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Some Fixed Point Results on Complex Valued S_b-Metric Spaces

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Abstract
More recently, the notion of a complex valued S_b-metric space has been introduced and studied. In this paper, we investigate some basic properties of this new space. We study some fixed point results on a complete complex valued S_b-metric space. A common fixed point theorem for two self-mappings on a complete complex valued S_b-metric space is also given.

Keywords: *Complex valued S_b-metric space; fixed point; common fixed point.*

AMS Subject Classification (2010): *Primary: 32C15; Secondary: 47H10, 54H25.*

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1. Introduction

Many authors have introduced some generalizations of metric spaces such as b-metric spaces, G-metric spaces, S-metric spaces, S_b-metric spaces etc. Bakhtin gave the notion of a b-metric space [4]. Mustafa and Sims introduced the concept of a G-metric space [12]. Sedghi, Shobe and Aliouche defined the notion of S-metric spaces and proved some fixed-point theorems on a complete S-metric space [20]. Then Aghajani, Abbas and Roshan studied a new type of metric which is called G_b-metric [1]. The notion of an S_b-metric space, as a generalization of metric and S-metric spaces, was presented and some properties of this space were investigated in two different studies (see [23] and [24] for more details).

Also many authors have proved some fixed-point theorems on complex valued metric spaces. Azam, Fisher and Khan introduced complex valued metric spaces and obtained common fixed-point theorems on a complex valued metric space [3]. Rao, Swamy and Prasad defined complex valued b-metric spaces [18]. Mlaiki obtained common fixed-point theorems on a complex S-metric space [8]. Ege studied complex valued G_b-metric spaces and proved the Banach's contraction principle on a complete complex valued G_b-metric space [7]. Also Priyobarta, Rohen and Mlaiki defined the concept of a complex valued S_b-metric space and proved some fixed point theorems using the topology of this space [17].

Since then, many authors investigate some fixed-point theorems on the above metric spaces (see [2], [5], [6], [9], [10], [11], [14], [15], [16], [17], [19], [21] and [22] for more details).

In this paper, we investigate some properties of the concept of a complex valued S_b-metric space and give a common fixed point result. In Section 2 we recall some known definitions. In Section 3 we investigate some properties of complex valued S_b-metric spaces and prove the Banach's contraction principle on a complete complex valued S_b-metric space. Then we give a generalization of this principle. In Section 4 we obtain a common fixed-point theorem for two self-mappings on a complete complex valued S_b-metric space. We expect that many mathematicians will study various fixed-point theorems using new expansive mappings (or contractive mappings) on a complex valued S_b-metric space.

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2. Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers and z_1, $z_2 \in \mathbb{C}$. The partial order $\precsim$ is defined on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2)$$

and

$$z_1 \prec z_2 \text{ if and only if } Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2).$$

Also we can write $z_1 \precsim z_2$ if one of the following conditions hold:

1. $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
2. $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
3. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Notice that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$$

and

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_2.$$

Now we recall the following definitions and lemma.

Definition 2.1. [26] The "max" function is defined for the partial order relation $\precsim$ as follow:

1. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.
2. $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.
3. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| < |z_2|$.

Lemma 2.1. *[26] Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\precsim$ be defined on $\mathbb{C}$. Then the following statements are satisfied:*

1. *If $z_1 \precsim \max\{z_2, z_3\}$ then $z_1 \precsim z_2$ if $z_3 \precsim z_2$,*
2. *If $z_1 \precsim \max\{z_2, z_3, z_4\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4\} \precsim z_2$,*
3. *If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4, z_5\} \precsim z_2$, and so on.*

Definition 2.2. [18] Let X be a nonempty set and $b \geq 1$ be a given real number. A complex valued b-metric on X is a function $d_C : X \times X \to \mathbb{C}$ which satisfies the following conditions for all $x, y, z \in X$.
($\mathcal{C}b1$) $0 \precsim d_C(x, y)$ and $d_C(x, y) = 0$ if and only if $x = y$,
($\mathcal{C}b2$) $d_C(x, y) = d_C(y, x)$,
($\mathcal{C}b3$) $d_C(x, z) \precsim b[d_C(x, y) + d_C(y, z)]$.
Then the pair (X, d) is called a complex valued b-metric space.

Definition 2.3. [8] Let X be a nonempty set. A complex valued S-metric on X is a function $S_C : X \times X \times X \to \mathbb{C}$ which satisfies the following conditions for all $x, y, z, a \in X$.
($\mathcal{C}S1$) $0 \precsim S_C(x, y, z)$,
($\mathcal{C}S2$) $S_C(x, y, z) = 0$ if and only if $x = y = z$,
($\mathcal{C}S3$) $S_C(x, y, z) \precsim S_C(x, x, a) + S_C(y, y, a) + S_C(z, z, a)$.
Then the pair (X, S) is called a complex valued S-metric space.

Definition 2.4. [23] Let X be a nonempty set and $b \geq 1$ be a given real number. An S_b-metric on X is a function $S_b : X \times X \times X \to [0, \infty)$ which satisfies the following conditions for each $x, y, z, a \in X$.
(S_b1) $S_b(x, y, z) = 0$ if and only if $x = y = z$,
(S_b2) $S_b(x, y, z) \leq b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.
Then the pair (X, S_b) is called an S_b-metric space.

Notice that every S-metric is an S_b-metric with $b = 1$.

Definition 2.5. [17] Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_{b_C} : X \times X \times X \to \mathbb{C}$ satisfies:

($\mathcal{C}S_b1$) $0 \prec S_{b_C}(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

($\mathcal{C}S_b2$) $S_{b_C}(x, y, z) = 0 \Leftrightarrow x = y = z$,

($\mathcal{C}S_b3$) $S_{b_C}(x, x, y) = S_{b_C}(y, y, x)$ for all $x, y \in X$,

($\mathcal{C}S_b4$) $S_{b_C}(x, y, z) \precsim b[S_{b_C}(x, x, a) + S_{b_C}(y, y, a) + S_{b_C}(z, z, a)]$ for all $x, y, z, a \in X$.

Then S_{b_C} is called a complex valued S_b-metric space and (X, S_{b_C}) is called a complex valued S_b-metric space.

3. Some Properties of Complex Valued S_b-Metric Spaces

In this section we redefine the notion of a complex valued S_b-metric space without the condition ($\mathcal{C}S_b3$) given in Definition 2.5 and some new fixed-point results on this space are given.

Definition 3.1. Let X be a nonempty set and $b \geq 1$ be a given real number. If the function $S_{b_C} : X \times X \times X \to \mathbb{C}$ satisfies the following conditions for each $x, y, z, a \in X$

($\mathcal{C}S_{b_C}1$) $0 \preceq S_{b_C}(x, y, z)$,

($\mathcal{C}S_{b_C}2$) $S_{b_C}(x, y, z) = 0$ if and only if $x = y = z$,

($\mathcal{C}S_{b_C}3$) $S_{b_C}(x, y, z) \precsim b[S_{b_C}(x, x, a) + S_{b_C}(y, y, a) + S_{b_C}(z, z, a)]$,

then the function S_{b_C} is called a complex valued S_b-metric and the pair (X, S_{b_C}) is called a complex valued S_b-metric space.

Notice that every complex valued S-metric is a complex valued S_b-metric with $b = 1$.

Example 3.1. Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to \mathbb{C}$ be defined as

$$S_C(x, y, z) = \frac{1}{2}\left(|x - y| + |y - z| + |x - z|\right),$$

for all $x, y, z \in X$. Then (X, S_C) is a complex valued S-metric space. Let us define the function $S_{b_C} : X \times X \times X \to \mathbb{C}$ as follows:

$$S_{b_C}(x, y, z) = S_C(x, y, z)^3,$$

for all $x, y, z \in X$. It can be easily verified that S_{b_C} is a complex valued S_b-metric on X with $b = 16$, but it is not a complex valued S-metric.

Lemma 3.1. *Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$. Then we have*

$$S_{b_C}(x, x, y) \precsim b S_{b_C}(y, y, x)$$

and

$$S_{b_C}(y, y, x) \precsim b S_{b_C}(x, x, y).$$

Proof. Using the condition ($\mathcal{C}S_{b_C}2$) and ($\mathcal{C}S_{b_C}3$) we find

$$S_{b_C}(x, x, y) \precsim b\left[2S_{b_C}(x, x, x) + S_{b_C}(y, y, x)\right] = b S_{b_C}(y, y, x)$$

and

$$S_{b_C}(y, y, x) \precsim b\left[2S_{b_C}(y, y, y) + S_{b_C}(x, x, y)\right] = b S_{b_C}(x, x, y).$$

□

By the above lemma, we have seen that a complex valued S_b-metric function is not symmetric. Then we give the following definition.

Definition 3.2. Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$. Then S_{b_C} is called symmetric if

$$S_{b_C}(x, x, y) = S_{b_C}(y, y, x), \tag{3.1}$$

for all $x, y \in X$ and the pair (X, S_{b_C}) is called a symmetric complex valued S_b-metric space.

The symmetry condition (3.1) coincides with the condition ($\mathcal{C}S_b3$) given in Definition 2.5 and hence Definition 3.2 and Definition 2.5 are coincide.

It is known that the symmetry condition (3.1) is satisfied for $b=1$ as seen in the following lemma.

Lemma 3.2. *[8] If (X, S_C) be a complex valued S-metric space, then we have*

$$S_C(x,x,y) = S_C(y,y,x),$$

for all $x, y \in X$.

Now we give the following definition similar to Definition 2.3 given in [17].

Definition 3.3. Let (X, S_{b_C}) be a complex valued S_b-metric space. Then

1. A sequence $\{x_n\}$ in X converges to x if and only if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n \geq n_0$, we have $S_{b_C}(x_n, x_n, x) \prec \varepsilon$. It is denoted by

$$\lim_{n\to\infty} x_n = x.$$

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n, m \geq n_0$, we have $S_{b_C}(x_n, x_n, x_m) \prec \varepsilon$.

3. A complex valued S_b-metric space (X, S_{b_C}) is called complete if every Cauchy sequence is convergent.

Lemma 3.3. *Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$. If the sequence $\{x_n\}$ in X converges to x then x is unique.*

Proof. Suppose that the sequence $\{x_n\}$ converges to both x and y with $x \neq y$. Then for each $0 \prec \varepsilon$ there exist $n_1, n_2 \in \mathbb{N}$ such that for all $n_1, n_2 \geq n_0$,

$$S_{b_C}(x_n, x_n, x) \prec \frac{\varepsilon}{4b^2}$$

and

$$S_{b_C}(x_n, x_n, y) \prec \frac{\varepsilon}{2b^2},$$

with $b \geq 1$. If we put $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$, using the condition ($\mathcal{C}S_{b_C}3$) and Lemma 3.1, we obtain

$$\begin{aligned} S_{b_C}(x,x,y) &\precsim b\left[2S_{b_C}(x,x,x_n) + S_{b_C}(y,y,x_n)\right] \\ &\precsim 2b^2 S_{b_C}(x_n,x_n,x) + b^2 S_{b_C}(x_n,x_n,y) \\ &\prec 2b^2\frac{\varepsilon}{4b^2} + b^2\frac{\varepsilon}{2b^2} = \varepsilon \end{aligned}$$

and so

$$|S_{b_C}(x,x,y)| \leq |\varepsilon|,$$

which implies $S_{b_C}(x,x,y) = 0$, that is, $x = y$. □

Lemma 3.4. *Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|S_{b_C}(x_n, x_n, x)| \to 0$.*

Proof. The proof is similar to the proof of Proposition 3.1 given in [17]. □

Lemma 3.5. *Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|S_{b_C}(x_n, x_n, x_m)| \to 0$ as $n, m \to \infty$.*

Proof. The proof is similar to the proof of Theorem 3.3 given in [17]. □

Lemma 3.6. *Let (X, S_{b_C}) be a complex valued S_b-metric space with $b \geq 1$. If the sequence $\{x_n\}$ in X converges to x then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since the sequence $\{x_n\}$ converges to x we have

$$|S_{b_C}(x_n, x_n, x)| \to 0, \tag{3.2}$$

as $n \to \infty$. Using the inequality ($\mathcal{C}S_{b_C}3$), we get

$$S_{b_C}(x_n, x_n, x_m) \precsim b[S_{b_C}(x_n, x_n, x) + S_{b_C}(x_n, x_n, x) + S_{b_C}(x_m, x_m, x)]$$

and

$$|S_{b_C}(x_n, x_n, x_m)| \leq b(2\,|S_{b_C}(x_n, x_n, x)| + |S_{b_C}(x_m, x_m, x)|).$$

If we take limit for $n, m \to \infty$ then using the condition (3.2), we obtain

$$|S_{b_C}(x_n, x_n, x_m)| \to 0.$$

Consequently, the sequence $\{x_n\}$ is a Cauchy sequence. □

In [17], the Banach's contraction principle was given using the condition ($\mathcal{C}S_b3$) (that is, symmetry condition) with three variables on a complete complex valued S_b-metric space (see Theorem 3.4 on page 16 in [17]). However, the symmetry condition is not necessary in the proof of the Banach's fixed point result. Hence, in the following theorem we prove the Banach's contraction principle without the symmetry condition using two variables on a complete complex valued S_b-metric space.

Theorem 3.1. *Let (X, S_{b_C}) be a complete complex valued S_b-metric space with $b \geq 1$ and $T : X \to X$ be a self-mapping satisfying*

$$S_{b_C}(Tx, Tx, Ty) \precsim \alpha S_{b_C}(x, x, y), \tag{3.3}$$

for all $x, y \in X$ where $0 \leq \alpha < \dfrac{1}{b^2}$. Then T has a unique fixed point x in X.

Proof. Let the self-mapping T satisfies the inequality (3.3) and $x_0 \in X$. Let us define the sequence $\{x_n\}$ as

$$x_n = T^n x_0.$$

Using the inequality (3.3) and mathematical induction, we get

$$S_{b_C}(x_n, x_n, x_{n+1}) \precsim \alpha^n S_{b_C}(x_0, x_0, x_1). \tag{3.4}$$

Since the inequalities ($\mathcal{C}S_{b_C}3$) and (3.4) are satisfied, using Lemma 3.1 we obtain

$$\begin{aligned} S_{b_C}(x_n, x_n, x_m) &\precsim b(2S_{b_C}(x_n, x_n, x_{n+1}) + S_{b_C}(x_m, x_m, x_{n+1})) \\ &\quad \cdots \\ &\precsim \frac{2b\alpha^n}{1 - b^2\alpha} S_{b_C}(x_0, x_0, x_1), \end{aligned}$$

for all $n, m \in \mathbb{N}$ with $m > n$. The above inequality implies

$$|S_{b_C}(x_n, x_n, x_m)| \leq \frac{2b\alpha^n}{1 - b^2\alpha} \left| S_{b_C}(x_0, x_0, x_1) \right|. \tag{3.5}$$

If we take limit for $n \to \infty$ we have

$$\frac{2b\alpha^n}{1 - b^2\alpha} \left| S_{b_C}(x_0, x_0, x_1) \right| \to 0,$$

since $\alpha \in \left[0, \dfrac{1}{b^2}\right)$ with $b \geq 1$. Hence using the inequality (3.5) we get

$$|S_{b_C}(x_n, x_n, x_m)| \to 0$$

and so $\{x_n\}$ is Cauchy. Since (X, S_{b_C}) is a complete complex valued S_b-metric space there exists $x \in X$ such that

$$\lim_{n \to \infty} x_n = x.$$

Suppose that $Tx \neq x$. Using the inequality (3.3), we get

$$S_{b_C}(Tx, Tx, x_{n+1}) \precsim \alpha S_{b_C}(x, x, x_n).$$

If we take limit for $n \to \infty$ then we have

$$S_{b_C}(Tx, Tx, x) \precsim \alpha S_{b_C}(x, x, x)$$

and

$$|S_{b_C}(Tx, Tx, x)| \leq \alpha |S_{b_C}(x, x, x)| = 0.$$

Hence we have $S_{b_C}(Tx, Tx, x) = 0$, that is, $Tx = x$.

Finally we show that the fixed point x is unique. Assume that $Tx = x$, $Ty = y$ and $x \neq y$. Therefore we obtain

$$S_{b_C}(Tx, Tx, Ty) = S_{b_C}(x, x, y) \precsim \alpha S_{b_C}(x, x, y)$$

and

$$|S_{b_C}(x, x, y)| \leq \alpha |S_{b_C}(x, x, y)|.$$

Since $\alpha \in \left[0, \frac{1}{b^2}\right)$ with $b \geq 1$, we get $x = y$. Consequently, T has a unique fixed point x in X. □

We can give the following corollary for a complete symmetric complex valued S_b-metric space.

Corollary 3.1. *Let (X, S_{b_C}) be a complete symmetric complex valued S_b-metric space with $b \geq 1$ and $T : X \to X$ be a self-mapping satisfying the inequality (3.3) for all $x, y \in X$ where $0 \leq \alpha < \frac{1}{b}$. Then T has a unique fixed point x in X.*

Corollary 3.1 coincides with Theorem 3.4 given in [17] for two variables on X.

Remark 3.1. If we take $b = 1$ in Theorem 3.1 we obtain the Banach's contraction principle on a complete complex valued S-metric space (see [8] for more details).

Example 3.2. Let $X = \mathbb{R}$ and the complex valued S_b-metric be defined as

$$S_{b_C}(x, y, z) = \frac{1}{4}\left(|x - y| + |y - z| + |x - z|\right)^2,$$

for all $x, y, z \in X$ with $b = 4$. Let us define the self-mapping T of X as follows:

$$Tx = \frac{x}{5},$$

for all $x \in X$. Therefore the inequality (3.3) is satisfied. Indeed, we obtain

$$S_{b_C}(Tx, Tx, Ty) = |Tx - Ty|^2 = \frac{|x - y|^2}{25} \leq \alpha S_{b_C}(x, x, y) = \frac{|x - y|^2}{20},$$

for all $x, y \in X$ and $\alpha = \frac{1}{20}$. Consequently, T has a unique fixed point $x = 0$ in X.

Now we give a generalization of the Banach's contraction principle on a complete complex valued S_b-metric space.

Theorem 3.2. *Let (X, S_{b_C}) be a complete complex valued S_b-metric space with $b \geq 1$ and $T : X \to X$ be a self-mapping satisfying the following condition:*

There exist real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying $\alpha_1 + b\alpha_2 + b\alpha_3 + \left(2b^2 + b\right)\alpha_4 < 1$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ such that

$$\begin{aligned} S_{b_C}(Tx, Tx, Ty) \precsim\ & \alpha_1 S_{b_C}(x, x, y) + \alpha_2 S_{b_C}(Tx, Tx, x) \\ & + \alpha_3 S_{b_C}(Ty, Ty, y) \\ & + \alpha_4 \max\{S_{b_C}(Tx, Tx, y), S_{b_C}(Ty, Ty, x)\}, \end{aligned} \tag{3.6}$$

for all $x, y \in X$.

Then T has a unique fixed point x in X.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as follows:

$$Tx_0 = x_1, Tx_1 = x_2, \ldots, Tx_n = x_{n+1}, \ldots.$$

Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (3.6), we obtain

$$\begin{array}{l} S_{b_C}(x_n, x_n, x_{n+1}) = S_{b_C}(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ \precsim \alpha_1 S_{b_C}(x_{n-1}, x_{n-1}, x_n) \\ +\alpha_2 S_{b_C}(x_n, x_n, x_{n-1}) + \alpha_3 S_{b_C}(x_{n+1}, x_{n+1}, x_n) \\ +\alpha_4 \max\{S_{b_C}(x_n, x_n, x_n), S_{b_C}(x_{n+1}, x_{n+1}, x_{n-1})\} \\ = \alpha_1 S_{b_C}(x_{n-1}, x_{n-1}, x_n) + \alpha_2 S_{b_C}(x_n, x_n, x_{n-1}) \\ +\alpha_3 S_{b_C}(x_{n+1}, x_{n+1}, x_n) + \alpha_4 S_{b_C}(x_{n+1}, x_{n+1}, x_{n-1}). \end{array} \tag{3.7}$$

By the condition ($\mathcal{C}S_{b_C}3$), we get

$$S_{b_C}(x_{n+1}, x_{n+1}, x_{n-1}) \precsim b[2S_{b_C}(x_{n+1}, x_{n+1}, x_n) + S_{b_C}(x_{n-1}, x_{n-1}, x_n)]. \tag{3.8}$$

Using the conditions (3.7), (3.8) and Lemma 3.1, we find

$$\begin{array}{rcl} S_{b_C}(x_n, x_n, x_{n+1}) & \precsim & \alpha_1 S_{b_C}(x_{n-1}, x_{n-1}, x_n) + b\alpha_2 S_{b_C}(x_{n-1}, x_{n-1}, x_n) \\ & & +b\alpha_3 S_{b_C}(x_n, x_n, x_{n+1}) \\ & & +2b^2\alpha_4 S_{b_C}(x_n, x_n, x_{n+1}) + b\alpha_4 S_{b_C}(x_{n-1}, x_{n-1}, x_n) \end{array}$$

and so

$$(1 - b\alpha_3 - 2b^2\alpha_4) S_{b_C}(x_n, x_n, x_{n+1}) \precsim (\alpha_1 + b\alpha_2 + b\alpha_4) S_{b_C}(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S_{b_C}(x_n, x_n, x_{n+1}) \precsim \frac{\alpha_1 + b\alpha_2 + b\alpha_4}{1 - b\alpha_3 - 2b^2\alpha_4} S_{b_C}(x_{n-1}, x_{n-1}, x_n). \tag{3.9}$$

Let $\alpha = \dfrac{\alpha_1 + b\alpha_2 + b\alpha_4}{1 - b\alpha_3 - 2b^2\alpha_4}$. Then $\alpha < 1$ since $\alpha_1 + b\alpha_2 + b\alpha_3 + \left(2b^2 + b\right)\alpha_4 < 1$. Repeating this process in the condition (3.9), we have

$$S_{b_C}(x_n, x_n, x_{n+1}) \precsim \alpha^n S_{b_C}(x_0, x_0, x_1). \tag{3.10}$$

So for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (3.10) and ($\mathcal{C}S_{b_C}3$) we get

$$S_{b_C}(x_n, x_n, x_m) \precsim \frac{2b\alpha^n}{1 - b^2\alpha} S_{b_C}(x_0, x_0, x_1).$$

The above inequality implies

$$|S_{b_C}(x_n, x_n, x_m)| \leq \frac{2b\alpha^n}{1 - b^2\alpha} |S_{b_C}(x_0, x_0, x_1)|. \tag{3.11}$$

If we take limit for $n \to \infty$ we have

$$\frac{2b\alpha^n}{1 - b^2\alpha} |S_{b_C}(x_0, x_0, x_1)| \to 0,$$

since $\alpha < 1$. Therefore using the inequality (3.11), we find

$$|S_{b_C}(x_n, x_n, x_m)| \to 0$$

and so $\{x_n\}$ is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that

$$\lim_{n \to \infty} x_n = x.$$

Now we show that x is a fixed point of T. Assume that $Tx \neq x$. Then using the inequality (3.6) we have

$$\begin{array}{rcl} S_{b_C}(x_n, x_n, Tx) & = & S_{b_C}(Tx_{n-1}, Tx_{n-1}, Tx) \\ & \precsim & \alpha_1 S_{b_C}(x_{n-1}, x_{n-1}, x) + \alpha_2 S_{b_C}(x_n, x_n, x_{n-1}) \\ & & +\alpha_3 S_{b_C}(Tx, Tx, x) \\ & & +\alpha_4 \max\{S_{b_C}(x_n, x_n, x), S_{b_C}(Tx, Tx, x_{n-1})\} \end{array}$$

and so taking limit for $n \to \infty$, using Lemma 3.1 we obtain

$$S_{b_C}(x,x,Tx) \precsim (\alpha_3+\alpha_4)S_{b_C}(Tx,Tx,x) \precsim b(\alpha_3+\alpha_4)S_{b_C}(x,x,Tx)$$

and

$$|S_{b_C}(x,x,Tx)| \leq b(\alpha_3+\alpha_4)\,|S_{b_C}(x,x,Tx)|,$$

which is a contradiction since $0 \leq b\,(\alpha_3+\alpha_4) < 1$. So we get $Tx = x$.

Finally we show that the fixed point x is unique. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the inequality (3.6) and Lemma 3.1, we have

$$\begin{aligned} S_{b_C}(Tx,Tx,Ty) &= S_{b_C}(x,x,y) \precsim \alpha_1 S_{b_C}(x,x,y) + \alpha_2 S_{b_C}(x,x,x) \\ &\quad +\alpha_3 S_{b_C}(y,y,y) + \alpha_4 \max\{S_{b_C}(x,x,y), S_{b_C}(y,y,x)\} \\ &= (\alpha_1 + b\alpha_4)S_{b_C}(x,x,y) \end{aligned}$$

and

$$|S_{b_C}(x,x,y)| \leq (\alpha_1+b\alpha_4)\,|S_{b_C}(x,x,y)|,$$

which implies $x = y$ since $\alpha_1 + b\alpha_4 < 1$. Consequently, T has a unique fixed point x in X. □

If we take $b = 1$ in Theorem 3.2 then we get the following corollary.

Corollary 3.2. *Let (X, S_C) be a complete complex valued S-metric space and $T : X \to X$ be a self-mapping satisfying the following condition*:

There exist real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying $\alpha_1+\alpha_2+\alpha_3+3\alpha_4 < 1$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ such that the condition (3.6) is satisfied for all $x, y \in X$.
Then T has a unique fixed point x in X.

We can give the following corollary for a complete symmetric complex valued S_b-metric space.

Corollary 3.3. *Let (X, S_{b_C}) be a complete symmetric complex valued S_b-metric space with $b \geq 1$ and $T : X \to X$ be a self-mapping satisfying the following condition*:

There exist real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying $\alpha_1 + b\alpha_2 + b\alpha_3 + 3b\alpha_4 < 1$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ such that the condition (3.6) is satisfied for all $x, y \in X$.
Then T has a unique fixed point x in X.

Remark 3.2. We note that Theorem 3.2 is a generalization of the Banach's contraction principle on a complete complex valued S_b-metric space. Indeed if we take $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\alpha_1 < \frac{1}{b^2}$ with $b \geq 1$ in Theorem 3.2 then we obtain the Banach's contraction principle on a complete complex valued S_b-metric space.

Now we give an example of a self-mapping satisfying the condition (3.6) such that the condition of the Banach's contraction principle (3.3) is not satisfied.

Example 3.3. Let $X = \mathbb{R}$ and the function $S_{b_C} : X \times X \times X \to \mathbb{C}$ be defined as

$$S_{b_C}(x,y,z) = \frac{1}{k}\left(|x-z| + |x+z-2y|\right),$$

for all $x, y, z \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. Then $(\mathbb{R}, S_{b_C})$ is a complete complex valued S_b-metric space with $b = 1$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} x+60 & ; \quad x \in \{0,4\} \\ 55 & ; \quad \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Therefore T satisfies the condition (3.6) for $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{3}$ and $\alpha_4 = 0$. Hence T has a unique fixed point $x = 55$. But T does not satisfy the condition of the Banach's contraction principle (3.3). Indeed, for $x = 0$, $y = 1$ we obtain

$$S_{b_C}(Tx,Tx,Ty) \leq \alpha S_{b_C}(x,x,y) = \alpha,$$

which is a contradiction since $\alpha < 1$.

4. A Common Fixed Point Theorem

In this section we give a common fixed point theorem on a complete complex valued S_b-metric space.

Theorem 4.1. *Let (X, S_{b_C}) be a complete complex valued S_b-metric space with $b \geq 1$ and $M, N : X \to X$ be two self-mappings satisfying*

$$\begin{aligned} S_{b_C}(Mx, Mx, Ny) &\precsim \alpha \max\{S_{b_C}(x,x,y), S_{b_C}(Mx,Mx,x), \\ &\quad S_{b_C}(Ny,Ny,y), S_{b_C}(Ny,Ny,x), \\ &\quad S_{b_C}(Mx,Mx,y)\}, \end{aligned} \tag{4.1}$$

for all $x, y \in X$ with $0 \leq \alpha < \dfrac{1}{2b^2+b}$. Then M and N have a unique common fixed point x in X.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as follows:

$$Mx_{2n} = x_{2n+1}, \; Nx_{2n+1} = x_{2n+2}. \tag{4.2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Using the conditions (4.1), (4.2), ($\mathcal{C}S_{b_C}3$) and Lemma 3.1, we have

$$\begin{aligned} S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k+2}) &= S_{b_C}(Mx_{2k}, M_{2k}, Nx_{2k+1}) \precsim \alpha \max\{S_{b_C}(x_{2k}, x_{2k}, x_{2k+1}), \\ &\quad S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k}), S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k+1}), \\ &\quad S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k}), S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k+1})\} \\ &= \alpha \max\{S_{b_C}(x_{2k}, x_{2k}, x_{2k+1}), S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k}), \\ &\quad S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k+1}), S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k})\} \\ &\precsim \alpha \max\{S_{b_C}(x_{2k}, x_{2k}, x_{2k+1}), S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k}), \\ &\quad S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k+1}), 2bS_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k+1}) \\ &\quad + bS_{b_C}(x_{2k}, x_{2k}, x_{2k+1})\} \\ &\precsim \alpha\gamma \end{aligned}$$

and so

$$\begin{aligned} |S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k+2})| &\leq \alpha |\gamma| \\ &\leq 2b\alpha |S_{b_C}(x_{2k+2}, x_{2k+2}, x_{2k+1})| + b\alpha |S_{b_C}(x_{2k}, x_{2k}, x_{2k+1})| \\ &\leq 2b^2\alpha |S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k+2})| + b\alpha |S_{b_C}(x_{2k}, x_{2k}, x_{2k+1})|, \end{aligned}$$

which implies

$$|S_{b_C}(x_{2k+1}, x_{2k+1}, x_{2k+2})| \leq \frac{b\alpha}{1-2b^2\alpha} |S_{b_C}(x_{2k}, x_{2k}, x_{2k+1})|.$$

Hence using the mathematical induction, we get

$$|S_{b_C}(x_n, x_n, x_{n+1})| \leq \beta^n |S_{b_C}(x_0, x_0, x_1)|,$$

where $\beta = \dfrac{b\alpha}{1-2b^2\alpha}$. Thus for all $m > n$, $m, n \in \mathbb{N}$ we obtain

$$|S_{b_C}(x_n, x_n, x_m)| \leq \frac{2b\beta^n}{1-b^2\beta} |S_{b_C}(x_0, x_0, x_1)|.$$

Consequently, we obtain $|S_{b_C}(x_n, x_n, x_m)| \to 0$ as $n, m \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in X. Since (X, S_{b_C}) is a complete complex valued S_b-metric space then $\{x_n\}$ converges to $x \in X$. Now we show that this point x is a fixed point of the self-mapping M. Assume that $Mx \neq x$. Using the inequalities (4.1) and ($\mathcal{C}S_{b_C}3$), we obtain

$$\begin{aligned} S_{b_C}(Mx, Mx, x) &\precsim b[2S_{b_C}(Mx, Mx, x_{2n+2}) + S_{b_C}(x, x, x_{2n+2})] \\ &\precsim 2bS_{b_C}(Mx, Mx, Nx_{2n+1}) + bS_{b_C}(x, x, x_{2n+2}) \\ &\precsim 2b\alpha \max\{S_{b_C}(x, x, x_{2n+1}), S_{b_C}(Mx, Mx, x), \\ &\quad S_{b_C}(x_{2n+2}, x_{2n+2}, x_{2n+1}), S_{b_C}(x_{2n+2}, x_{2n+2}, x), \\ &\quad S_{b_C}(Mx, Mx, x_{2n+1})\} + bS_{b_C}(x, x, x_{2n+2}). \end{aligned}$$

If we take limit for $n \to \infty$ we have

$$S_{b_C}(Mx, Mx, x) \precsim 2b\alpha S_{b_C}(Mx, Mx, x)$$

and so

$$|S_{b_C}(Mx, Mx, x)| \leq 2b\alpha |S_{b_C}(Mx, Mx, x)|,$$

which is a contradiction since $0 \leq \alpha < \frac{1}{2b^2 + b}$. Consequently, we have $Mx = x$.

Assume that there exists $y \in X$ such that $Nx = y$. Now we show that $x = y$. Suppose that $x \neq y$. Using the inequality (4.1) and Lemma 3.1, we get

$$\begin{aligned} S_{b_C}(x, x, y) &= S_{b_C}(Mx, Mx, Nx) \preceq \alpha \max\{S_{b_C}(x, x, x), S_{b_C}(Mx, Mx, x), \\ &\quad S_{b_C}(Nx, Nx, x), S_{b_C}(Nx, Nx, x), S_{b_C}(Mx, Mx, x)\} \\ &= \alpha \max\{0, 0, S_{b_C}(y, y, x), S_{b_C}(y, y, x), 0\} \\ &= \alpha S_{b_C}(y, y, x) \precsim b\alpha S_{b_C}(x, x, y) \end{aligned}$$

and

$$|S_{b_C}(x, x, y)| \leq b\alpha |S_{b_C}(x, x, y)|,$$

which is a contradiction. Hence we have $x = y$ and x is the common fixed point of the self-mappings M and N.

Now we prove that the fixed point x is unique. Suppose that z is another common fixed point of the self-mappings M and N. Using the inequality (4.1) and Lemma 3.1, we have

$$\begin{aligned} S_{b_C}(x, x, z) &= S_{b_C}(Mx, Mx, Nz) \precsim \alpha \max\{S_{b_C}(x, x, z), S_{b_C}(Mx, Mx, x), \\ &\quad S_{b_C}(Nz, Nz, z), S_{b_C}(Nz, Nz, x), S_{b_C}(Mx, Mx, z)\} \\ &= \alpha \max\{S_{b_C}(x, x, z), 0, 0, S_{b_C}(z, z, x), S_{b_C}(x, x, z)\} \\ &\precsim b\alpha S_{b_C}(x, x, z) \end{aligned}$$

and so

$$|S_{b_C}(x, x, z)| \leq b\alpha |S_{b_C}(x, x, z)|,$$

which is a contradiction. Consequently, M and N have a unique common fixed point x in X. □

Corollary 4.1. *Let (X, S_{b_C}) be a complete symmetric complex valued S_b-metric space with $b \geq 1$ and $M, N : X \to X$ be two self-mapping satisfying the condition (4.1) for all $x, y \in X$ with $0 \leq \alpha < \frac{1}{3b}$. Then M and N have a unique common fixed point x in X.*

Example 4.1. Let $X = \mathbb{R}^+ \cup \{0\}$ and the function $S_{b_C} : X \times X \times X \to \mathbb{C}$ be defined as

$$S_{b_C}(x, y, z) = i(|x - z| + |x + z - 2y|),$$

for all $x, y, z \in X$. Then (X, S_{b_C}) is a complete complex valued S_b-metric space with $b = 1$. Let us define the self-mappings $M, N : X \to X$ as follows:

$$Mx = \frac{x}{6}$$

and

$$Nx = 0,$$

for all $x \in X$. It can be easily seen that M and N satisfy the condition (4.1) for $\alpha = \frac{1}{4}$. So M and N have a unique common fixed point $x = 0$ in X.

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