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Complete lifts of vector fields to the special class of semi-tensor bundle

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Abstract

Using projection (submersion) of the cotangent bundle T*M over a manifold M, we define a semi-tensor (pull-back) bundle tM of type (p,q). The present paper is devoted to some results concerning with the complete lifts of vector fields from manifold M to its (p,q)-semitensor bundle.

Keywords: *Vector field, complete lift, pull-back bundle, cotangent bundle, semi-tensor bundle.*
2010 Mathematics Subject Classification: *53A45, 55R10, 57R25.*

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and let $(T^*(M_n), \pi_1, M_n)$ be a cotangent bundle over M_n. We use the notation $(x^i) = \left(x^{\overline{\alpha}}, x^{\alpha}\right)$, where the indices $i, j, \ldots$ run from 1 to $2n$, the indices $\overline{\alpha}, \overline{\beta}, \ldots$ from 1 to n and the indices $\alpha, \beta, \ldots$ from $n+1$ to $2n$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = p_\alpha$ are fibre coordinates of the cotangent bundle $T^*(M_n)$ [12].
Let now $\left(T_q^p(M_n), \widetilde{\pi}, M_n\right)$ be a tensor bundle [1], [4], [[5], p.118] with base space M_n, and let $T^*(M_n)$ be cotangent bundle determined by a natural projection (submersion) $\pi_1 : T^*(M_n) \to M_n$. The semi-tensor bundle (induced, pull-back [2], [3], [6], [7], [10], [13], [14]) of the tensor bundle $\left(T_q^p(M_n), \widetilde{\pi}, M_n\right)$ is the bundle $\left(t_q^p(M_n), \pi_2, T^*(M_n)\right)$ over cotangent bundle $T^*(M_n)$ with a total space

$$\begin{aligned} t_q^p(M_n) &= \left\{ \left(\left(x^{\overline{\alpha}}, x^{\alpha}\right), x^{\overline{\overline{\alpha}}}\right) \in T^*(M_n) \times \left(T_q^p\right)_x(M_n) : \pi_1\left(x^{\overline{\alpha}}, x^{\alpha}\right) = \widetilde{\pi}\left(x^{\alpha}, x^{\overline{\overline{\alpha}}}\right) = (x^\alpha) \right\} \\ &\subset T^*(M_n) \times \left(T_q^p\right)_x(M_n) \end{aligned}$$

and with the projection map $\pi_2 : t_q^p(M_n) \to T^*(M_n)$ defined by $\pi_2(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}}) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where $\left(T_q^p\right)_x(M_n)$ $\left(x = \pi_1(\widetilde{x}), \widetilde{x} = \left(x^{\overline{\alpha}}, x^\alpha\right) \in T^*(M_n)\right)$ is the tensor space at a point x of M_n, where $x^{\overline{\overline{\alpha}}} = t_{\alpha_1 \ldots \alpha_q}^{\beta_1 \ldots \beta_p}$ $\left(\overline{\overline{\alpha}}, \overline{\overline{\beta}}, \ldots = 2n+1, \ldots, 2n+n^{p+q}\right)$ are fiber coordinates of the tensor bundle $T_q^p(M_n)$ [12].
If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'}, x^{\overline{\overline{\alpha}}'})$ is another system of local adapted coordinates in the semi-tensor bundle $t_q^p(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} p_\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\overline{\alpha}}'} = t_{\alpha_1' \ldots \alpha_q'}^{\beta_1' \ldots \beta_p'} = A_{\alpha_1 \ldots \alpha_p}^{\beta_1' \ldots \beta_p'} A_{\alpha_1' \ldots \alpha_q'}^{\beta_1 \ldots \beta_q} t_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p} = A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} x^{\overline{\overline{\beta}}}. \end{cases} \tag{1.1}$$

The jacobian of (1.1) has components

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_{\alpha'}^{\beta} & p_\sigma A_\beta^{\beta'} A_{\beta'\alpha'}^{\sigma} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & t_{(\sigma)}^{(\alpha)} \partial_\beta A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\sigma)} & A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} \end{pmatrix}, \tag{1.2}$$

where $I = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $J = (\overline{\beta}, \beta, \overline{\overline{\beta}})$, $I, J \ldots = 1, \ldots, 2n+n^{p+q}$, $t_{(\sigma)}^{(\alpha)} = t_{\sigma_1 \ldots \sigma_q}^{\alpha_1 \ldots \alpha_p}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$, $A_{\alpha'}^{\beta} = \frac{\partial x^\beta}{\partial x^{\alpha'}}$, $A_{\beta'\alpha'}^{\sigma} = \frac{\partial^2 x^\sigma}{\partial x^{\beta'} \partial x^{\alpha'}}$.

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It is easily verified that the condition $Det\bar{A} \neq 0$ is equivalent to the condition:

$Det(A^{\beta}_{\alpha'}) \neq 0, Det(A^{\alpha'}_{\beta}) \neq 0, Det(A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')}) \neq 0.$

Also, $\dim t^p_q(M_n) = 2n + n^{p+q}$ [12].

We note that the special classes of semi-tensor bundles were examined in [9], [11], [12]. The present paper is devoted to some results concerning with the complete lifts of vector fields from manifold M to its (p,q)-semitensor bundle.

We denote by $\Im^p_q(T^*(M_n))$ and $\Im^p_q(M_n)$ the modules over $F(T^*(M_n))$ and $F(M_n)$ of all tensor fields of type (p,q) on $T^*(M_n)$ and M_n, respectively, where $F(T^*(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ $-$functions on $T^*(M_n)$ and M_n, respectively.

2. Vertical lifts of tensor fields and $\gamma-$operator

Let $A \in \Im^p_q(T^*(M_n))$. On putting

$$^{vv}A = \begin{pmatrix} ^{vv}A^{\overline{\alpha}} \\ ^{vv}A^{\alpha} \\ ^{vv}A^{\overline{\overline{\alpha}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}, \tag{2.1}$$

from (1.2), we easily see that with $^{vv}A' = \bar{A}(^{vv}A)$. The vector field $^{vv}A \in \Im^1_0(t^p_q(M_n))$ is called the vertical lift of $A \in \Im^p_q(T^*(M_n))$ to the semi-tensor bundle $t^p_q(M_n)$ [12].

On the other hand, ^{vv}f the vertical lift of function $f \in \Im^0_0(M_n)$ on $t^p_q(M_n)$ is defined by:

$$^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{2.2}$$

Let $\varphi \in \Im^1_1(M_n)$. We define a vector field $\gamma\varphi$ in $\pi^{-1}(U)$ by

$$\begin{cases} \gamma\varphi = \left(\sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q} \varphi^{\alpha_\lambda}_{\varepsilon}\right) \frac{\partial}{\partial x^{\overline{\overline{\beta}}}}, & (p \geq 1, q \geq 0) \\ \widetilde{\gamma}\varphi = \left(\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q} \varphi^{\varepsilon}_{\beta_\mu}\right) \frac{\partial}{\partial x^{\overline{\overline{\beta}}}}, & (p \geq 0, q \geq 1) \end{cases} \tag{2.3}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^p_q(M_n)$. From (1.2) we easily see that the vector fields $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ defined in each $\pi^{-1}(U) \subset t^p_q(M_n)$ determine respectively global vertical vector fields on $t^p_q(M_n)$. We call $\gamma\varphi$ (or $\widetilde{\gamma}\varphi$) the vertical-vector lift of the tensor field $\varphi \in \Im^1_1(M_n)$ to $t^p_q(M_n)$. For any $\varphi \in \Im^1_1(M_n)$, if we take account of (1.2) and (2.3), we can prove that $(\gamma\varphi)' = \bar{A}(\gamma\varphi)$ where $\gamma\varphi$ is a vector field defined by [12]:

$$\gamma\varphi = (\gamma\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q} \varphi^{\alpha_\lambda}_{\varepsilon} \end{pmatrix}. \tag{2.4}$$

Let $\varphi \in \Im^1_1(M_n)$. On putting

$$\widetilde{\gamma}\varphi = (\widetilde{\gamma}\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q} \varphi^{\varepsilon}_{\beta_\mu} \end{pmatrix}, \tag{2.5}$$

we easily see that $(\widetilde{\gamma}\varphi)' = \bar{A}(\widetilde{\gamma}\varphi)$ [12].

3. Complete lifts of vector fields

Let $X \in \Im^1_0(T^*(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. The complete lift ^cX of X to cotangent bundle is defined by $^cX = X^\alpha \partial_\alpha - p_\beta(\partial_\alpha X^\beta)\partial_{\overline{\alpha}}$ [[8], p.236]. On putting

$$^{cc}X = \begin{pmatrix} ^{cc}X^{\overline{\beta}} \\ ^{cc}X^{\beta} \\ ^{cc}X^{\overline{\overline{\beta}}} \end{pmatrix} = \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) \\ X^\beta \\ \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q} \partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q} \partial_{\beta_\mu} X^\varepsilon \end{pmatrix}, \tag{3.1}$$

from (1.2), we easily see that $^{cc}X' = \bar{A}(^{cc}X)$. The vector field ^{cc}X is called the complete lift of $^cX \in \Im^1_0(T^*(M_n))$ to $t^p_q(M_n)$ [12].

Theorem 3.1. *For any vector fields X, Y on $T^*(M_n)$ and $f \in \Im^0_0(M_n)$, we have*
(i) $^{cc}(X+Y) = {}^{cc}X + {}^{cc}Y,$
(ii) $^{cc}X\, {}^{vv}f = {}^{vv}(Xf).$

Proof. (i) This immediately follows from (3.1).(ii) Let $X \in \Im^1_0(T^*(M_n))$. Then we get by (2.2) and (3.1):

$$\begin{aligned} ^{cc}X\,{}^{vv}f &= {}^{cc}X^I \partial_I(^{vv}f) \\ &= {}^{cc}X^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}}(^{vv}f)}_{0} + {}^{cc}X^\alpha \partial_\alpha(^{vv}f) + {}^{cc}X^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}}(^{vv}f)}_{0} \\ &= X^\alpha \partial_\alpha(^{vv}f) \\ &= {}^{vv}(Xf), \end{aligned}$$

which gives (ii) of Theorem 3.1. □

Theorem 3.2. *If $\varphi \in \Im_1^1(M_n)$, $f \in \Im_0^0(M_n)$ and $A \in \Im_q^p(T^*(M_n))$, then*
(i) $({}^{vv}A)({}^{vv}f) = 0$,
(ii) $(\gamma\varphi)({}^{vv}f) = 0$,
(iii) $(\widetilde{\gamma}\varphi)({}^{vv}f) = 0$.

Proof. (i) If $A \in \Im_q^p(T^*(M_n))$, then, by (2.1) and (2.2), we find

$$\begin{aligned} ({}^{vv}A)^{vv}f &= ({}^{vv}A)^I \partial_I({}^{vv}f) \\ &= ({}^{vv}A)^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} + \underbrace{({}^{vv}A)^{\alpha}}_{0} \partial_\alpha({}^{vv}f) + ({}^{vv}A)^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}}({}^{vv}f)}_{0} \\ &= 0. \end{aligned}$$

Thus, we have (i) of Theorem 3.2.
(ii) If $\varphi \in \Im_1^1(M_n)$, then we have by (2.2) and (2.4):

$$\begin{aligned} (\gamma\varphi)({}^{vv}f) &= (\gamma\varphi)^I \partial_I({}^{vv}f) \\ &= (\gamma\varphi)^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} + \underbrace{(\gamma\varphi)^{\alpha}}_{0} \partial_\alpha({}^{vv}f) + (\gamma\varphi)^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}}({}^{vv}f)}_{0} \\ &= 0. \end{aligned}$$

Thus, we have (ii) of Theorem 3.2.
(iii) If $\varphi \in \Im_1^1(M_n)$, then we have by (2.2) and (2.5):

$$\begin{aligned} (\widetilde{\gamma}\varphi)({}^{vv}f) &= (\widetilde{\gamma}\varphi)^I \partial_I({}^{vv}f) \\ &= (\widetilde{\gamma}\varphi)^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} + \underbrace{(\widetilde{\gamma}\varphi)^{\alpha}}_{0} \partial_\alpha({}^{vv}f) + (\widetilde{\gamma}\varphi)^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}}({}^{vv}f)}_{0} \\ &= 0. \end{aligned}$$

Thus, we have (iii) of Theorem 3.2. □

Theorem 3.3. *Let $X, Y \in \Im_0^1(T^*(M_n))$. For the Lie product, we have*

$[{}^{cc}X, {}^{cc}Y] = {}^{cc}[X,Y] (i.e. L_{{}^{cc}X}{}^{cc}Y = {}^{cc}(L_X Y))$.

Proof. If $X, Y \in \Im_0^1(T^*(M_n))$ and $\begin{pmatrix} [{}^{cc}X, {}^{cc}Y]^{\overline{\beta}} \\ [{}^{cc}X, {}^{cc}Y]^{\beta} \\ [{}^{cc}X, {}^{cc}Y]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X, {}^{cc}Y]^J$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t_q^p(M_n)$, then we have

$[{}^{cc}X, {}^{cc}Y]^J = ({}^{cc}X)^I \partial_I ({}^{cc}Y)^J - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^J$.

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} [{}^{cc}X, {}^{cc}Y]^{\overline{\beta}} &= ({}^{cc}X)^I \partial_I ({}^{cc}Y)^{\overline{\beta}} - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^{\overline{\beta}} \\ &= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}Y)^{\overline{\beta}} + ({}^{cc}X)^{\alpha} \partial_{\alpha} ({}^{cc}Y)^{\overline{\beta}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} ({}^{cc}Y)^{\overline{\beta}} \\ &\quad - ({}^{cc}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}X)^{\overline{\beta}} - ({}^{cc}Y)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\overline{\beta}} - ({}^{cc}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} ({}^{cc}X)^{\overline{\beta}} \\ &= -({}^{cc}X)^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}} p_\varepsilon}_{\delta_\varepsilon^\alpha} (\partial_\beta Y^\varepsilon) - ({}^{cc}X)^{\alpha} \partial_\alpha p_\varepsilon (\partial_\beta Y^\varepsilon) \\ &\quad - ({}^{cc}X)^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}} p_\varepsilon}_{0} (\partial_\beta Y^\varepsilon) + ({}^{cc}Y)^{\overline{\alpha}} \underbrace{\partial_{\overline{\alpha}} p_\varepsilon}_{\delta_\varepsilon^\alpha} (\partial_\beta X^\varepsilon) \\ &\quad + ({}^{cc}Y)^{\alpha} \partial_\alpha p_\varepsilon (\partial_\beta X^\varepsilon) + ({}^{cc}Y)^{\overline{\overline{\alpha}}} \underbrace{\partial_{\overline{\overline{\alpha}}} p_\varepsilon}_{0} (\partial_\beta X^\varepsilon) \\ &= -({}^{cc}X)^{\overline{\alpha}} (\partial_\beta Y^\alpha) - ({}^{cc}X)^{\alpha} \partial_\alpha p_\varepsilon (\partial_\beta Y^\varepsilon) \\ &\quad + ({}^{cc}Y)^{\overline{\alpha}} (\partial_\beta X^\alpha) + ({}^{cc}Y)^{\alpha} \partial_\alpha p_\varepsilon (\partial_\beta X^\varepsilon) \\ &= +p_\varepsilon \partial_\alpha X^\varepsilon (\partial_\beta Y^\alpha) - X^\alpha \partial_\alpha p_\varepsilon (\partial_\beta Y^\varepsilon) \\ &\quad - p_\varepsilon \partial_\alpha Y^\varepsilon (\partial_\beta X^\alpha) + Y^\alpha \partial_\alpha p_\varepsilon (\partial_\beta X^\varepsilon) \\ &= p_\varepsilon(-X^\alpha \partial_\alpha \partial_\beta Y^\varepsilon + \partial_\beta Y^\alpha \partial_\alpha X^\varepsilon + Y^\alpha \partial_\alpha \partial_\beta X^\varepsilon - \partial_\beta X^\alpha \partial_\alpha Y^\varepsilon) \\ &= -p_\varepsilon(\partial_\beta \underbrace{(X^\alpha \partial_\alpha Y^\varepsilon - Y^\alpha \partial_\alpha X^\varepsilon)}_{[X,Y]^\varepsilon}) \\ &= -p_\varepsilon(\partial_\beta [X,Y]^\varepsilon) \end{aligned}$$

by virtue of (3.1). Secondly, if $J=\beta$, we have

$$
\begin{aligned}
[{}^{cc}X,{}^{cc}Y]^{\beta} &= ({}^{cc}X)^{I}\partial_{I}({}^{cc}Y)^{\beta}-({}^{cc}Y)^{I}\partial_{I}({}^{cc}X)^{\beta}\\
&= ({}^{cc}X)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}Y)^{\beta}}_{0}+({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{cc}Y)^{\beta}+({}^{cc}X)^{\overline{\overline{\alpha}}}\underbrace{\partial_{\overline{\overline{\alpha}}}({}^{cc}Y)^{\beta}}_{0}\\
&\quad -({}^{cc}Y)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}X)^{\beta}}_{0}-({}^{cc}Y)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\beta}-({}^{cc}Y)^{\overline{\overline{\alpha}}}\underbrace{\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\beta}}_{0}\\
&= ({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{cc}Y)^{\beta}-({}^{cc}Y)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\beta}\\
&= X^{\alpha}\partial_{\alpha}Y^{\beta}-Y^{\alpha}\partial_{\alpha}X^{\beta}\\
&= [X,Y]^{\beta}
\end{aligned}
$$

by virtue of (3.1). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$
\begin{aligned}
[{}^{cc}X,{}^{cc}Y]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^{I}\partial_{I}({}^{cc}Y)^{\overline{\overline{\beta}}}-({}^{cc}Y)^{I}\partial_{I}({}^{cc}X)^{\overline{\overline{\beta}}}\\
&= ({}^{cc}X)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}Y)^{\overline{\overline{\beta}}}}_{0}+({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{cc}Y)^{\overline{\overline{\beta}}}+({}^{cc}X)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}Y)^{\overline{\overline{\beta}}}\\
&\quad -({}^{cc}Y)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\overline{\beta}}}}_{0}-({}^{cc}Y)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\overline{\overline{\beta}}}-({}^{cc}Y)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}}
\end{aligned}
$$

$$
\begin{aligned}
&= X^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\varepsilon}\Big)\\
&\quad +\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\alpha_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\gamma}\Big)\partial_{\overline{\overline{\alpha}}}\\
&\quad \Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}\partial_{\sigma}Y^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\gamma}\Big)\\
&\quad -Y^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}X^{\varepsilon}\Big)\\
&\quad -\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\alpha_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\varepsilon}\Big)\partial_{\overline{\overline{\alpha}}}\\
&\quad \Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}\partial_{\sigma}X^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}X^{\gamma}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= X^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\beta_\lambda}\Big)-X^{\alpha}\partial_{\alpha}\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}Y^{\varepsilon})\\
&\quad +\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\alpha_\lambda}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}}_{\delta_{\alpha_\lambda}^{\sigma}}\partial_{\sigma}Y^{\beta_\lambda}\\
&\quad -Y^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\beta_\lambda}\Big)+Y^{\alpha}\partial_{\alpha}\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}X^{\varepsilon})\\
&\quad -\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\alpha_\lambda}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}}_{\delta_{\alpha_\lambda}^{\sigma}}\partial_{\sigma}X^{\beta_\lambda}\\
&\quad \underbrace{+\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\varepsilon}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}}_{\delta_{\gamma}^{\alpha}}\partial_{\beta_\mu}Y^{\gamma}}_{\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\varepsilon}(\partial_{\beta_\mu}Y^{\alpha})}\\
&\quad \underbrace{-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}Y^{\varepsilon}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}}_{\delta_{\gamma}^{\alpha}}\partial_{\beta_\mu}X^{\gamma}}_{\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}Y^{\varepsilon}(\partial_{\beta_\mu}X^{\alpha})}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(\partial_\varepsilon X^\sigma)\left(\partial_\sigma Y^{\beta_\lambda}\right) + \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} X^\alpha \partial_\alpha \partial_\varepsilon Y^{\beta_\lambda} \\
&\quad - \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(\partial_\varepsilon Y^\sigma)\left(\partial_\sigma X^{\beta_\lambda}\right) - \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} Y^\alpha \partial_\alpha \partial_\varepsilon X^{\beta_\lambda} \\
&\quad \underbrace{+ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(-X^\alpha \partial_{\alpha_\mu}\partial_{\beta_\mu} Y^\varepsilon + \partial_{\beta_\mu} Y^\alpha \partial_{\alpha_\mu} X^\varepsilon + Y^\alpha \partial_{\alpha_\mu}\partial_{\beta_\mu} X^\varepsilon - \partial_{\beta_\mu} X^\alpha \partial_{\alpha_\mu} Y^\varepsilon)}_{-\sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}\underbrace{(X^\alpha\partial_{\alpha_\mu}Y^\varepsilon - Y^\alpha\partial_{\alpha_\mu}X^\varepsilon)}_{[X,Y]^\varepsilon})} \\
&= \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_\varepsilon [X,Y]^{\beta_\lambda} - \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}[X,Y]^\varepsilon)
\end{aligned}
$$

by virtue of (3.1). On the other hand, we know that ${}^{cc}[X,Y]$ have components

$$
{}^{cc}[X,Y] = \begin{pmatrix} -p_\varepsilon(\partial_\beta [X,Y]^\varepsilon) \\ [X,Y]^\beta \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_\varepsilon [X,Y]^{\beta_\lambda} - \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}[X,Y]^\varepsilon \end{pmatrix}
$$

with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ on $t_q^p(M_n)$. Thus Theorem 3.3 is proved. □

Theorem 3.4. *Let $X \in \Im_0^1(T^*(M_n))$. For the Lie product, we have*

$[{}^{cc}X, {}^{vv}A] = {}^{vv}(L_X A)$

for any $A \in \Im_q^p(T^*(M_n))$.

Proof. If $X \in \Im_0^1(T^*(M_n))$, $A \in \Im_q^p(T^*(M_n))$ and $\begin{pmatrix} [{}^{cc}X, {}^{vv}A]^{\overline{\beta}} \\ [{}^{cc}X, {}^{vv}A]^{\beta} \\ [{}^{cc}X, {}^{vv}A]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X, {}^{vv}A]^J$ with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ on $t_q^p(M_n)$, then we have

$[{}^{cc}X, {}^{vv}A]^J = ({}^{cc}X)^I \partial_I ({}^{vv}A)^J - ({}^{vv}A)^I \partial_I ({}^{cc}X)^J.$

Firstly, if $J = \overline{\beta}$, we have

$$
\begin{aligned}
[{}^{cc}X, {}^{vv}A]^{\overline{\beta}} &= ({}^{cc}X)^I \partial_I \underbrace{({}^{vv}A)^{\overline{\beta}}}_{0} - ({}^{vv}A)^I \partial_I ({}^{cc}X)^{\overline{\beta}} \\
&= -\underbrace{({}^{vv}A)^{\overline{\alpha}}}_{0}\partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\beta}} - \underbrace{({}^{vv}A)^{\alpha}}_{0}\partial_{\alpha}({}^{cc}X)^{\overline{\beta}} - ({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\beta}} \\
&= A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\underbrace{\partial_{\overline{\overline{\alpha}}} p_\varepsilon}_{0}(\partial_\beta X^\varepsilon) \\
&= 0
\end{aligned}
$$

by virtue of (2.1) and (3.1). Secondly, if $J = \beta$, we have

$$
\begin{aligned}
[{}^{cc}X, {}^{vv}A]^{\beta} &= ({}^{cc}X)^I \partial_I \underbrace{({}^{vv}A)^{\beta}}_{0} - ({}^{vv}A)^I \partial_I ({}^{cc}X)^{\beta} \\
&= -\underbrace{({}^{vv}A)^{\overline{\alpha}}}_{0}\partial_{\overline{\alpha}}({}^{cc}X)^{\beta} - \underbrace{({}^{vv}A)^{\alpha}}_{0}\partial_{\alpha}({}^{cc}X)^{\beta} - ({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\beta} \\
&= A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\overline{\overline{\alpha}}} X^\beta \\
&= 0
\end{aligned}
$$

by virtue of (2.1) and (3.1). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$\begin{aligned}
[{}^{cc}X,{}^{vv}A]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^I\partial_I({}^{vv}A)^{\overline{\overline{\beta}}}-({}^{vv}A)^I\partial_I({}^{cc}X)^{\overline{\overline{\beta}}}\\
&= ({}^{cc}X)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{vv}A)^{\overline{\overline{\beta}}}}_{0}+({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{vv}A)^{\overline{\overline{\beta}}}+({}^{cc}X)^{\overline{\overline{\alpha}}}\underbrace{\partial_{\overline{\overline{\alpha}}}({}^{vv}A)^{\overline{\overline{\beta}}}}_{0}\\
&\quad-\underbrace{({}^{vv}A)^{\overline{\alpha}}}_{0}\partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\overline{\beta}}}-\underbrace{({}^{vv}A)^{\alpha}}_{0}\partial_{\alpha}({}^{cc}X)^{\overline{\overline{\beta}}}-({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}}\\
&= ({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{vv}A)^{\overline{\overline{\beta}}}-({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}}\\
&= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}\Big(\sum_{\lambda=1}^{p}t^{\alpha_1\ldots\sigma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_\sigma X^{\beta_\lambda}-\sum_{\mu=1}^{q}t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\partial_{\beta_\mu}X^{\gamma}\Big)\\
&= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}\sum_{\lambda=1}^{p}t^{\alpha_1\ldots\sigma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_\sigma X^{\beta_\lambda}\\
&\quad+({}^{vv}A)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}\sum_{\mu=1}^{q}t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\partial_{\beta_\mu}X^{\gamma}\\
&= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-A^{\alpha_1\ldots\alpha_\lambda\ldots\alpha_p}_{\beta_1\ldots\beta_q}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\lambda=1}^{p}t^{\alpha_1\ldots\sigma\ldots\alpha_p}_{\beta_1\ldots\beta_q}}_{\delta^{\sigma}_{\alpha_\lambda}}\partial_\sigma X^{\beta_\lambda}\\
&\quad\underbrace{+A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\alpha\ldots\beta_q}\underbrace{\partial_{\overline{\overline{\alpha}}}\sum_{\mu=1}^{q}t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}}_{\delta^{\alpha}_{\gamma}}\partial_{\beta_\mu}X^{\gamma}}_{\sum_{\mu=1}^{q}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\alpha\ldots\beta_q}\partial_{\beta_\mu}X^{\alpha}}\\
&= X^{\alpha}\partial_{\alpha}A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-\sum_{\lambda=1}^{p}\left(\partial_\sigma X^{\beta_\lambda}\right)A^{\alpha_1\ldots\sigma\ldots\alpha_p}_{\beta_1\ldots\beta_q}+\sum_{\mu=1}^{q}\left(\partial_{\beta_\mu}X^{\alpha}\right)A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\alpha\ldots\beta_q}\\
&= L_XA^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}
\end{aligned}$$

by virtue of (2.1) and (3.1). On the other hand, we know that ${}^{vv}(L_XA)$ have components

$${}^{vv}(L_XA)=\begin{pmatrix}0\\0\\(L_XA)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t^p_q(M_n)$. Thus, Theorem 3.4 is proved. □

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