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ON NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY GENERALIZED SĂLĂGEAN DIFFERENTIAL OPERATOR

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Abstract. The aim of this investigation is to introduce two new subclasses of the class σ related with the generalized Sălăgean differential operator and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Moreover, we give some interesting results by using the relationship between Sălăgean's differential operator and generalized Sălăgean differential operator.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$.

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [5]).

For $\delta \geq 1$ and $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following differential operator:

$$\begin{aligned}
&\mathcal{D}_\delta^0 f(z) = f(z),\\
&\mathcal{D}_\delta^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = \mathcal{D}_\delta f(z),\\
&\quad\vdots\\
&\mathcal{D}_\delta^n f(z) = (1-\delta)\mathcal{D}_\delta^{n-1} f(z) + \delta z \left(\mathcal{D}_\delta^{n-1} f(z)\right)' = \mathcal{D}\left(\mathcal{D}_\delta^{n-1} f(z)\right), \; z \in \mathbb{U}, \; n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}.
\end{aligned} \tag{1.2}$$

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From (1.2), we see that

$$\mathcal{D}_\delta^n f(z) = z + \sum_{k\geq 2} [1+(k-1)\delta]^n a_k z^k$$

with $\mathcal{D}_\delta^n f(0) = 0$. It is worth mentioning that when $\delta = 1$ in (1.2), we have the Sălăgean's differential operator.

Koebe one-quarter theorem ensures that the image of $\mathbb{U}$ under every univalent functions $f \in \mathcal{A}$ contains an open disk centered at origin and with radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{U}) \to \mathbb{U}$ satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ..., \quad w \in f(\mathbb{U}). \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse $g = f^{-1}$ are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions which are (or which are not in) the class σ, together with various other properties of the bi-univalent function class σ one can refer recent works [2], [4], [6], [9].

Two of the most important subclasses of univalent functions are the class $\mathcal{S}^\star(\beta)$ of starlike functions of order β and the class $\mathcal{C}(\beta)$ of convex functions of order β. By definition, we get

$$\mathcal{S}^\star(\beta) = \left\{ f \in \mathcal{S} \,\middle|\, \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta; \quad z \in \mathbb{U} \text{ and } 0 \leq \beta < 1 \right\}$$

and

$$\mathcal{C}(\beta) = \left\{ f \in \mathcal{S} \,\middle|\, \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta; \quad z \in \mathbb{U} \text{ and } 0 \leq \beta < 1 \right\}.$$

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $\mathcal{S}_\sigma^\star(\beta)$ of bi-starlike function of order β, or $\mathcal{C}_\sigma(\beta)$ of bi-convex function of order β if both f and its inverse f^{-1} are, respectively, starlike and convex function of order β. These classes were introduced and investigated by Brannan and Taha [3]. Moreover, Brannan and Taha [3] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes $\mathcal{S}_\sigma^\star(\beta)$ and $\mathcal{C}_\sigma(\beta)$.

The aim of this investigation is to present two new subclasses of the class σ related with the generalized Sălăgean differential operator and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

In order to prove our main results, we require the following lemma.

Lemma 1.1. *(see [8]) If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \quad (z \in \mathbb{U}), \tag{1.4}$$

then

$$|c_i| \leq 2 \quad (i \in \mathbb{N}) \tag{1.5}$$

where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$ for which $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$.

2. Coefficient bounds for the function class $\mathcal{H}_\sigma^{m,n,\delta}(\alpha)$

Definition 2.1. *A function $f(z)$ defined by (1.1) is said to be in the class $\mathcal{H}_\sigma^{m,n,\delta}(\alpha)$ $\left(0 < \alpha \leq 1, \quad \delta \geq 1 \quad m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m > n\right)$ if the following conditions are satisfied*

$$f \in \sigma \quad \text{and} \quad \left| \arg\left(\frac{\mathcal{D}_\delta^m f(z)}{\mathcal{D}_\delta^n f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| \arg\left(\frac{\mathcal{D}_\delta^m g(w)}{\mathcal{D}_\delta^n g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (w \in f(\mathbb{U})) \tag{2.2}$$

where the function $g = f^{-1}$ is defined by (1.3).

For the functions in the class $\mathcal{H}_\sigma^{m,n,\delta}(\alpha)$, we start by finding the estimates on the coefficients $|a_2|$ and $|a_3|$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\sigma^{m,n,\delta}(\alpha)$ $\left(0 < \alpha \leq 1, \quad \delta \geq 1 \quad m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m > n\right)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha\left((1+2\delta)^m - (1+2\delta)^n\right) + \left((1+\delta)^m - (1+\delta)^n\right)^2 - \alpha\left((1+\delta)^{2m} - (1+\delta)^{2n}\right)}} \tag{2.3}$$

and

$$|a_3| \leq \frac{2\alpha}{(1+2\delta)^m - (1+2\delta)^n} + \frac{4\alpha^2}{\left((1+\delta)^m - (1+\delta)^n\right)^2}. \tag{2.4}$$

Proof. It can be said that the inequalities (2.1) and (2.2) are equivalent to

$$\frac{\mathcal{D}_\delta^m f(z)}{\mathcal{D}_\delta^n f(z)} = \left(p(z)\right)^\alpha \tag{2.5}$$

and

$$\frac{\mathcal{D}_\delta^m g(w)}{\mathcal{D}_\delta^n g(w)} = \left(q(w)\right)^\alpha, \tag{2.6}$$

where $p(z)$ and $q(w)$ belong to the class $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + ... \tag{2.8}$$

By considering (2.1), (2.7) and (2.2), (2.8) we get that

$$\big((1+\delta)^m-(1+\delta)^n\big)a_2=\alpha p_1, \tag{2.9}$$

$$\big((1+2\delta)^m-(1+2\delta)^n\big)a_3-(1+\delta)^n\big((1+\delta)^m-(1+\delta)^n\big)a_2^2=\frac{\alpha(\alpha-1)}{2}p_1^2+\alpha p_2, \tag{2.10}$$

$$\big((1+\delta)^m-(1+\delta)^n\big)a_2=-\alpha q_1 \tag{2.11}$$

and

$$\big((1+2\delta)^m-(1+2\delta)^n\big)(2a_2^2-a_3)-(1+\delta)^n\big((1+\delta)^m-(1+\delta)^n\big)a_2^2=\frac{\alpha(\alpha-1)}{2}q_1^2+\alpha q_2. \tag{2.12}$$

From (2.9) and (2.11) we obtain

$$p_1=-q_1 \tag{2.13}$$

and

$$2\big((1+\delta)^m-(1+\delta)^n\big)^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14) we get that

$$\begin{aligned}2\Big[\big((1+2\delta)^m-(1+2\delta)^n\big)-(1+\delta)^n\big((1+\delta)^m-(1+\delta)^n\big)\Big]a_2^2&\\ =\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)&\\ =\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\frac{2\big((1+\delta)^m-(1+\delta)^n\big)^2a_2^2}{\alpha^2}&.\end{aligned}$$

Therefore, we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{2\alpha\big((1+2\delta)^m-(1+2\delta)^n\big)+\big((1+\delta)^m-(1+\delta)^n\big)^2-\alpha\big((1+\delta)^{2m}-(1+\delta)^{2n}\big)}.$$

And also, with the help of Lemma 1.1, we arrive at

$$|a_2|\le\frac{2\alpha}{\sqrt{2\alpha\big((1+2\delta)^m-(1+2\delta)^n\big)+\big((1+\delta)^m-(1+\delta)^n\big)^2-\alpha\big((1+\delta)^{2m}-(1+\delta)^{2n}\big)}}.$$

This inequality is the desired estimate for $|a_2|$ as asserted (2.3).

Now, in order to find the bound on $|a_3|$ by subtracting (2.12) from (2.10), we obtain

$$2\big((1+2\delta)^m-(1+2\delta)^n\big)(a_3-a_2^2)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

and

$$a_3=\frac{\alpha(p_2-q_2)}{2\big((1+2\delta)^m-(1+2\delta)^n\big)}+\frac{\alpha^2(p_1^2+q_1^2)}{2\big((1+\delta)^m-(1+\delta)^n\big)^2}. \tag{2.15}$$

Considering Lemma 1.1 again for the coefficients p_1, p_2, q_1 and q_2, we arrive at

$$|a_3|\le\frac{2\alpha}{\big((1+2\delta)^m-(1+2\delta)^n\big)}+\frac{4\alpha^2}{\big((1+\delta)^m-(1+\delta)^n\big)^2}.$$

This completes the proof of the Theorem 2.1. □

3. Coefficient bounds for the function class $\mathcal{H}_{\sigma}^{m,n,\delta}(\beta)$

Definition 3.1. *A function $f(z)$ defined by 1.1 is said to be in the class $\mathcal{H}_{\sigma}^{m,n,\delta}(\beta)$ $\big(0 \leq \beta < 1, \quad \delta \geq 1 \quad m,n \in \mathbb{N}_0, m > n\big)$ if the following conditions are satisfied:*

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}\left(\frac{\mathcal{D}_{\delta}^{m} f(z)}{\mathcal{D}_{\delta}^{n} f(z)}\right) > \beta \quad (z \in \mathbb{U}) \tag{3.1}$$

and

$$\operatorname{Re}\left(\frac{\mathcal{D}_{\delta}^{m} g(w)}{\mathcal{D}_{\delta}^{n} g(w)}\right) > \beta \quad (w \in f(\mathbb{U})) \tag{3.2}$$

where the function $g = f^{-1}$ is defined by (1.3).

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\sigma}^{m,n,\delta}(\beta)$ $\big(0 \leq \beta < 1, \quad \delta \geq 1 \quad m,n \in \mathbb{N}_0, m > n\big)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\big((1+2\delta)^m - (1+2\delta)^n\big) - (1+\delta)^n\big((1+\delta)^m - (1+\delta)^n\big)}} \tag{3.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1+\delta)^m - (1+\delta)^n]^2} + \frac{2(1-\beta)}{(1+2\delta)^m - (1+2\delta)^n}. \tag{3.4}$$

Proof. It can be written that the inequalities (3.1) and (3.2) are equivalent to

$$\frac{\mathcal{D}_{\delta}^{m} f(z)}{\mathcal{D}_{\delta}^{n} f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{\mathcal{D}_{\delta}^{m} g(w)}{\mathcal{D}_{\delta}^{n} g(w)} = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ belong to the class $\mathcal{P}$ have the forms (2.7) and (2.8), respectively. Equating coefficients in (3.5) and (3.6) yields

$$\big((1+\delta)^m - (1+\delta)^n\big)a_2 = (1-\beta)p_1, \tag{3.7}$$

$$\big((1+2\delta)^m - (1+2\delta)^n\big)a_3 - (1+\delta)^n\big((1+\delta)^m - (1+\delta)^n\big)a_2^2 = (1-\beta)p_2, \tag{3.8}$$

$$\big((1+\delta)^m - (1+\delta)^n\big)a_2 = -(1-\beta)q_1 \tag{3.9}$$

and

$$\big((1+2\delta)^m - (1+2\delta)^n\big)(2a_2^2 - a_3) - (1+\delta)^n\big((1+\delta)^m - (1+\delta)^n\big)a_2^2 = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$p_1 = -q_1 \tag{3.11}$$

and

$$2\big((1+\delta)^m - (1+\delta)^n\big)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

And also from (3.8) and (3.10), we obtain

$$2\big[\big((1+2\delta)^m-(1+2\delta)^n\big)-(1+\delta)^n\big((1+\delta)^m-(1+\delta)^n\big)\big]a_2^2=(1-\beta)(p_2+q_2). \quad (3.13)$$

Hence, we arrive at

$$|a_2|^2 \leq \frac{2(1-\beta)}{\big((1+2\delta)^m-(1+2\delta)^n\big)-(1+\delta)^n\big((1+\delta)^m-(1+\delta)^n\big)}$$

which is the desired inequality as given in the (3.3).

Now, by subtracting (3.10) from (3.8), we have

$$2\big[(1+2\delta)^m-(1+2\delta)^n\big](a_3-a_2^2)=(1-\beta)(p_2-q_2)$$

or equivalently

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{2\big[(1+2\delta)^m-(1+2\delta)^n\big]}.$$

Upon substituting the value of a_2^2 from (3.12) we arrive at

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{2\big[(1+\delta)^m-(1+\delta)^n\big]^2}+\frac{(1-\beta)(p_2-q_2)}{2\big[(1+2\delta)^m-(1+2\delta)^n\big]}.$$

Next, with the help of the Lemma 1.1, we get that

$$|a_3| \leq \frac{4(1-\beta)^2}{\big[(1+\delta)^m-(1+\delta)^n\big]^2}+\frac{2(1-\beta)}{(1+2\delta)^m-(1+2\delta)^n}$$

which is the bound on $|a_3|$ as asserted in (3.4). □

4. Some particular cases of the main results

It is important to mention that the Al-Oboudi's differential operator is actually a generalization of the Sălăgean's differential operator. Namely, we have the relation

$$\mathcal{D}_1^n f(z)=\mathcal{D}^n f(z),$$

where $\mathcal{D}_\delta^n$ stands for the Al-Oboudi's differential operator. Taking into account this, it is clear that Theorem 2.1 and Theorem 3.1, in particular for $\delta=1$, reduce to some interesting results.

Now, taking $\delta=1$ in the Theorem 2.1 we attain the following result given by [10].

Corollary 4.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\sigma^{m,n,1}(\alpha)=\mathcal{H}_\sigma^{m,n}(\alpha)$ $\big(0<\alpha\leq 1, \quad m,n\in\mathbb{N}_0, m>n\big)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(3^m-3^n)+(2^m-2^n)^2-\alpha(2^{2m}-2^{2n})}}$$

and

$$|a_3| \leq \frac{2\alpha}{3^m-3^n}+\frac{4\alpha^2}{\big(2^m-2^n\big)^2}.$$

Setting $\delta = 1$ in the Theorem 3.1 we arrive at the following result given by [10].

Corollary 4.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\sigma}^{m,n,1}(\beta) = \mathcal{H}_{\sigma}^{m,n}(\beta)$ $\big(0 \leq \beta < 1, \quad m, n \in \mathbb{N}_0, m > n\big)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\big(3^m - 3^n\big) - 2^n\big(2^m - 2^n\big)}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(2^m - 2^n)^2} + \frac{2(1-\beta)}{3^m - 3^n}.$$

Putting $m = 1$ and $n = 0$ in Corollary 4.2 we obtain the following result

Corollary 4.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\sigma}^{1,0,1}(\beta) = \mathcal{S}_{\sigma}^{\star}(\beta)$. Then, we get*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq (1-\beta)(5-4\beta).$$

Taking $m = 2$ and $n = 1$ in Corollary 4.2 we have the following result

Corollary 4.4. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\sigma}^{2,1,1}(\beta) = \mathcal{C}_{\sigma}(\beta)$. Then, we get*

$$|a_2| \leq \sqrt{1-\beta}$$

and

$$|a_3| \leq \frac{(1-\beta)(4-3\beta)}{3}.$$

Moreover, it is worth mentioning that if we take $\delta = 1$ and $m = n + 1$ in the Definition 3.1, we obtain the class $\mathcal{S}_{\sigma,n}(\beta)$ of n-bi-starlike functions of order β, which is introduced and studied by Orhan et al [7].

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