dunkl's q-wavelet packet.*
ii) The function g_j^P, $j \in \mathbb{Z}$, is called Dunkl's q-wavelet packet's member of step j.

We have the following immediate properties.

Theorem 4.3. *For all $\lambda \in \mathbb{R}_q$, we have*
$0 \le \mathscr{F}_D^{\alpha,q}(g_k^P)(\lambda) \le 1,\quad k \in \mathbb{Z}$ *and* $\sum_{j=-\infty}^{\infty}[\mathscr{F}_D^{\alpha,q}(g_j^P)(\lambda)]^2 = 1.$

Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Dunkl's q-wavelet packet. We introduce for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_q$, the function $g_{j,x}^P$ as

$$g_{j,x}^P(y) = T_y^{\alpha,q}(g_j^P)(x),\ \ y \in \widetilde{\mathbb{R}}_q. \tag{4.2}$$

Some properties of these functions are summarized in the following result that its proof follows easily from the properties of the q-Dunkl translation operator and the definition of the Dunkl's q-wavelet packets.

Theorem 4.4. *For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_q$, the function $g_{j,x}^P$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$ and we have for all $\lambda \in \widetilde{\mathbb{R}}_q$,*

- $\mathscr{F}_D^{\alpha,q}(g_{j,x}^P)(\lambda) = \psi_\lambda^{\alpha,q}(x)\mathscr{F}_D^{\alpha,q}(g_j^P)(\lambda)$.
- $\|g_{j,x}^P\|_{2,\alpha,q} \le \frac{4\|g_j^P\|_{2,\alpha,q}}{(q;q)_\infty}$.

Definition 4.5. *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Dunkl's q-wavelet packet. We define the Dunkl's q-wavelet packet transform $\Psi_{q,g}^P$ by*

$$\Psi_{q,g}^P(f)(j,y) = c_{\alpha,q}\int_{-\infty}^{\infty} f(x)\overline{g_{j,y}^P}(x)|x|^{2\alpha+1} d_q x,\ j \in \mathbb{Z},\ \ y \in \widetilde{\mathbb{R}}_q \quad \text{and} \quad f \in L^2_{\alpha,q}(\mathbb{R}_q), \tag{4.3}$$

where $c_{\alpha,q}$ is given by the relation (3.2).

Remark 4.6. *The equality (4.3) is equivalent to*

$$\Psi_{q,g}^P(f)(j,y) = \check{f} *_D \overline{g_j^P}(y) = \mathscr{F}_D^{\alpha,q}(\mathscr{F}_D^{\alpha,q}(\check{f} *_D \overline{g_j^P}))(-y) = \mathscr{F}_D^{\alpha,q}[\mathscr{F}_D^{\alpha,q}(\check{f}).\mathscr{F}_D^{\alpha,q}(\overline{g_j^P})](-y), \tag{4.4}$$

where $\check{f}(x) = f(-x)$.

The following theorem provides some useful properties of $\Psi^P_{q,g}$.

Theorem 4.7. *Let $(g^P_j)_{j\in\mathbb{Z}}$ be a Dunkl's q-wavelet packet and $f\in L^2_{\alpha,q}(\mathbb{R}_q)$. Then,*

1. *for all $j\in\mathbb{Z}$, $b\in\widetilde{\mathbb{R}}_q$, we have*
$$|\Psi^P_{q,g}(f)(j,b)|\leq\frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{2,\alpha,q}\left\|g^P_j\right\|_{2,\alpha,q};$$
2. *for all $j\in\mathbb{Z}$, the mapping $b\mapsto\Psi^P_{q,g}(f)(j,b)$ is continuous on $\widetilde{\mathbb{R}}_q$ and we have $\lim_{b\to\infty}\Psi^P_{q,g}(f)(j,b)=0$.*

Proof. (1) From the relation (4.3), Theorem 4.4 and the Cauchy-Schwarz inequality, we have for $j\in\mathbb{Z}$ and $b\in\mathbb{R}_q$

$$|\Psi^P_{q,g}(f)(j,b)|=c_{\alpha,q}\left|\int_{-\infty}^{\infty}f(x)\overline{g^P_{j,b}}(x)|x|^{2\alpha+1}d_qx\right|\leq c_{\alpha,q}||f||_{2,\alpha,q}||g^P_{j,b}||_{2,\alpha,q}\leq\frac{4c_{\alpha,q}}{(q;q)_\infty}||f||_{2,\alpha,q}||g^P_j||_{2,\alpha,q}.$$

(2) Let $j\in\mathbb{Z}$ and $f\in L^2_{\alpha,q}(\mathbb{R}_q)$. From Theorem 3.4, we have $\mathscr{F}^{\alpha,q}_D(\check{f})$ and $\mathscr{F}^{\alpha,q}_D(\overline{g^P_j})$ are in $L^2_{\alpha,q}(\mathbb{R}_q)$ and the product $\mathscr{F}^{\alpha,q}_D(\check{f})\mathscr{F}^{\alpha,q}_D(\overline{g^P_j})$ is in $L_{1,\alpha,q}(\mathbb{R}_q)$. So, the relation (4.4) together with Theorem 3.2 achieve the proof. □

The following result shows Plancheral and Parseval formulas for the Dunkl's q-wavelet packet transform $\Psi^P_{q,g}$.

Theorem 4.8. *Let $(g^P_j)_{j\in\mathbb{Z}}$ be a Dunkl's q-wavelet packet.*
(1) **Plancherel formula for $\Psi^P_{q,g}$**
For $f\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}|\ \Psi^P_{q,g}(f)(j,b)\ |^2\ |b|^{2\alpha+1}d_qb=\|f\|^2_{2,\alpha,q}. \tag{4.5}$$

(2) **Parseval formula for $\Psi^P_{q,g}$**
For $f_1,f_2\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty}f_1(x)\overline{f_2(x)}|x|^{2\alpha+1}d_qx=\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f_1)(j,b)\overline{\Psi^P_{q,g}(f_2)(j,b)}|b|^{2\alpha+1}d_qb. \tag{4.6}$$

Proof. (1) From the relations (3.21) and (4.4), we obtain

$$\int_{-\infty}^{\infty}|\Psi^P_{q,g}(f)(j,b)|^2|b|^{2\alpha+1}d_qb=\int_{-\infty}^{\infty}|\mathscr{F}^{\alpha,q}_D(\check{f})(a)|^2[\mathscr{F}^{\alpha,q}_D(g^P_j(a)]^2|a|^{2\alpha+1}d_qa.$$

So, the use of the Fubini's theorem and the fact that

$$\sum_{j=-\infty}^{\infty}[\mathscr{F}^{\alpha,q}_D(g^P_j)(\lambda)]^2=1,$$

give

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi^P_{q,g}(f)(j,b)|^2|b|^{2\alpha+1}d_qb &= \int_{-\infty}^{\infty}|\mathscr{F}^{\alpha,q}_D(\check{f})(a)|^2\sum_{j=-\infty}^{\infty}[\mathscr{F}^{\alpha,q}_D(g^P_j(a))]^2|a|^{2\alpha+1}d_qa\\ &= \int_{-\infty}^{\infty}|\mathscr{F}^{\alpha,q}_D(\check{f})(a)|^2|a|^{2\alpha+1}d_qa.\end{aligned}$$

Thus, (4.5) follows from Theorem 3.4.
(2) The result is a direct consequence of assertion (1). □

Theorem 4.9. *Let $(g^P_j)_{j\in\mathbb{Z}}$ be a Dunkl's q-wavelet packet. For $f\in L^2_{\alpha,q}(\mathbb{R}_q)$, one has the following reconstruction formula :*

$$f(x)=c_{\alpha,q}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f)(j,b)g^P_{j,b}(x)|b|^{2\alpha+1}d_qb,\quad x\in\mathbb{R}_q.$$

Proof. For $x\in\mathbb{R}_q$, we have $h=\delta_x$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$. Then, according to the relation (4.6), the definition of $\Psi^P_{q,g}$ and the definition of the q-Jackson's integral, we have

$$\begin{aligned}(1-q)|x|^{2\alpha+2}f(x) &= \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f)(j,b)\overline{\Psi^P_{q,g}(h)}(j,b)|b|^{2\alpha+1}d_qb\\ &= c_{\alpha,q}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f)(j,b)\left(\int_{-\infty}^{\infty}\overline{h}(t)g^P_{j,b}(t)|t|^{2\alpha+1}d_qt\right)|b|^{2\alpha+1}d_qb\\ &= (1-q)|x|^{2\alpha+2}c_{\alpha,q}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f)(j,b)g^P_{j,b}(x)|b|^{2\alpha+1}d_qb,\end{aligned}$$

which is equivalent to

$$f(x)=c_{\alpha,q}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^P_{q,g}(f)(j,b)g^P_{j,b}(x)|b|^{2\alpha+1}d_qb.$$

□

5. Dunkl's q-Scale discrete scaling function

In this section, we consider a Dunkl's q-wavelet packet $(g_j^P)_{j\in\mathbb{Z}}$.

Theorem 5.1. *1. For all $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, we have*

$$\sum_{j=-\infty}^{m-1}[\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2=\frac{1}{C_g}\int_{r_m}^{\infty}|\mathscr{F}_D^{\alpha,q}(H_a(g))(x)|^2\frac{d_qa}{a}. \tag{5.1}$$

2. *For all $m\in\mathbb{Z}$, the function $x\mapsto\left(\sum_{j=-\infty}^{m-1}[\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2\right)^{\frac{1}{2}}$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$.*
3. *For all $m\in\mathbb{Z}$ there exists a function G_m^P in $L^2_{\alpha,q}(\mathbb{R}_q)$ such that for all $x\in\mathbb{R}_q$,*

$$\mathscr{F}_D^{\alpha,q}(G_m^P)(x)=\left(\sum_{j=-\infty}^{m-1}[\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2\right)^{\frac{1}{2}}. \tag{5.2}$$

Proof. (1) follows from the definition of g_j^P.
(2) From the Fubini's theorem, the relation (5.1) and Theorem 3.10, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\sum_{j=-\infty}^{m-1}[\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2|x|^{2\alpha+1}d_qx &= \frac{1}{C_g}\int_{-\infty}^{\infty}\int_{r_m}^{\infty}|\mathscr{F}_D^{\alpha,q}(H_a(g))(x)|^2\frac{d_qa}{a}|x|^{2\alpha+1}d_qx\\ &= \frac{1}{C_g}\int_{r_m}^{\infty}\left(\int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(g)(ax)|^2|x|^{2\alpha+1}d_qx\right)\frac{d_qa}{a}.\end{aligned}$$

By the change of variables $u=ax$ and Theorem 3.4, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{1}{C_g}\int_{r_m}^{\infty}|\mathscr{F}_D^{\alpha,q}(H_a(g))(x)|^2\frac{d_qa}{a}\right)|x|^{2\alpha+1}d_qx &= \frac{1}{C_g}\int_{r_m}^{\infty}\left(\int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(g)(x)|^2|x|^{2\alpha+1}d_qx\right)\frac{d_qa}{a^{2\alpha+3}}\\ &= \frac{\|g\|_{2,\alpha,q}}{C_g}\int_{r_m}^{\infty}\frac{d_qa}{a^{2\alpha+3}}<\infty.\end{aligned}$$

This completes the proof of (2).
(3) We deduce the result from the previous assertion and Theorem 3.4. □

Definition 5.2. *The sequence $(G_m^P)_{m\in\mathbb{Z}}$ is called Dunkl's q-scale discrete scaling function.*

The sequence $(G_m^P)_{m\in\mathbb{Z}}$ verifies the following trivial and easily proved properties.

Theorem 5.3.

(i) For all $m\in\mathbb{Z}$ and $\lambda\in\mathbb{R}_q$, we have

$$0\le\mathscr{F}_D^{\alpha,q}(G_m^P)(\lambda)\le 1. \tag{5.3}$$

(ii) For all $\lambda\in\mathbb{R}_q$, we have

$$\lim_{m\to\infty}\mathscr{F}_D^{\alpha,q}(G_m^P)(\lambda)=1. \tag{5.4}$$

Theorem 5.4. *For $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the following relations*

(i)

$$\left[\mathscr{F}_D^{\alpha,q}(G_m^P)(x)\right]^2+\sum_{j=m}^{\infty}[\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2=1, \tag{5.5}$$

(ii)

$$\left[\mathscr{F}_D^{\alpha,q}(g_m^P)(x)\right]^2=\left[\mathscr{F}_D^{\alpha,q}(G_{m+1}^P)(x)\right]^2-\left[\mathscr{F}_D^{\alpha,q}(G_m^P)(x)\right]^2, \tag{5.6}$$

(iii)

$$\sum_{m=-\infty}^{\infty}\left(\left[\mathscr{F}_D^{\alpha,q}(G_{m+1}^P)(x)\right]^2-\left[\mathscr{F}_D^{\alpha,q}(G_m^P)(x)\right]^2\right)=1 \tag{5.7}$$

hold.

Proof.

(i) Follows immediately from (5.2) and Theorem 4.3.

(ii) We deduce the result from the relation (5.2).

(iii) The relation is a consequence of (5.6) and Theorem 4.3. □

Now, let $(G_m^P)_{m\in\mathbb{Z}}$ be a Dunkl's q-scale discrete scaling function and consider for all $m\in\mathbb{Z}$, $x\in\mathbb{R}_q$, the function $G_{m,x}^P$ given by

$$G_{m,x}^P(y)=T_y^{\alpha,q}(G_m^P)(x),\ \ \forall y\in\mathbb{R}_q. \tag{5.8}$$

From the properties of the Dunkl's q-translation, one can prove easily the following result giving some properties of the function $G_{m,x}^P$.

Theorem 5.5. *For all $m\in\mathbb{Z}$ and $x\in\mathbb{R}_q$, the function $G_{m,x}^P$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$ and we have*

- $\mathscr{F}_D^{\alpha,q}(G_{m,x}^P)(\lambda)=\psi_\lambda^{\alpha,q}(x)\mathscr{F}_D^{\alpha,q}(G_m^P)(\lambda),\ \ \lambda\in\mathbb{R}_q,$
- $\|G_{m,x}^P\|_{2,\alpha,q}\le\frac{4\|G_m^P\|_{2,\alpha,q}}{(q;q)_\infty}.$

Definition 5.6. *Let $(G_m^P)_{m\in\mathbb{Z}}$ be a Dunkl's q-scale discrete scaling function. We define the Dunkl's q-scale discrete scaling transform $\Theta_{q,G}^P$ on $L^2_{\alpha,q}(\mathbb{R}_q)$, by*

$$\Theta_{q,G}^P(f)(m,x)=c_{\alpha,q}\int_{-\infty}^{\infty}f(b)\overline{G_{m,x}^P(b)}|b|^{2\alpha+1}d_qb,\quad m\in\mathbb{Z},\ \text{and}\ \ x\in\mathbb{R}_q. \tag{5.9}$$

Remark 5.7. *The relation (5.9) is equivalent to*

$$\Theta_{q,G}^P(f)(m,x)=\check{f}*_D\overline{G_m^P}(x). \tag{5.10}$$

In the two following results, we will provide a Plancheral and a Parseval formulas for $\Theta_{q,G}^P$.

Theorem 5.8. *Let $(G_m^P)_{m\in\mathbb{Z}}$ be a Dunkl's q-scale discrete scaling function.*
(1) **Plancherel formula for $\Theta_{q,G}^P$**
For $f\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\|f\|^2_{2,\alpha,q}=\lim_{m\to\infty}\int_{-\infty}^{\infty}|\Theta_{q,G}^P(f)(m,b)|^2|b|^{2\alpha+1}d_qb. \tag{5.11}$$

(2) **Parseval formula for $\Theta_{q,G}^P$**
For $f_1,f_2\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty}f_1(x)\overline{f}_2(x)|x|^{2\alpha+1}d_qx=\lim_{m\to\infty}\int_{-\infty}^{\infty}\Theta_{q,G}^P(f_1)(m,b)\overline{\Theta_{q,G}^P(f_2)(m,b)}|b|^{2\alpha+1}d_qb. \tag{5.12}$$

Proof. (1) Due to the relations (5.10) and (3.21), we have for all $m\in\mathbb{Z}$,

$$\int_{-\infty}^{\infty}|\Theta_{q,G}^P(f)(m,b)|^2|b|^{2\alpha+1}d_qb=\int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(\check{f})(x)|^2[\mathscr{F}_D^{\alpha,q}(G_m^P)(x)]^2|x|^{2\alpha+1}d_qx. \tag{5.13}$$

The relations (5.3) and (5.4), and the Lebesgue's theorem yield to

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}|\Theta_{q,G}^P(f)(m,b)|^2|b|^{2\alpha+1}d_qb=\left\|\mathscr{F}_D^{\alpha,q}(\check{f})\right\|^2_{2,\alpha,q}.$$

Finally, Theorem 3.4 achieves the proof of (1).
(2) The result follows from (5.11). □

Using the Dunkl's q-scale discrete scaling function $(G_m^P)_{m\in\mathbb{Z}}$ and the Dunkl's q-wavelet packet transform $\Psi_{q,g}^P$, one can obtain another Plancheral formula for $\Theta_{q,G}^P$. This is the aim of the following result.

Theorem 5.9. *(1)* **Plancherel formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For all $f\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have for all $m\in\mathbb{Z}$,

$$\|f\|^2_{2,\alpha,q}=\int_{-\infty}^{\infty}|\Theta_{q,G}^P(f)(m,b)|^2d_qb+\sum_{j=m}^{\infty}\int_{-\infty}^{\infty}|\Psi_{q,g}^P(f)(j,b)|^2|b|^{2\alpha+1}d_qb. \tag{5.14}$$

(2) **Parseval formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For $f_1,f_2\in L^2_{\alpha,q}(\mathbb{R}_q)$, we have for all $m\in\mathbb{Z}$,

$$\int_{-\infty}^{\infty}f_1(x)\overline{f}_2(x)|x|^{2\alpha+1}d_qx=\int_{-\infty}^{\infty}\Theta_{q,G}^P(f_1)(m,b)\overline{\Theta_{q,G}^P(f_2)(m,b)}|b|^{2\alpha+1}d_qb\quad+$$
$$\sum_{j=m}^{\infty}\int_{-\infty}^{\infty}\Psi_{q,g}^P(f_1)(j,b)\overline{\Psi_{q,g}^P(f_2)(j,b)}|b|^{2\alpha+1}d_qb.$$

Proof. (1) On the one hand, from the relations (5.13) and (5.2), we have for all $m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} |\Theta_{q,G}^{P}(f)(m,b)|^2 |b|^{2\alpha+1} d_q b = \int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(\check{f})(x)|^2 \left(\sum_{j=-\infty}^{m-1} \left[\mathscr{F}_D^{\alpha,q}(g_j^P)(x) \right]^2 \right) |x|^{2\alpha+1} d_q x.$$

On the other hand, using the relations (3.21) and (4.4), and the Fubini's theorem, we obtain

$$\sum_{j=m}^{\infty} \int_{-\infty}^{\infty} |\Psi_{q,g}^{P}(f)(j,b)|^2 |b|^{2\alpha+1} d_q b = \int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(\check{f})(x)|^2 \left(\sum_{j=m}^{\infty} \left[\mathscr{F}_D^{\alpha,q}(g_j^P)(x) \right]^2 \right) |x|^{2\alpha+1} d_q x.$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} |\Theta_{q,G}^{P}(f)(m,b)|^2 |b|^{2\alpha+1} d_q b \quad + \quad & \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} |\Psi_{q,g}^{P}(f)(j,b)|^2 |b|^{2\alpha+1} d_q b = \\ & \int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(\check{f})(x)|^2 \left(\sum_{j=-\infty}^{\infty} [\mathscr{F}_D^{\alpha,q}(g_j^P)(x)]^2 \right) |x|^{2\alpha+1} d_q x. \end{aligned}$$

The result follows then from Theorem 4.3 and Theorem 3.4.
(2) The assertion (2) follows from (1). □

Theorem 5.10. *For $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have the following reconstruction formulas.*
(1) For all $x \in \mathbb{R}_q$,

$$f(x) = c_{\alpha,q} \lim_{m \to \infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) |b|^{2\alpha+1} d_q b. \tag{5.15}$$

(2) For all $x \in \mathbb{R}_q$ and all $m \in \mathbb{Z}$,

$$f(x) = c_{\alpha,q} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) |b|^{2\alpha+1} d_q b + c_{\alpha,q} \sum_{j=m}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b. \tag{5.16}$$

Proof. (1) Let $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, fix $x \in \mathbb{R}_q$ and put $h = \delta_x$. By using the relation (5.12), we get

$$\begin{aligned} (1-q)|x|^{2\alpha+2} f(x) &= \lim_{m \to \infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) \overline{\Theta_{q,G}^{P}(h)(m,b)} |b|^{2\alpha+1} d_q b \\ &= \lim_{m \to \infty} c_{\alpha,q} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) \left(\int_{-\infty}^{\infty} \bar{h}(t) G_{m,b}^{P}(t) |t|^{2\alpha+1} d_q t \right) |b|^{2\alpha+1} d_q b \\ &= \lim_{m \to \infty} c_{\alpha,q} (1-q) |x|^{2\alpha+2} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) |b|^{2\alpha+1} d_q b. \end{aligned}$$

Thus,

$$f(x) = c_{\alpha,q} \lim_{m \to \infty} \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(m,b) G_{m,b}^{P}(x) |b|^{2\alpha+1} d_q b.$$

(2) The technique of the proof is similar to (1). □

Theorem 5.11. *For $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, one has for all $j \in \mathbb{Z}$,*

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b = \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(j+1,b) G_{j+1,b}^{P}(x) |b|^{2\alpha+1} d_q b \quad - \\ \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(j,b) G_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b. \end{aligned}$$

Proof. Using the relations (5.16) and (5.6), and Theorem 3.4, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(j+1,b) G_{j+1,b}^{P}(x) |b|^{2\alpha+1} d_q b - \int_{-\infty}^{\infty} \Theta_{q,G}^{P}(f)(j,b) G_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b \\ = & \int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q} [\mathscr{F}_D^{\alpha,q}(\check{f} *_D \overline{G_{j+1}^P})](-b) G_{j+1,b}^{P}(x) |b|^{2\alpha+1} d_q b - \int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q} [\mathscr{F}_D^{\alpha,q}(\check{f} *_D \overline{G_j^P})](-b) G_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b \\ = & \int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q}(\check{f})(b) \left(\left[\mathscr{F}_D^{\alpha,q}(G_{j+1}^P) \right]^2 - \left[\mathscr{F}_D^{\alpha,q}(G_j^P) \right]^2 \right) (b) \psi_b^{\alpha,q}(x) |b|^{2\alpha+1} d_q b \\ = & \int_{-\infty}^{\infty} \mathscr{F}_D^{\alpha,q}(\check{f})(b) \left[\mathscr{F}_D^{\alpha,q}(g_j^P) \right]^2 (b) \psi_b^{\alpha,q}(x) |b|^{2\alpha+1} d_q b \\ = & \int_{-\infty}^{\infty} \Psi_{q,g}^{P}(f)(j,b) g_{j,b}^{P}(x) |b|^{2\alpha+1} d_q b. \end{aligned}$$

□

Acknowledgement

This is a text of acknowledgements. Do not forget people who have assisted you on your work. Do not exaggerate with thanks. If your work has been paid by a Grant, mention the Grant name and number here.

References

[1] N. Bettaibi and R. H. Bettaieb, *q-Analogue of the Dunkl transform on the real line*, Tamsui Oxford Journal of Mathematical Sciences, 25(2)(2007), 117-205

[2] N. Bettaibi, R. H. Bettaieb and S. Bouaziz, *Wavelet transform associated with the q-Dunkl operator*, Tamsui Oxford Journal of Mathematical Sciences, 26(1) (2010) 77-101.

[3] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, V. 160, SIAM, Philadelfia, PA, 1992.

[4] A. Fitouhi, N. Bettaibi, *Wavelet Transform in Quantum Calculus*. J. Non. Math. Phys. 13, (2006), 492-506.

[5] A. Fitouhi and R. H. Bettaieb, Wavelet Transform in the q^2-Analogue Fourier Analysis, Math. Sci. Res. J. 12 (2008), no. 9, 202–214.

[6] Grossman A and Morlet J, *Decomposition of Hardy functions into square integrable wavelets of constant shape*, SIAM J. Math. Anal. 15 (1984), 723–736.

[7] F. H. Jackson, *On a q-Definite Integrals*. Quarterly Journal of Pure and Applied Mathematics 41, 1910, 193-203.

[8] T. H. Koornwinder, *The continuous Wavelet Transform*, Series in Approximations and decompositions, Vol. 1, Wavelets: An Elementary Treatment of Theory and Applications. Edited by T. H. Koornwinder, World Scientific, 1993, $27-48$.

[9] T. H. Koornwinder and R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. 333, 1992, 445-461.

[10] R. L. Rubin, *A q^2-Analogue Operator for q^2- analogue Fourier Analysis*, J. Math. Analys. App. 212, 1997, $571-582$.

[11] R. L. Rubin, *Duhamel Solutions of non-Homogenous q^2- Analogue Wave Equations*, Proc. of Amer. Maths. Soc. V135, Nr 3, 2007, $777-785$.

[12] F. Soltani, *Fock spaces for the q-Dunkl kernel*, The Advances in Pure Mathematics (APM), 2(3) (2012) pp. 169-176 DOI: 10.4236/apm.2012.23023

[13] K. Trimèche, *Generalized harmonic analysis and wavelet packets*, Gordon and Breach Science Publishers, 2001.

[14] O'Neill, B., *Semi Riemannian geometry with applications to rela*tivity, Academic Press, Inc. New York, 1983.

[15] Hacısalihoğlu, H. H., *Diferensiyel geometri*, Cilt I-II, Ankara Üniversitesi, Fen Fakültesi Yayınları, 2000.

[16] A. Görgülü and A. C. Çöken, The Euler theorem for parallel pseudo-Euclidean hypersurfaces in pseudo-Euclideanspace $\mathbb{E}_1^{n+1}$, Journ. Inst. Math. and Comp. Sci. (Math. Series) Vol:6, No.2 (1993), 161-165.