Konuralp Journal of Mathematics, 6 (2) (2018) 321-331

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

An Extension of Lowen's Uniformity to the Fuzzy Soft Sets

İzzettin Demir[1] **and Oya Bedre Özbakır**[2*]

[1]*Department of Mathematics, Duzce University, 81620, Duzce-Turkey*
[2]*Department of Mathematics, Ege University, 35100, Izmir-Turkey*
[*]*Corresponding author E-mail: oya.ozbakir@ege.edu.tr*

Abstract

In this paper, first we define the notion of a saturated fuzzy soft filter. Afterwards, we introduce the notion of a fuzzy soft uniformity as a generalization of uniformity in the sense of Lowen [22]. Also, we show how a fuzzy soft topology is derived from a fuzzy soft uniformity. Then, we give a new kind of fuzzy soft neighborhood system and investigate its relationship with a fuzzy soft uniformity. Finally, we show that a fuzzy soft uniformly continuous mapping is a fuzzy soft continuous.

Keywords: *Fuzzy soft set, fuzzy soft topology, saturated fuzzy soft filter, fuzzy soft uniformity, fuzzy soft uniformly continuous mapping.*
2010 Mathematics Subject Classification: *06D72, 54A40, 54E15.*

1. Introduction

In 1999, Molodtsov [29] introduced the concept of a soft set theory as a new mathematical tool for dealing with uncertainties. This theory provides a very general framework with the involvement of parameters. The parameters can be expressed in the form of words, sentences, real numbers and so on. Hence, soft set theory has attractive applications in other disciplines and real life problems, most of these applications was shown by Molodtsov [29]. Recently, researchers are contributing a lot regarding soft set theory and its applications [14,16,26,28,33,40].

Maji et.al. [27] combined the concept of fuzzy set and soft set and introduced the new notion of the fuzzy soft set. Roy and Maji [32] presented some results on an application of fuzzy soft sets in decision making problem. Then, Tanay and Kandemir [36] initiated the concept of a fuzzy soft topology and gave the some basic properties of it by following Chang [8]. Also, the fuzzy soft topology in Lowen's sense [21] was given by Varol and Aygün [38]. In recent years, there have been considerable advances in fuzzy soft sets and their applications [1,3,11–13,19,30].

Uniformity is a very important concept close to topology and a suitable tool for an investigating of topology. Many authors have obtained the concept of fuzzy uniformity in different approaches. Using the entourage approach of Weil [39], Lowen [22] and Höhle [15] defined the fuzzy uniformities in many-valued setting and lattice-valued setting, respectively. Fuzzy uniformity in Lowen's sense has been accepted by many authors and has attracted wide attention in the literature [2,4–7,18,23–25,35]. It should also be pointed out that there are other approaches such as the uniform covering approach of Tukey [37] and uniform operator approach articulated in Kotze [20], which was first given in the fuzzy setting by Hutton [17]. Then, Shi [34] introduced the theory of pointwise uniformities on fuzzy sets. Zhang [41] presented a comparison of various uniformities in fuzzy topology.

Extensions of uniform structures to the soft sets and also fuzzy soft sets have been studied by some authors. By using soft sets, Çetkin and Aygün [9] defined soft uniform spaces. As distinct from Çetkin and Aygün's definition, Demir and Özbakır [31] introduced a new concept of soft uniformity and studied its basic properties. Then, Çetkin and Aygün [10] proposed Shi's uniformity in the context of fuzzy soft sets.
In this work, with the help of fuzzy soft sets, we define the concept of a saturated fuzzy soft filter and investigate some of its properties. By using this, we introduce the notion of a fuzzy soft uniformity on the base of the axioms suggested by Lowen [22] and obtain some results analogous to those that hold for Lowen's uniformity. Also, we compare it to a fuzzy soft topology and a fuzzy soft neighborhood system. Moreover, we establish that a fuzzy soft uniformly continuous mapping is a fuzzy soft continuous.

2. Preliminaries

In this section, we recall some basic notions regarding fuzzy soft sets which will be used in the sequel. Throughout this work, let X be an initial universe, I^X be the set of all fuzzy subsets of X and E be the set of all parameters for X.

Email addresses: **, izzettindemir@duzce.edu.tr (İzzettin Demir), oya.ozbakir@ege.edu.tr (Oya Bedre Özbakır)**

Definition 2.1 ([27]). *A fuzzy soft set f on the universe X with the set E of parameters is defined by the set of ordered pairs*

$f = \{(e, f(e)) : e \in E, f(e) \in I^X\}$

where f is a mapping given by $f : E \to I^X$.

Throughout this paper, the family of all fuzzy soft sets over X is denoted by $FS(X,E)$.

Definition 2.2 ([27]). *Let $f, g \in FS(X,E)$. Then,*
(i) The fuzzy soft set f is called a null fuzzy soft set, denoted by $\widetilde{\emptyset}$, if $f(e) = 0_X$ for all $e \in E$.
(ii) The fuzzy soft set f is called an absolute fuzzy soft set, denoted by $\widetilde{X}$, if $f(e) = 1_X$ for all $e \in E$.
(iii) f is a fuzzy soft subset of g if $f(e) \leq g(e)$ for all $e \in E$. It is denoted by $f \sqsubseteq g$.
(iv) The union of f and g is a fuzzy soft set h defined by $h(e) = f(e) \vee g(e)$ for all $e \in E$. h is denoted by $f \sqcup g$.

Definition 2.3 ([1]). *Let $f, g \in FS(X,E)$. Then, the intersection of f and g is a fuzzy soft set h defined by $h(e) = f(e) \wedge g(e)$ for all $e \in E$. h is denoted by $f \sqcap g$.*

Definition 2.4 ([38]). *Let $f \in FS(X,E)$. Then, the complement of f is denoted by f^c, where $f^c : E \to I^X$ is a mapping defined by $f^c(e) = 1_X - f(e)$ for all $e \in E$. Clearly, $(f^c)^c = f$.*

Definition 2.5 ([1]). *Let J be an arbitrary index set and let $\{f_i\}_{i\in J}$ be a family of fuzzy soft sets over X. Then,*
(i) The union of these fuzzy soft sets is the fuzzy soft set h defined by $h(e) = \bigvee_{i\in J} f_i(e)$ for every $e \in E$ and this fuzzy soft set is denoted by $\bigsqcup_{i\in J} f_i$.
(ii) The intersection of these fuzzy soft sets is the fuzzy soft set h defined by $h(e) = \bigwedge_{i\in J} f_i(e)$ for every $e \in E$ and this fuzzy soft set is denoted by $\bigsqcap_{i\in J} f_i$.

Theorem 2.6 ([38]). *Let J be an index set and f, g, $f_i \in FS(X,E)$ for all $i \in J$. Then, the following statements are satisfied.*
(i) $(\bigsqcap_{i\in J} f_i)^c = \bigsqcup_{i\in J} f_i^c$.
(ii) $(\bigsqcup_{i\in J} f_i)^c = \bigsqcap_{i\in J} f_i^c$.
(iii) If $f \sqsubseteq g$, then $g^c \sqsubseteq f^c$.

Definition 2.7 ([19]). *Let $FS(X,E)$ and $FS(Y,K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two mappings. Then, the mapping φ_ψ is called a fuzzy soft mapping from X to Y, denoted by $\varphi_\psi : FS(X,E) \to FS(Y,K)$.*

(i) *Let $f \in FS(X,E)$. Then $\varphi_\psi(f)$ is the fuzzy soft set over Y defined as follows:*

$$\varphi_\psi(f)(k)(y) = \begin{cases} \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(k)} f(e)\right)(x), & \text{if } \psi^{-1}(k) \neq \emptyset,\ \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for all $k \in K$ and all $y \in Y$.
$\varphi_\psi(f)$ is called a image of a fuzzy soft set f.

(ii) *Let $g \in FS(Y,K)$. Then $\varphi_\psi^{-1}(g)$ is the soft set over X defined as follows:*

$$\varphi_\psi^{-1}(g)(e)(x) = g(\psi(e))(\varphi(x))$$

for all $e \in E$ and all $x \in X$.
$\varphi_\psi^{-1}(g)$ is called a pre-image of a fuzzy soft set g.

The fuzzy soft mapping φ_ψ is called injective, if φ and ψ are injective. The fuzzy soft mapping φ_ψ is called surjective, if φ and ψ are surjective.

Theorem 2.8 ([19]). *Let $f_1, f_2, f_i \in FS(X,E)$ and $g_1, g_2, g_i \in FS(Y,K)$ for all $i \in J$ where J is an index set. Then, for a fuzzy soft mapping $\varphi_\psi : FS(X,E) \to FS(Y,K)$, the following conditions are satisfied.*
(i) If $f_1 \sqsubseteq f_2$, then $\varphi_\psi(f_1) \sqsubseteq \varphi_\psi(f_2)$.
(ii) If $g_1 \sqsubseteq g_2$, then $\varphi_\psi^{-1}(g_1) \sqsubseteq \varphi_\psi^{-1}(g_2)$.
(iii) $\varphi_\psi(\bigsqcup_{i\in J} f_i) = \bigsqcup_{i\in J} \varphi_\psi(f_i)$.
(iv) $\varphi_\psi^{-1}(\bigsqcup_{i\in J} g_i) = \bigsqcup_{i\in J} \varphi_\psi^{-1}(g_i)$.
(v) $\varphi_\psi^{-1}(\bigsqcap_{i\in J} g_i) = \bigsqcap_{i\in J} \varphi_\psi^{-1}(g_i)$.

Theorem 2.9 ([38]). *Let $f \in FS(X,E)$ and let $g \in FS(Y,K)$. Then, for a fuzzy soft mapping $\varphi_\psi : FS(X,E) \to FS(Y,K)$, the following conditions are satisfied.*
(i) $f \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(f))$.
(ii) $\varphi_\psi(\varphi_\psi^{-1}(g)) \sqsubseteq g$.

Definition 2.10 ([36]). *Let τ be the collection of fuzzy soft sets over X. Then, τ is said to be a fuzzy soft topology on X if*
(FST1) $\widetilde{\emptyset}, \widetilde{X}$ belong to τ.

(FST2) the union of any number of fuzzy soft sets in τ belongs to τ.
(FST3) the intersection of any two fuzzy soft sets in τ belongs to τ.
(X,τ,E) is called a fuzzy soft topological space. The members of τ are called fuzzy soft open sets in X. A fuzzy soft set f over X is called a fuzzy soft closed in X if $f^c \in \tau$.

Definition 2.11 ([36]). *Let (X,τ,E) be a fuzzy soft topological space and $f \in FS(X,E)$. The fuzzy soft interior of f is the fuzzy soft set $f^o = \bigsqcup\{g : g\ is\ a\ fuzzy\ soft\ open\ set\ and\ g \sqsubseteq f\}$.*

Definition 2.12 ([30, 38]). *Let (X,τ,E) be a fuzzy soft topological space and $f \in FS(X,E)$. The fuzzy soft closure of f is the fuzzy soft set $\overline{f} = \bigsqcap\{g : g\ is\ a\ fuzzy\ soft\ closed\ set\ and\ f \sqsubseteq g\}$.*

Theorem 2.13 ([38]). *Let us consider an operator associating with each fuzzy soft set f on X another fuzzy soft set $\overline{f}$ such that the following properties hold:*
(FSO1) $f \sqsubseteq \overline{f}$,
(FSO2) $\overline{\overline{f}} = \overline{f}$,
(FSO3) $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$,
(FSO4) $\overline{\widetilde{\emptyset}} = \widetilde{\emptyset}$.

Then, the family

$$\tau = \{ f \in FS(X,E) : \overline{f^c} = f^c \}$$

defines a fuzzy soft topology on X and for every $f \in FS(X,E)$, the fuzzy soft set $\overline{f}$ is the fuzzy soft closure of f in the fuzzy soft topological space (X,τ,E).

This operator is called the fuzzy soft closure operator.

Definition 2.14 ([38]). *Let (X,τ_1,E) and (Y,τ_2,K) be two fuzzy soft topological spaces. A fuzzy soft mapping $\varphi_\psi : (X,\tau_1,E) \to (Y,\tau_2,K)$ is called fuzzy soft continuous if $\varphi_\psi^{-1}(g) \in \tau_1$ for every $g \in \tau_2$.*

Theorem 2.15 ([38]). *Let (X,τ_1,E) and (Y,τ_2,K) be two fuzzy soft topological spaces and $\varphi_\psi : (X,\tau_1,E) \to (Y,\tau_2,K)$ be a fuzzy soft mapping. Then, the following conditions are equivalent:*
(i) φ_ψ is fuzzy soft continuous.
(ii) For every $f \in FS(X,E)$, $\varphi_\psi(\overline{f}) \sqsubseteq \overline{\varphi_\psi(f)}$.

Definition 2.16 ([11]). *A fuzzy soft filter $\mathcal{F}$ on X is a nonempty collection of subsets of $FS(X,E)$ with the following properties:*
(FSF1) $\widetilde{\emptyset} \notin \mathcal{F}$.
(FSF2) If $f,g \in \mathcal{F}$, then $f \sqcap g \in \mathcal{F}$.
(FSF3) If $f \in \mathcal{F}$ and $f \sqsubseteq g$, then $g \in \mathcal{F}$.
If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fuzzy soft filters on X, we say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is coarser than $\mathcal{F}_1$) iff $\mathcal{F}_1 \supseteq \mathcal{F}_2$.

Definition 2.17 ([12]). *A collection $\mathcal{B}$ of subsets of $FS(X,E)$ is called a base for a fuzzy soft filter on X if the following two conditions are satisfied :*
(FSB1) $\mathcal{B} \neq \emptyset$ and $\widetilde{\emptyset} \notin \mathcal{B}$,
(FSB2) If $f,g \in \mathcal{B}$, then there is an $h \in FS(X,E)$ such that $h \sqsubseteq f \sqcap g$.

If $\mathcal{B}$ is a base for a fuzzy soft filter on X, then the collection

$$\langle \mathcal{B} \rangle = \{f \in FS(X,E) : there\ exists\ an\ h \in \mathcal{B}\ such\ that\ h \sqsubseteq f\}$$

is a fuzzy soft filter on X. We say that the fuzzy soft filter $\langle \mathcal{B} \rangle$ is generated by $\mathcal{B}$

Theorem 2.18 ([12]). *Let $\varphi_\psi : FS(X,E) \to FS(Y,K)$ be a fuzzy soft mapping and let $\mathcal{F}$ be a fuzzy soft filter on X. Then, $\{\varphi_\psi(f) : f \in \mathcal{F}\}$ is a base for a fuzzy soft filter $\varphi_\psi(\mathcal{F})$ on Y.*

3. Saturated Fuzzy Soft Filters

Now, we define the concept of a saturated fuzzy soft filter which will play a crucial role in the next section and obtain its some related properties.
Throughout this paper, $[0,1]$ and $(0,1]$ are denoted by I and I_0, respectively.

Definition 3.1. *Let $f \in FS(X,E)$ and $\epsilon \in I$. Then, the mappings $(f+\epsilon) : E \to I^X$ and $(f-\epsilon) : E \to I^X$ is the fuzzy soft sets on X defined as follows:*

$$(f+\epsilon)(e)(x) = (f(e)(x)+\epsilon) \wedge 1 \ \ and \ \ (f-\epsilon)(e)(x) = (f(e)(x)-\epsilon) \vee 0$$

for all $e \in E$ and all $x \in X$.

Let $\mathcal{F}$ be a fuzzy soft filter base on X. We define

$$\widehat{\mathcal{F}} = \{\bigsqcup_{\epsilon\in I_0}(f_\epsilon - \epsilon) : f_\epsilon \in \mathcal{F} \ for\ all\ \epsilon \in I_0\}$$

Then, we get the following properties.

Theorem 3.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two fuzzy soft filter bases on X. Then,*
(i) $\mathcal{F} \subseteq \widehat{\mathcal{F}}$.
(ii) If for all $\epsilon \in I_0$ we have $(f+\epsilon) \in \mathcal{F}$, then $f \in \widehat{\mathcal{F}}$.
(iii) $\widehat{\mathcal{F}}$ is a fuzzy soft filter base on X.
(iv) If $\mathcal{F} \subseteq \mathcal{G}$, then $\widehat{\mathcal{F}} \subseteq \widehat{\mathcal{G}}$.
(v) $\widetilde{\mathcal{F}} = \widehat{\langle\mathcal{F}\rangle}$, where $\widetilde{\mathcal{F}} = \langle\widehat{\mathcal{F}}\rangle$.
(vi) $\widehat{\widehat{\mathcal{F}}} \subseteq \widetilde{\mathcal{F}}$.
(vii) If $\mathcal{F} \subseteq \mathcal{G}$, then $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{G}}$.
(viii) If $\mathcal{F}$ is a fuzzy soft filter on X, then $\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}$.

Proof. (i) Let $f \in \mathcal{F}$ and take $f_\epsilon = f$ for all $\epsilon \in I_0$.Then, we get

$$\bigsqcup_{\epsilon\in I_0}(f_\epsilon - \epsilon) = \bigsqcup_{\epsilon\in I_0}(f - \epsilon) = f \in \widehat{\mathcal{F}}.$$

(ii) Let $(f+\epsilon) \in \mathcal{F}$ for all $\epsilon \in I_0$. Thus,

$$\bigsqcup_{\epsilon\in I_0}((f+\epsilon) - \epsilon) = f \in \widehat{\mathcal{F}}.$$

(iii) We shall show that $\widehat{\mathcal{F}}$ satisfies conditions (FSB1)-(FSB2).
(FSB1) is obvious.
(FSB2) Let $f, g \in \widehat{\mathcal{F}}$. Then, there exist two families $\{f_\epsilon : \epsilon \in I_0\}$ and $\{g_\epsilon : \epsilon \in I_0\}$ of $\mathcal{F}$ such that $f = \bigsqcup_{\epsilon\in I_0}(f_\epsilon - \epsilon)$ and $g = \bigsqcup_{\epsilon\in I_0}(g_\epsilon - \epsilon)$. Therefore, for all $e \in E$ and $x \in X$, we have

$$\begin{aligned}(f \sqcap g)(e)(x) &= (\textstyle\bigvee_{\epsilon\in I_0}(f_\epsilon - \epsilon)(e)(x)) \wedge (\textstyle\bigvee_{\epsilon\in I_0}(g_\epsilon - \epsilon)(e)(x))\\ &= \textstyle\bigvee_{\epsilon\in I_0}((f_\epsilon - \epsilon)(e)(x) \wedge (g_\epsilon - \epsilon)(e)(x))\\ &\geq \textstyle\bigvee_{\epsilon\in I_0}((f_\epsilon \sqcap g_\epsilon) - \epsilon)(e)(x).\end{aligned}$$

Because $\mathcal{F}$ is a fuzzy soft filter base on X, there exists a family $\{h_\epsilon : \epsilon \in I_0\}$ of $\mathcal{F}$ such that $h_\epsilon \sqsubseteq f_\epsilon \sqcap g_\epsilon$ for all $\epsilon \in I_0$. Thus, $\bigsqcup_{\epsilon\in I_0}(h_\epsilon - \epsilon) \in \widehat{\mathcal{F}}$ and we obtain

$$f \sqcap g \sqsupseteq \bigsqcup_{\epsilon\in I_0}((f_\epsilon \sqcap g_\epsilon) - \epsilon) \sqsupseteq \bigsqcup_{\epsilon\in I_0}(h_\epsilon - \epsilon).$$

(iv) It is clear from the definition of $\widehat{\mathcal{F}}$.
(v) Let $f \in \widetilde{\mathcal{F}} = \langle\widehat{\mathcal{F}}\rangle$. Then, for some $g \in \widehat{\mathcal{F}}$, we have $g \sqsubseteq f$. Therefore, there exists a family $\{g_\epsilon : \epsilon \in I_0\}$ of $\mathcal{F}$ such that $g = \bigsqcup_{\epsilon\in I_0}(g_\epsilon - \epsilon) \sqsubseteq f$. For all $\epsilon \in I_0$, we get $g_\epsilon \sqsubseteq (f+\epsilon)$ and hence $(f+\epsilon) \in \langle\mathcal{F}\rangle$. Thus,

$$\bigsqcup_{\epsilon\in I_0}((f+\epsilon) - \epsilon) = f \in \widehat{\langle\mathcal{F}\rangle}.$$

Conversely, let $f \in \widehat{\langle\mathcal{F}\rangle}$. Then, there exists a family $\{f_\epsilon : \epsilon \in I_0\}$ of $\langle\mathcal{F}\rangle$ such that $f = \bigsqcup_{\epsilon\in I_0}(f_\epsilon - \epsilon)$. Therefore, there exists a family $\{g_\epsilon : \epsilon \in I_0\}$ of $\mathcal{F}$ satisfying $g_\epsilon \sqsubseteq f_\epsilon$ for all $\epsilon \in I_0$. Thus, from the fact that $\bigsqcup_{\epsilon\in I_0}(g_\epsilon - \epsilon) \sqsubseteq f$ it follows that $f \in \langle\widehat{\mathcal{F}}\rangle$.

(vi) Let $f \in \widehat{\widehat{\mathcal{F}}}$. Then, there exists a family $\{f_\epsilon : \epsilon \in I_0\}$ of $\widehat{\mathcal{F}}$ such that $f = \bigsqcup_{\epsilon\in I_0}(f_\epsilon - \epsilon)$. Similarly, for all $\epsilon \in I_0$, there exists a family $\{f_\epsilon^\delta : \delta \in I_0\}$ of $\mathcal{F}$ such that $f_\epsilon = \bigsqcup_{\delta\in I_0}(f_\epsilon^\delta - \delta)$. Therefore, we have

$$\begin{aligned}f &= \textstyle\bigsqcup_{\epsilon\in I_0}(\bigsqcup_{\delta\in I_0}(f_\epsilon^\delta - \delta) - \epsilon)\\ &= \textstyle\bigsqcup_{\epsilon\in I_0}\bigsqcup_{\delta\in I_0}((f_\epsilon^\delta - \delta) - \epsilon)\\ &= \textstyle\bigsqcup_{\epsilon\in I_0}\bigsqcup_{\delta\in I_0}(f_\epsilon^\delta - (\delta+\epsilon)).\end{aligned}$$

Now, take a $g_\alpha = \bigsqcup_{\epsilon,\delta\in I_0} f_\epsilon^\delta \in \langle\mathcal{F}\rangle$ where $\alpha = \delta+\epsilon$. Since

$$f = \bigsqcup_{\alpha\in I_0}(g_\alpha - \alpha)$$

we get $f \in \widehat{\langle\mathcal{F}\rangle} = \widetilde{\mathcal{F}}$.
(vii) and (viii) are clear from (iv) and (v), respectively.

Definition 3.3. *A fuzzy soft filter $\mathcal{F}$ on X is called a saturated fuzzy soft filter if $\mathcal{F} = \widehat{\mathcal{F}}$.*

Theorem 3.4. *Let $\mathcal{F}$ be a fuzzy soft filter on X. Then, the following conditions are equivalent:*
(i) $\mathcal{F}$ is a saturated fuzzy soft filter.
(ii) If $(f+\epsilon)\in\mathcal{F}$ for all $\epsilon\in I_0$, then $f\in\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii) It is clear by Theorem 3.2(ii).
(ii) $\Rightarrow$ (i) It follows from Theorem 3.2(i) that $\mathcal{F}\subseteq\widehat{\mathcal{F}}$. Conversely, let $f\in\widehat{\mathcal{F}}$. Then, there exists a family $\{f_\epsilon:\epsilon\in I_0\}$ of $\mathcal{F}$ such that $f=\bigsqcup_{\epsilon\in I_0}(f_\epsilon-\epsilon)$. Therefore, we have $f_\epsilon\sqsubseteq(f+\epsilon)$ for all $\epsilon\in I_0$. Since $\mathcal{F}$ is a fuzzy soft filter, we obtain $(f+\epsilon)\in\mathcal{F}$ for all $\epsilon\in I_0$. Thus, from hypothesis it follows that $f\in\mathcal{F}$.

In view of Theorem 3.4, we shall give an example of a saturated fuzzy soft filter.

Example 3.5. *Let $\mathcal{F}_c$ be a filter on X in classical meaning. Then, for any $\alpha\in I_0$*

$$\mathcal{F}_c^\alpha=\{f\in FS(X,E):f^\beta\in\mathcal{F}_c\ for\ all\ \beta<\alpha\}$$

where

$$f^\beta=\{x\in X:f(e)(x)\geq\beta\ for\ all\ e\in E\}$$

is a saturated fuzzy soft filter on X.

Theorem 3.6. *Let $\mathcal{F}$ be a fuzzy soft filter on X. Then, the following statements are satisfied.*
(i) $\widehat{\mathcal{F}}=\{f\in FS(X,E):(f+\epsilon)\in\mathcal{F}\ for\ all\ \epsilon\in I_0\}$.
(ii) $\widehat{\mathcal{F}}$ is a saturated fuzzy soft filter on X.

Proof. (i) Let $f\in\widehat{\mathcal{F}}$. Then, there exists a family $\{f_\epsilon:\epsilon\in I_0\}$ of $\mathcal{F}$ such that $f=\bigsqcup_{\epsilon\in I_0}(f_\epsilon-\epsilon)$. Therefore, we get $f_\epsilon\sqsubseteq(f+\epsilon)$ for all $\epsilon\in I_0$. Hence, since $\mathcal{F}$ is a fuzzy soft filter, we obtain $(f+\epsilon)\in\mathcal{F}$ for all $\epsilon\in I_0$. On the other hand, let $(f+\epsilon)\in\mathcal{F}$ for all $\epsilon\in I_0$. It follows from Theorem 3.2(ii) that $f\in\widehat{\mathcal{F}}$.

(ii) By Theorem 3.2(v) and (viii), it is clear that $\widehat{\mathcal{F}}$ is a fuzzy soft filter on X. Now, we shall show that $\widehat{\mathcal{F}}=\widehat{\widehat{\mathcal{F}}}$. It follows from Theorem 3.2(i) that $\widehat{\mathcal{F}}\subseteq\widehat{\widehat{\mathcal{F}}}$. To prove the converse inclusion, let $f\in\widehat{\widehat{\mathcal{F}}}$. By (i), we have $(f+\epsilon)\in\widehat{\mathcal{F}}$ for all $\epsilon\in I_0$. Again from (i), we get $(f+\epsilon)+\delta=f+(\epsilon+\delta)\in\mathcal{F}$ for all $\epsilon\in I_0$ and all $\delta\in I_0$. Thus, $f\in\widehat{\mathcal{F}}$.

Theorem 3.7. *If $\mathcal{F}$ is a fuzzy soft filter base on X, then $\widetilde{\mathcal{F}}=\widehat{\widetilde{\mathcal{F}}}$.*

Proof. It is clear from Theorem 3.2(v), (viii) and Theorem 3.6(ii).

4. Fuzzy Soft Uniformity

In this section, on the basis of the notion of saturated fuzzy soft filters, we introduce a fuzzy soft uniformity, which is a fuzzy soft version of uniformity which is proposed by Lowen [22]. Also, we present the concept of a fuzzy soft neighborhood system and investigate its relation with fuzzy soft uniformity.

Definition 4.1. *(i) Let $f\in FS(X\times X,E)$. A mapping $f_s:E\to I^{X\times X}$ is a fuzzy soft set on $X\times X$ defined as follows:*

$$f_s(e)(x,y)=f(e)(y,x)$$

for all $e\in E$ and all $(x,y)\in X\times X$.
(ii) Let $f,g\in FS(X\times X,E)$. A mapping $f\circ g:E\to I^{X\times X}$ is a fuzzy soft set on $X\times X$ defined as follows:

$$(f\circ g)(e)(x,y)=\bigvee_{z\in X}(g(e)(x,z)\wedge f(e)(z,y))$$

for all $e\in E$ and all $(x,y)\in X\times X$.

Definition 4.2. *The non-empty family $\mathcal{U}\subseteq FS(X\times X,E)$ is called a fuzzy soft uniformity on X if the following axioms are satisfied:*
(FSU1) $\mathcal{U}$ is a saturated fuzzy soft filter.
(FSU2) If $f\in\mathcal{U}$, then we have $f(e)(x,x)=1$ for all $e\in E$ and $x\in X$.
(FSU3) If $f\in\mathcal{U}$, then we have $f_s\in\mathcal{U}$.
(FSU4) If $f\in\mathcal{U}$ and $\epsilon\in I_0$, then there exists a $g\in\mathcal{U}$ such that $(g\circ g)\sqsubseteq(f+\epsilon)$.
The triplet $(X,\mathcal{U},E)$ is called a fuzzy soft uniform space on X.

Remark 4.3. *The above definition coincides with Lowen's uniformity when the parameter set E is a singleton set.*

Example 4.4. *Let X be a non-empty set and E be a set of parameters for X. Then, the family*

$$\mathcal{U} = \{f \in FS(X\times X,E) : f(e)(x,x) = 1 \text{ for all } e \in E \text{ and } x \in X\}$$

is a fuzzy soft uniformity on X.
It is enough to show that $\mathcal{U}$ satisfies (FSU4), since the other axioms are readily verified.
(FSU4) Let $f \in \mathcal{U}$ and $\epsilon \in I_0$. Then, since for all $e \in E$ and $(x,y) \in X\times X$

$$(f\circ f)(e)(x,y) = \bigvee_{z\in X}(f(e)(x,z)\wedge f(e)(z,y)) \le f(e)(x,y) \le (f(e)(x,y)+\epsilon)\wedge 1$$

it follows that $(f\circ f) \sqsubseteq (f+\epsilon)$.

Definition 4.5. *The non-empty family $\mathcal{B} \subseteq FS(X\times X,E)$ is called a fuzzy soft uniform base on X if the following axioms are satisfied:*
(FSUB1) $\mathcal{B}$ is a fuzzy soft filter base.
(FSUB2) If $f \in \mathcal{B}$, then we have $f(e)(x,x) = 1$ for all $e \in E$ and $x \in X$.
(FSUB3) If $f \in \mathcal{B}$ and $\epsilon \in I_0$, then there exists a $g \in \mathcal{B}$ such that $g \sqsubseteq (f_s+\epsilon)$.
(FSUB4) If $f \in \mathcal{B}$ and $\epsilon \in I_0$, then there exists a $g \in \mathcal{B}$ such that $(g\circ g) \sqsubseteq (f+\epsilon)$.

Example 4.6. *Let X be a non-empty set and E be a set of parameters for X. Then, the family*

$$\mathcal{B} = \{f_\epsilon \in FS(X\times X,E) : \epsilon > 0\}$$

is a fuzzy soft uniform base on X, where

$$f_\epsilon(e)(x,y) = \begin{cases} 1, & \text{if } x = y; \\ (|x-y|-\epsilon), & \text{if } x \neq y \end{cases}$$

for all $e \in E$ and $(x,y) \in X\times X$.

Definition 4.7. *A family $\mathcal{B}$ of subsets of $FS(X\times X,E)$ is called a base for a fuzzy soft unifomity $\mathcal{U}$ on X if the following two conditions are satisfied :*
(i) $\mathcal{B}$ is a fuzzy soft filter base.
(ii) $\widetilde{\mathcal{B}} = \mathcal{U}$.

Example 4.8. *Let X be a non-empty set and E be a set of parameters for X. Let us define an $f \in FS(X\times X,E)$ by*

$$f(e)(x,y) = \begin{cases} 1, & \text{if } x = y; \\ \frac{1}{2}, & \text{if } x \neq y \end{cases}$$

for all $e \in E$ and $(x,y) \in X\times X$. Then, the family $\mathcal{B} = \{f\}$ is a base for some fuzzy soft uniformity on X. Indeed, it is easily verified that $\mathcal{B}$ is a fuzzy soft filter base. Now, we shall show that the family

$$\widetilde{\{f\}} = \widehat{\langle f\rangle} = \mathcal{U} = \{g \in FS(X\times X,E) : \bigsqcup_{\epsilon\in I_0}(f-\epsilon) \sqsubseteq g\}$$

is a fuzzy soft uniformity on X.
(FSU1) It is clear that $\mathcal{U}$ is a fuzzy soft filter. Let $(g+\delta) \in \mathcal{U}$ for all $\delta \in I_0$. Then, since $(f-\epsilon) \sqsubseteq (g+\delta)$ for all $\epsilon \in I_0$ and all $\delta \in I_0$, we have $f \sqsubseteq g$. Therefore, from the fact that $f \in \mathcal{U}$ and $\mathcal{U}$ is a fuzzy soft filter it follows that $g \in \mathcal{U}$. Thus, by Theorem 3.4, $\mathcal{U}$ is a saturated fuzzy soft filter.
(FSU2) and (FSU3) are obvious.
(FSU4) If $g \in \mathcal{U}$, then we have $\bigsqcup_{\epsilon\in I_0}(f-\epsilon) \sqsubseteq g$. Thus, since for all $\epsilon \in I_0$ we get $(f\circ f) = f \sqsubseteq (g+\epsilon)$, this axiom is also satisfied.

Theorem 4.9. *If $\mathcal{B}$ is a fuzzy soft uniform base on X, then $\widetilde{\mathcal{B}}$ is a fuzzy soft uniformity on X with $\mathcal{B}$ as a base.*

Proof. The last part of the theorem is obvious. We shall show that $\widetilde{\mathcal{B}}$ satisfies axioms (FSU1)-(FSU4).
(FSU1) and (FSU2) are clear.
(FSU3) Let $f \in \widetilde{\mathcal{B}}$ and $\epsilon \in I_0$. Then, there exists a family $\{h_\delta : \delta \in I_0\}$ of $\mathcal{B}$ such that

$$\bigsqcup_{\delta\in I_0}(h_\delta-\delta) \sqsubseteq f.$$

In particular, take an $h_{\epsilon/2} \in \mathcal{B}$ satisfying $(h_{\epsilon/2}-\frac{\epsilon}{2}) \sqsubseteq f$. By hypothesis, there exists a $g_{\epsilon/2} \in \mathcal{B}$ such that $g_{\epsilon/2} \sqsubseteq (h_{\epsilon/2})_s + \frac{\epsilon}{2}$. Therefore, we obtain

$$g_{\epsilon/2} \sqsubseteq \left((h_{\epsilon/2})_s + \frac{\epsilon}{2}\right) \sqsubseteq (f_s+\epsilon).$$

From the fact that $\mathcal{B} \subseteq \widetilde{\mathcal{B}}$ it follows that $(f_s+\epsilon) \in \widetilde{\mathcal{B}}$. Thus, by Theorem 3.4, we get $f_s \in \widetilde{\mathcal{B}}$.

(FSU4) Let $f \in \widetilde{\mathcal{B}}$ and $\epsilon \in I_0$. Then, there exists a family $\{h_\delta : \delta \in I_0\}$ of $\mathcal{B}$ such that

$$\bigsqcup_{\delta \in I_0} (h_\delta - \delta) \sqsubseteq f.$$

Choose an $h_{\epsilon/2} \in \mathcal{B}$ with $(h_{\epsilon/2} - \frac{\epsilon}{2}) \sqsubseteq f$. By hypothesis, there exists a $g_{\epsilon/2} \in \mathcal{B}$ such that $(g_{\epsilon/2} \circ g_{\epsilon/2}) \sqsubseteq (h_{\epsilon/2} + \frac{\epsilon}{2})$. Also, since $\mathcal{B} \subseteq \widetilde{\mathcal{B}}$, we have $g_{\epsilon/2} \in \widetilde{\mathcal{B}}$. Thus, there exists a $g_{\epsilon/2} \in \widetilde{\mathcal{B}}$ such that

$$(g_{\epsilon/2} \circ g_{\epsilon/2}) \sqsubseteq \left(h_{\epsilon/2} + \frac{\epsilon}{2}\right) \sqsubseteq (f + \epsilon),$$

which completes the proof.

Theorem 4.10. *If $\mathcal{B}$ is a base for a fuzzy soft uniformity $\mathcal{U}$ on X, then $\mathcal{B}$ is a fuzzy soft uniform base on X.*

Proof. Since the other axioms are easily verified, it suffices to show that $\mathcal{B}$ satisfies (FSUB3) and (FSUB4).
(FSUB3) Let $f \in \mathcal{B}$ and $\epsilon \in I_0$. Then, we have $(f + \frac{\epsilon}{2}) \in \widetilde{\mathcal{B}} = \mathcal{U}$ and so that $(f_s + \frac{\epsilon}{2}) \in \widetilde{\mathcal{B}}$. Therefore, there exists an $h_{\epsilon/2} \in \mathcal{B}$ such that $(h_{\epsilon/2} - \frac{\epsilon}{2}) \sqsubseteq (f_s + \frac{\epsilon}{2})$, which shows that $h_{\epsilon/2} \sqsubseteq (f_s + \epsilon)$ as required.
(FSUB4) Let $f \in \mathcal{B}$ and $\epsilon \in I_0$. Then, we have $(f + \frac{\epsilon}{3}) \in \widetilde{\mathcal{B}} = \mathcal{U}$. By the definition of $\mathcal{U}$, there exists a $g \in \widetilde{\mathcal{B}}$ such that $(g \circ g) \sqsubseteq (f + \frac{\epsilon}{3}) + \frac{\epsilon}{3}$. Also, for an $h_{\epsilon/3} \in \mathcal{B}$, we get $(h_{\epsilon/3} - \frac{\epsilon}{3}) \sqsubseteq g$, that is, $h_{\epsilon/3} \sqsubseteq (g + \frac{\epsilon}{3})$. Thus, we obtain

$$(h_{\epsilon/3} \circ h_{\epsilon/3}) \sqsubseteq \left((g \circ g) + \frac{\epsilon}{3}\right) \sqsubseteq (f + \epsilon),$$

which proves (FSUB4).

Theorem 4.11. *Let $(X, \mathcal{U}, E)$ be a fuzzy soft uniform space. Then, the family*

$$\mathcal{B} = \{f \in \mathcal{U} : f = f_s\}$$

is a base for $\mathcal{U}$.

Proof. It is easy to verify that $\mathcal{B}$ is a fuzzy soft filter base. Now, let us show that $\widetilde{\mathcal{B}} = \mathcal{U}$. Let $f \in \widetilde{\mathcal{B}}$. Then, there exists a family $\{h_\epsilon : \epsilon \in I_0\}$ of $\mathcal{B}$ such that $\bigsqcup_{\epsilon \in I_0} (h_\epsilon - \epsilon) \sqsubseteq f$. Since $\mathcal{U}$ is a saturated fuzzy soft filter, it follows from $\mathcal{B} \subseteq \mathcal{U}$ that $f \in \mathcal{U}$.
Conversely, let $f \in \mathcal{U}$. By (FSU3), we have $f_s \in \mathcal{U}$ and so $(f \sqcap f_s) \in \mathcal{U}$. Because $(f \sqcap f_s)_s = (f \sqcap f_s)$, we obtain $(f \sqcap f_s) \in \mathcal{B}$. Also, since $(f \sqcap f_s) \sqsubseteq f$, this means that $f \in \langle \mathcal{B} \rangle$. Thus, from the fact that $\langle \mathcal{B} \rangle \subseteq \widetilde{\langle \mathcal{B} \rangle} = \widetilde{\mathcal{B}}$ it follows that $f \in \widetilde{\mathcal{B}}$.

Let $f \in FS(X \times X, E)$ and $h \in FS(X, E)$. Then, we define a fuzzy soft set $f\langle h \rangle \in FS(X, E)$ by

$$f\langle h \rangle(e)(x) = \bigvee_{y \in X} (h(e)(y) \wedge f(e)(y, x))$$

for all $e \in E$ and all $x \in X$.

Lemma 4.12. *Let $(X, \mathcal{U}, E)$ be a fuzzy soft uniform space and let $f, g \in \mathcal{U}$ and $h, h' \in FS(X, E)$. Then, the following properties are satisfied.*
(i) $h \sqsubseteq f\langle h \rangle$.
(ii) For all $\epsilon \in I$, $(f + \epsilon)\langle h \rangle \sqsubseteq f\langle h \rangle + \epsilon$.
(iii) $f\langle h \sqcup h' \rangle = f\langle h \rangle \sqcup f\langle h' \rangle$.
(iv) $f\langle h \rangle \sqcup g\langle h' \rangle \sqsupseteq (f \sqcap g)\langle h \sqcup h' \rangle$.
(v) $f\langle g\langle h \rangle \rangle = (f \circ g)\langle h \rangle$.

Proof. (i) It is clear from the definition of $f\langle h \rangle$.
(ii) Let $\epsilon \in I$. For all $e \in E$ and $x \in X$, we obtain

$$\begin{aligned}
(f + \epsilon)\langle h \rangle(e)(x) &= \textstyle\bigvee_{y \in X} (h(e)(y) \wedge (f + \epsilon)(e)(y, x)) \\
&\leq \textstyle\bigvee_{y \in X} (h(e)(y) + \epsilon) \wedge (f(e)(y, x) + \epsilon) \\
&= \left(\textstyle\bigvee_{y \in X} (h(e)(y) \wedge f(e)(y, x))\right) + \epsilon \\
&= f\langle h \rangle(e)(x) + \epsilon,
\end{aligned}$$

which proves $(f + \epsilon)\langle h \rangle \sqsubseteq f\langle h \rangle + \epsilon$.
(iii) For all $e \in E$ and $x \in X$, we have

$$\begin{aligned}
f\langle h \sqcup h' \rangle(e)(x) &= \textstyle\bigvee_{y \in X} ((h \sqcup h')(e)(y) \wedge f(e)(y, x)) \\
&= \textstyle\bigvee_{y \in X} \Big((h(e)(y) \wedge f(e)(y, x)) \vee (h'(e)(y) \wedge f(e)(y, x))\Big) \\
&= \left(\textstyle\bigvee_{y \in X} (h(e)(y) \wedge f(e)(y, x))\right) \vee \left(\textstyle\bigvee_{y \in X} (h'(e)(y) \wedge f(e)(y, x))\right)
\end{aligned}$$

$$
\begin{aligned}
&= f\langle h\rangle(e)(x) \vee f\langle h'\rangle(e)(x)\\
&= (f\langle h\rangle \sqcup f\langle h'\rangle)(e)(x).
\end{aligned}
$$

This proves our assertion.

(iv) Let $e \in E$ and $x \in X$. Then,

$$
\begin{aligned}
f\langle h\rangle(e)(x) \vee g\langle h'\rangle(e)(x) &= \Big(\bigvee_{y\in X} h(e)(y) \wedge f(e)(y,x)\Big) \vee \Big(\bigvee_{y\in X} h'(e)(y) \wedge g(e)(y,x)\Big)\\
&\geq \bigvee_{y\in X} (h(e)(y) \vee h'(e)(y)) \wedge (f(e)(y,x) \wedge g(e)(y,x))\\
&= \bigvee_{y\in X} (h \sqcup h')(e)(y) \wedge (f \sqcap g)(e)(y,x)\\
&= (f \sqcap g)\langle h \sqcup h'\rangle(e)(x),
\end{aligned}
$$

which shows that $f\langle h\rangle \sqcup g\langle h'\rangle \sqsupseteq (f \sqcap g)\langle h \sqcup h'\rangle$.

(v) For all $e \in E$ and $x \in X$, we have

$$
\begin{aligned}
f\langle g\langle h\rangle\rangle(e)(x) &= \bigvee_{y\in X} (g\langle h\rangle(e)(y) \wedge f(e)(y,x))\\
&= \bigvee_{y\in X} \Big(\bigvee_{z\in X} (h(e)(z) \wedge g(e)(z,y))\Big) \wedge f(e)(y,x)\\
&= \bigvee_{y\in X} \bigvee_{z\in X} h(e)(z) \wedge g(e)(z,y) \wedge f(e)(y,x)
\end{aligned}
$$

On the other hand, for all $e \in E$ and $x \in X$, we have

$$
\begin{aligned}
(f \circ g)\langle h\rangle(e)(x) &= \bigvee_{z\in X} (h(e)(z) \wedge (f \circ g)(e)(z,x))\\
&= \bigvee_{z\in X} h(e)(z) \wedge \Big(\bigvee_{y\in X} (g(e)(z,y) \wedge f(e)(y,x))\Big)\\
&= \bigvee_{z\in X} \bigvee_{y\in X} h(e)(z) \wedge g(e)(z,y) \wedge f(e)(y,x)
\end{aligned}
$$

Thus, we obtain $f\langle g\langle h\rangle\rangle = (f \circ g)\langle h\rangle$ and the proof is concluded.

We induce a fuzzy soft topology from a given fuzzy soft uniformity by using the fuzzy soft closure operator as shown in the following theorem.

Theorem 4.13. *Let $(X,\mathcal{U},E)$ be a fuzzy soft uniform space and $h \in FS(X,E)$. Then, the mapping $h \to \overline{h}$ satisfies the conditions (FSO1)-(FSO4), where*

$$\overline{h} = \bigsqcap_{f\in\mathcal{U}} f\langle h\rangle.$$

Therefore, the family

$$\tau_{\mathcal{U}} = \{h \in FS(X,E) : \overline{h^c} = h^c\}$$

is a fuzzy soft topology on X.

Proof. We shall show that the mapping $h \to \overline{h}$ has the properties (FSO1)-(FSO4).

(FSO1) By Lemma 4.12(i), it is clear.

(FSO2) It follows immediately from (FSO1) that $\overline{h} \sqsubseteq \overline{\overline{h}}$. To prove the reverse inclusion, let $f \in \mathcal{U}$ and $\epsilon \in I_0$. Then, there exists a $g \in \mathcal{U}$ such that $(g \circ g) \sqsubseteq (f + \epsilon)$. Therefore, for all $e \in E$ and $x \in X$,

$$
\begin{aligned}
f\langle h\rangle(e)(x) &= \bigvee_{y\in X} h(e)(y) \wedge f(e)(y,x)\\
&\geq \bigvee_{y\in X} h(e)(y) \wedge ((g \circ g)(e)(y,x) - \epsilon)\\
&= \bigvee_{y\in X} h(e)(y) \wedge \Big(\bigvee_{z\in X} g(e)(y,z) \wedge g(e)(z,x) - \epsilon\Big)\\
&= \bigvee_{y\in X} \bigvee_{z\in X} h(e)(y) \wedge g(e)(y,z) \wedge g(e)(z,x) - \epsilon\\
&\geq \bigvee_{z\in X} g(e)(z,x) \wedge \Big(\bigwedge_{g'\in\mathcal{U}} \bigvee_{y\in X} h(e)(y) \wedge g'(e)(y,z)\Big) - \epsilon\\
&= \bigvee_{z\in X} g(e)(z,x) \wedge \overline{h}(e)(z) - \epsilon\\
&= g\langle\overline{h}\rangle(e)(x) - \epsilon,
\end{aligned}
$$

which shows that

$$\bigwedge_{f\in\mathcal{U}} f\langle h\rangle(e)(x) \geq \bigwedge_{g\in\mathcal{U}} g\langle\overline{h}\rangle(e)(x) - \epsilon.$$

Since the above result holds for all $\epsilon \in I_0$ it follows that

$$\bigwedge_{f\in\mathcal{U}} f\langle h\rangle(e)(x) \geq \bigwedge_{g\in\mathcal{U}} g\langle\overline{h}\rangle(e)(x),$$

that is, $\overline{h}(e)(x) \geq \overline{\overline{h}}(e)(x)$. Thus, we get $\overline{h} \sqsupseteq \overline{\overline{h}}$.

(FSO3) Let $h, h' \in FS(X,E)$. Then, for all $e \in E$ and $x \in X$, we have

$$(\overline{h \sqcup h'})(e)(x) = \bigwedge_{f\in\mathcal{U}} f\langle h \sqcup h'\rangle(e)(x)$$

$$= \bigwedge_{f\in\mathcal{U}} (f\langle h\rangle(e)(x) \vee f\langle h'\rangle(e)(x))$$
$$\geq \Big(\bigwedge_{f\in\mathcal{U}} f\langle h\rangle(e)(x)\Big) \vee \Big(\bigwedge_{f\in\mathcal{U}} f\langle h'\rangle(e)(x)\Big)$$
$$= \overline{h}(e)(x) \vee \overline{h'}(e)(x) = (\overline{h} \sqcup \overline{h'})(e)(x),$$

which shows that $\overline{h \sqcup h'} \sqsupseteq \overline{h} \sqcup \overline{h'}$. Conversely, we know that

$$(\overline{h} \sqcup \overline{h'})(e)(x) = \Big(\bigwedge_{f\in\mathcal{U}} f\langle h\rangle(e)(x)\Big) \vee \Big(\bigwedge_{g\in\mathcal{U}} g\langle h'\rangle(e)(x)\Big)$$
$$= \bigwedge_{f,g\in\mathcal{U}} (f\langle h\rangle(e)(x) \vee g\langle h'\rangle(e)(x)).$$

Since $f' = (f \sqcap g) \in \mathcal{U}$, by Lemma 4.12(iv), we have

$$(\overline{h} \sqcup \overline{h'})(e)(x) \geq \bigwedge_{f'\in\mathcal{U}} f'\langle\, h \sqcup h'\rangle(e)(x) = (\overline{h \sqcup h'})(e)(x).$$

Thus, we obtain $\overline{h} \sqcup \overline{h'} \sqsupseteq \overline{h \sqcup h'}$.

(FSO4) It is clear that $\overline{\widetilde{\emptyset}} = \widetilde{\emptyset}$.

Then, it is called the fuzzy soft uniform topology induced by the fuzzy soft uniformity $\mathcal{U}$.

Example 4.14. *It is easily verify that the fuzzy soft uniform space defined in Example 4.4 induces the fuzzy soft topological space $\tau = FS(X,E)$ on X.*

Theorem 4.15. *If $\mathcal{B}$ is a base for a fuzzy soft uniformity $\mathcal{U}$, then for all $h \in FS(X,E)$, we have*

$$\overline{h} = \bigsqcap_{f\in\widehat{\mathcal{B}}} f\langle h\rangle = \bigsqcap_{f\in\langle\mathcal{B}\rangle} f\langle h\rangle = \bigsqcap_{f\in\mathcal{B}} f\langle h\rangle.$$

Proof. Since $\mathcal{B} \subseteq \widehat{\mathcal{B}} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \langle\mathcal{B}\rangle \subseteq \mathcal{U}$, it is sufficient to show the following. Let $h \in FS(X,E)$ and $\epsilon \in I_0$. Then, since for all $f \in \mathcal{U}$ there exists a $g \in \mathcal{B}$ such that $(g-\epsilon) \sqsubseteq f$, it follows that

$$\overline{h} = \bigsqcap_{f\in\mathcal{U}} f\langle h\rangle \sqsupseteq \bigsqcap_{g\in\mathcal{B}} (g-\epsilon)\langle h\rangle \sqsupseteq \bigsqcap_{g\in\mathcal{B}} g\langle h\rangle - \epsilon,$$

which proves that $\overline{h} \sqsupseteq \bigsqcap_{g\in\mathcal{B}} g\langle h\rangle$.

Definition 4.16. *A collection $\{\mathcal{N}_x\}_{x\in X}$, where $\mathcal{N}_x \subseteq FS(X,E)$ for all $x \in X$, is called a fuzzy soft neighborhood system on X if the following axioms are satisfied for all $x \in X$.*

(FSNS1) $\mathcal{N}_x$ is a saturated fuzzy soft filter.

(FSNS2) For all $h \in \mathcal{N}_x$ and $e \in E$, we have $h(e)(x) = 1$.

(FSNS3) For all $h \in \mathcal{N}_x$ and $\epsilon \in I_0$, there exists a family $\{h_z : z \in X\}$ such that $h_z \in \mathcal{N}_z$ for all $z \in X$ and such that

$$\bigvee_{z\in X} (h_x(e)(z) \wedge h_z(e)(y)) \leq h(e)(y) + \epsilon$$

for all $e \in E$ and $y \in X$.

$\mathcal{N}_x$ is called a fuzzy soft neighborhood filter on X and the elements of $\mathcal{N}_x$ are called fuzzy soft neighborhoods of the point x

Let $f \in FS(X\times X,E)$ and $x \in X$. Then, we define a fuzzy soft set $f\langle x\rangle \in FS(X,E)$ by

$$f\langle x\rangle(e)(y) = f(e)(y,x)$$

for all $e \in E$ and all $y \in X$. Then, we get the following theorem.

Theorem 4.17. *Let $(X,\mathcal{U},E)$ be a fuzzy soft uniform space. If for all $x \in X$ we define*

$$\mathcal{U}_x = \{f\langle x\rangle : f \in \mathcal{U}\},$$

then $\{\mathcal{U}_x\}_{x\in X}$ is a fuzzy soft neighborhood system on X.

Proof. We shall show that $\{\mathcal{U}_x\}_{x\in X}$ satisfies axioms (FSNS1)-(FSNS3).

(FSNS1) Let $x \in X$. First, let us show that $\mathcal{U}_x$ is a fuzzy soft filter.

It is clear that $\mathcal{U}_x \neq \emptyset$ and $\widetilde{\emptyset} \notin \mathcal{U}_x$.

Let $f\langle x\rangle, g\langle x\rangle \in \mathcal{U}_x$. Then, we get $f,g \in \mathcal{U}$. Thus, from the fact that $f \sqcap g \in \mathcal{U}$ and $f\langle x\rangle \sqcap g\langle x\rangle = (f \sqcap g)\langle x\rangle$ it follows that $f\langle x\rangle \sqcap g\langle x\rangle \in \mathcal{U}_x$.

Let $f\langle x\rangle \in \mathcal{U}_x$ and $f\langle x\rangle \sqsubseteq h$. Then, we have $f \in \mathcal{U}$ and $f(e)(y,x) \leq h(e)(y)$ for all $e \in E$ and $y \in X$. Now, take a fuzzy soft set $g \in FS(X\times X,E)$ such that

$$g(e)(x_1,x_2) = \begin{cases} f(e)(x_1,x_2), & \text{if } x_2 \neq x; \\ h(e)(x_1), & \text{if } x_2 = x \end{cases}$$

for all $e \in E$ and all $(x_1,x_2) \in X \times X$. Therefore, $f \sqsubseteq g$ and so that $g \in \mathcal{U}$. Thus, by $g\langle x\rangle = h$, we obtain $h \in \mathcal{U}_x$.

Next, we prove that $\mathcal{U}_x$ is a saturated fuzzy soft filter. To see this, let $(f\langle x\rangle + \epsilon) \in \mathcal{U}_x$ for all $\epsilon \in I_0$. Because $(f\langle x\rangle + \epsilon) = (f+\epsilon)\langle x\rangle$ for all $\epsilon \in I_0$, we have $(f+\epsilon) \in \mathcal{U}$ for all $\epsilon \in I_0$. From (FSU1) it follows that $f \in \mathcal{U}$ and therefore we obtain $f\langle x\rangle \in \mathcal{U}_x$. Thus, by Theorem 3.4, $\mathcal{U}_x$ is a saturated fuzzy soft filter.

(FSNS2) is clear.

(FSNS3) Let $x \in X$ and let $f\langle x\rangle \in \mathcal{U}_x$ and $\epsilon \in I_0$. Since $f \in \mathcal{U}$, by (FSU4), there exists a $g \in \mathcal{U}$ such that $(g \circ g) \sqsubseteq (f+\epsilon)$. Now, let us take a family $\{g\langle z\rangle : z \in X\}$ such that $g\langle z\rangle \in \mathcal{U}_z$ for all $z \in X$. Therefore, for all $e \in E$ and $y \in X$, we have

$$\begin{aligned}\bigvee_{z\in X}(g\langle x\rangle(e)(z) \wedge g\langle z\rangle(e)(y)) &= \bigvee_{z\in X}(g(e)(z,x) \wedge g(e)(y,z))\\ &= (g\circ g)(e)(y,x)\\ &\le f(e)(y,x)+\epsilon\\ &= f\langle x\rangle(e)(y)+\epsilon.\end{aligned}$$

Thus, $\{\mathcal{U}_x\}_{x\in X}$ is a fuzzy soft neighborhood system on X.

5. Fuzzy Soft Uniformly Continuous Mapping

Our task in this section is to define the notion of a fuzzy soft uniformly continuous mapping and investigate its connection with fuzzy soft continuous mapping.

Definition 5.1. *Let $(X,\mathcal{U},E)$ and $(Y,\mathcal{V},K)$ be two fuzzy soft uniform spaces. A fuzzy soft mapping $\varphi_\psi : (X,\mathcal{U},E) \to (Y,\mathcal{V},K)$ is called a fuzzy soft uniformly continuous if for all $f \in \mathcal{V}$, there exists a $g \in \mathcal{U}$ such that $(\varphi\times\varphi)_\psi(g) \sqsubseteq f$ (Here, the mapping $\varphi\times\varphi : X\times X \to Y\times Y$ is defined by $(\varphi\times\varphi)(x_1,x_2) = (\varphi(x_1),\varphi(x_2))$ for all $(x_1,x_2) \in X\times X$).*

This is obviously equivalent to saying that for all $f \in \mathcal{V}$, $(\varphi\times\varphi)^{-1}_\psi(f) \in \mathcal{U}$.

Example 5.2. *Let $(X,\mathcal{U},E)$ be a fuzzy soft uniform space which is defined in Example 4.4. Then, the fuzzy soft mapping $\varphi_\psi : (X,\mathcal{U},E) \to (X,\mathcal{U},E)$ is a fuzzy soft uniformly continuous, where $\varphi : X \to X$ and $\psi : E \to E$ are arbitrary mappings.*

Proposition 5.3. *The composition of two fuzzy soft uniformly continuous mappings is a fuzzy soft uniformly continuous mapping.*

Proof. We can easily prove it by using the definition of fuzzy soft uniformly continuous mapping.

Theorem 5.4. *Let $(X,\mathcal{U},E)$ and $(Y,\mathcal{V},K)$ be two fuzzy soft uniform spaces and let $\mathcal{B}$ and $\mathcal{C}$ be two bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, a fuzzy soft mapping $\varphi_\psi : (X,\mathcal{U},E) \to (Y,\mathcal{V},K)$ is a fuzzy soft uniformly continuous if and only if for all $f \in \mathcal{V}$ and $\epsilon \in I_0$ there exists a $g \in \mathcal{B}$ such that $(g-\epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f)$.*

Proof. Let $f \in \mathcal{V}$ and $\epsilon \in I_0$. By applying Theorem 3.6(i), we get $(f+\frac{\epsilon}{2}) \in \mathcal{V}$. From the hypothesis it follows that there exists a $g' \in \mathcal{U} = \widetilde{\mathcal{B}}$ such that

$$g' \sqsubseteq (\varphi\times\varphi)^{-1}_\psi\left(f+\frac{\epsilon}{2}\right) = (\varphi\times\varphi)^{-1}_\psi(f)+\frac{\epsilon}{2}.$$

Therefore, we obtain a $g \in \mathcal{B}$ satisfying $(g-\frac{\epsilon}{2}) \sqsubseteq g'$. Thus, since $(g-\epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f)$, the necessity is proved.

To prove the sufficiency, let $f \in \mathcal{V} = \widetilde{\mathcal{C}}$. Then, there exists a family $\{f_\delta : \delta \in I_0\}$ of $\mathcal{C}$ such that $(f_\delta - \delta) \sqsubseteq f$ for all $\delta \in I_0$. Let $\epsilon \in I_0$. By hypothesis, we obtain $(g_\epsilon - \epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f_\epsilon)$ for some $g_\epsilon \in \mathcal{B}$. Therefore,

$$g_\epsilon \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f_\epsilon)+\epsilon \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f+\epsilon)+\epsilon = (\varphi\times\varphi)^{-1}_\psi(f)+2\epsilon,$$

that is, we have

$$(g_\epsilon - 2\epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f).$$

As the above result holds for all $\epsilon \in I_0$ it follows that $g = \bigsqcup_{\epsilon\in I_0}(g_\epsilon - 2\epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f)$. Thus, since $g \in \widehat{\mathcal{B}} \subseteq \mathcal{U}$, the proof is concluded.

Corollary 5.5. *Let $(X,\mathcal{U},E)$ and $(Y,\mathcal{V},K)$ be two fuzzy soft uniform spaces. Then, a fuzzy soft mapping $\varphi_\psi : (X,\mathcal{U},E) \to (Y,\mathcal{V},K)$ is a fuzzy soft uniformly continuous if and only if for all $f \in \mathcal{V}$ and $\epsilon \in I_0$ there exists a $g \in \mathcal{U}$ such that $(g-\epsilon) \sqsubseteq (\varphi\times\varphi)^{-1}_\psi(f)$.*

Proof. It follows immediately from Theorem 5.4 and the fact that every fuzzy soft uniformity is a base for itself.

Theorem 5.6. *If a fuzzy soft mapping $\varphi_\psi : (X,\mathcal{U},E) \to (Y,\mathcal{V},K)$ is fuzzy soft uniformly continuous, then the fuzzy soft mapping $\varphi_\psi : (X,\tau_{\mathcal{U}},E) \to (Y,\tau_{\mathcal{V}},K)$ is fuzzy soft continuous.*

Proof. By Theorem 2.15, it suffices to show that $\varphi_\psi(\overline{h}) \sqsubseteq \overline{\varphi_\psi(h)}$ for all $h \in FS(X,E)$. Let $h \in FS(X,E)$. For all $k \in K$ and $y \in Y$, where $\psi^{-1}(k) \neq \emptyset$ and $\varphi^{-1}(y) \neq \emptyset$, we have

$$\overline{\varphi_\psi(h)}(k)(y) = \bigwedge_{f\in\mathcal{V}} f\langle\varphi_\psi(h)\rangle(k)(y)$$

$$= \bigwedge_{f\in\mathcal{V}}\Big(\bigvee_{z\in Y}\varphi_\psi(h)(k)(z)\wedge f(k)(z,y)\Big)$$
$$= \bigwedge_{f\in\mathcal{V}}\bigvee_{z\in Y}\Big(\bigvee_{x\in\varphi^{-1}(z)}\bigvee_{e\in\psi^{-1}(k)}h(e)(x)\wedge f(k)(z,y)\Big).$$

Then, for all $e\in\psi^{-1}(k)$, we get

$$\overline{\varphi_\psi(h)}(k)(y)\geq\bigwedge_{f\in\mathcal{V}}\bigvee_{x\in X}\Big(h(e)(x)\wedge f(\psi(e))(\varphi(x),y)\Big).$$

On the other hand, since φ_ψ is fuzzy soft uniformly continuous, for all $f\in\mathcal{V}$, there exists a $g\in\mathcal{U}$ such that $g\sqsubseteq(\varphi\times\varphi)_\psi^{-1}(f)$. Therefore, for all $x'\in\varphi^{-1}(y)$, we obtain

$$\begin{aligned}\overline{\varphi_\psi(h)}(k)(y) &\geq \bigwedge_{f\in\mathcal{V}}\bigvee_{x\in X}\Big(h(e)(x)\wedge f(\psi(e))(\varphi(x),\varphi(x'))\Big)\\ &\geq \bigwedge_{g\in\mathcal{U}}\bigvee_{x\in X}\left(h(e)(x)\wedge g(e)(x,x')\right)\\ &= \overline{h}(e)(x').\end{aligned}$$

Thus, from the fact that

$$\overline{\varphi_\psi(h)}(k)(y)\geq\bigvee_{x'\in\varphi^{-1}(y)}\bigvee_{e\in\psi^{-1}(k)}\overline{h}(e)(x')$$

it follows that $\overline{\varphi_\psi(h)}\sqsupseteq\varphi_\psi(\overline{h})$.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] Ahmad B, Kharal A. On fuzzy soft sets. Adv Fuzzy Syst 2009; Article ID 586507: 6 pages.
[2] Artico G, Moresco R. Fuzzy proximities compatible with Lowen uniformities. Fuzzy Sets Syst 1987; 21: 85-98.
[3] Aygünoğlu A, Aygün H. Introduction to fuzzy soft groups. Comput Math Appl 2009; 58: 1279-1286.
[4] Bayoumi F. On initial and final fuzzy uniform structures. Fuzzy Sets Syst 2003; 133: 299-319.
[5] Bayoumi F. On initial and final fuzzy uniform structures Part II. Fuzzy Sets Syst 2006; 157: 1970-1982.
[6] Burton MH. Boundedness in uniform spaces and fuzzy uniform spaces. Fuzzy Sets Syst 1993; 58: 195-207.
[7] Burton MH. Completeness in fuzzy uniform spaces. Quaestiones Math 1993; 16: 13-36.
[8] Chang CL. Fuzzy topological spaces. J Math Anal Appl 1968; 24(1): 182-190.
[9] Çetkin V, Aygün H. Uniformity structure in the context of soft set. Ann Fuzzy Math Inform 2013; 6(1): 69-76.
[10] Çetkin V, Aygün H. Extension of Shi's quasi-uniformity to the fuzzy soft sets. Math Sci Appl E-Notes, 2013; 1(2): 42-50.
[11] Çetkin V, Aygün H. On convergence of fuzzy soft filters. 3rd International Eurasian Conference on Mathematical Sciences and Applications; 25-28 August 2014; Vienna, Austria.
[12] Demir İ, Özbakır OB. Some properties of fuzzy soft proximity spaces. Sci World J 2015; Article ID 752634: 10 pages.
[13] Demir İ, Özbakır OB, Yıldız İ. Fuzzy soft ultrafilters and convergence properties of fuzzy soft filters. J New Results Sci 2015; 8: 92-107.
[14] Güler AÇ, Kale G. Regularity and normality on soft ideal topological spaces. Ann Fuzzy Math Inform 2015; 9(3): 373-383.
[15] Höhle U. Probabilistic uniformization of fuzzy topologies. Fuzzy Sets Syst 1978; 1: 311-332.
[16] Hussain S. A note on soft connectedness. J Egypt Math Soc 2015; 23: 6-11.
[17] Hutton B. Uniformities on fuzzy topological spaces. J Math Anal Appl 1977; 58: 557-571.
[18] Katsaras AK. Linear fuzzy neighborhood spaces. J Fuzzy Sets Syst 1984; 16: 143-154.
[19] Kharal A, Ahmad B. Mappings on fuzzy soft classes. Adv Fuzzy Syst 2009; Article ID 407890: 6 pages.
[20] Kotze W. Uniform spaces. In: Höhle U, Rodabaugh SE, editors. Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory; Boston, Dordrecht. London: Kluwer Academic Publishers, 1999, pp. 553-580.
[21] Lowen R. Fuzzy topological spaces and fuzzy compactness. J Math Anal Appl 1976; 56: 621-633.
[22] Lowen R. Fuzzy uniform spaces. J Math Anal Appl 1981; 82: 370-385.
[23] Lowen R. Fuzzy neighbourhood spaces. Fuzzy Sets Syst 1982; 7: 165-189.
[24] Lowen R, Wuyts P. Completeness, compactness and precompactness in fuzzy uniform spaces: Part 1. J Math Anal Appl 1982; 90(2): 563-581.
[25] Lowen R, Wuyts P. Completeness, compactness and precompactness in fuzzy uniform spaces: Part 2. J Math Anal Appl 1983; 92(2): 342-371.
[26] Ma X, Sulaiman N, Qin H, Herawan T, Zain JM. A new efficient normal parameter reduction algorithm of soft sets. Comput Math Appl 2011; 62: 588-598.
[27] Maji PK, Biswas R, Roy AR. Fuzzy soft sets. J Fuzzy Math 2001; 9(3): 589-602.
[28] Maji PK, Biswas R, Roy AR. Soft set theory. Comput Math Appl 2003; 45: 555-562.
[29] Molodtsov D. Soft set theory-first results. Comput Math Appl 1999; 37: 19-31.
[30] Neog TJ, Sut DK, Hazarika GC. Fuzzy soft topological spaces. Int J Latest Trend Math 2012; 2: 54-67.
[31] Özbakır OB, Demir İ. On the soft uniformity and its some properties. J Math Comput Sci 2015; 5: 762-779.
[32] Roy AR, Maji PK. A fuzzy soft set theoretic approach to decision making problems. J Comput Appl Math 2007; 203: 412-418.
[33] Shabir M, Naz M. On soft topological spaces. Comput Math Appl 2011; 61: 1786-1799.
[34] Shi FG. Pointwise uniformities in fuzzy set theory. Fuzzy Sets Syst 1998; 98: 141-146.
[35] Soetens E, Wuyts P. A characterisation of fuzzy uniform spaces by coverings. J Math Anal Appl 1993; 180: 275-302.
[36] Tanay B, Kandemir MB. Topological structures of fuzzy soft sets. Comput Math Appl 2011; 61: 412-418.
[37] Tukey JW. Convergence and Uniformity in Topology. Ann of Math Stud vol 2, Princeton University Press, 1940.
[38] Varol BP, Aygün H. Fuzzy soft topology. Hacet J Math Stat 2012; 41(3): 407-419.
[39] Weil A. Sur les Espaces a Structure Uniforme et sur la Topologie Generale. Hermann, Paris, 1937.
[40] Yıldırım ED, Güler AÇ, Özbakır OB. On soft $\widetilde{I}$-baire spaces. Ann Fuzzy Math Inform 2015; 10(1): 109-121.
[41] Zhang D. A comparison of various uniformities in fuzzy topology. Fuzzy Sets Syst 2003; 140: 399-422.