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On Ordered Hyperspace Topologies in the Setting of Čech Closure Ordered Spaces

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Abstract
In this work, we introduce some possible ordered hyperspace topologies on families of subsets constructed in the setting of a Čech closure operator.

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1. Introduction

Generalizations of topological spaces are frequently used in many branches of mathematics and computer science. Some of these generalizations are obtained by omitting some axioms of Kuratowski closure. Closure operators which are grounded, extensive and additive were first studied by E. Čech [6]. Čech closure operators have numerous applications, for example they were used for solving problems related to digital image processing in [12]. Slapal [12] showed that more general structures can be suitable to study topological properties of digital images. The relation between Čech closure space and structural configuration of proteins were studied in [13]. The interested reader may find more details in [1], [3], [6], [11], [16], [24].

There are several topologies defined on closed subsets of a topological space or a metric space. Such topologies are called hyperspace topologies. Subbasic open sets for a hyperspace topology are the closed sets which hit a particular set or the closed sets which miss a particular set. One of the well-studied hit-and-miss hyperspace topologies is the well known Vietoris Topology [7]. The setting of such topologies give rise to modifications by using closure operators. In [17], hyperspaces of Čech closure spaces were introduced and in [3] the authors starts with a Čech closure space and defines a modification of the Vietoris topology.

We face up order and ordered structures in daily life and they have many applications, especially, in computer science and social sciences. The cooperation between topology and order was studied by Leopolda Nachbin [14] in the 1950's and he developed the theory of topological ordered space which is a triple $(X, \tau, \preceq)$, where (X, τ) is a topological space endowed with a preorder "$\preceq$". The fundemental properties of the theory of the ordered topological spaces can be found in [5], [20, 22]. Topological ordered spaces have important applications in dynamical systems [4], computer science [9], game theory [19] and microeconomics [2]. A preorder "$\preceq$"on a topological space (X, τ) is said to have a continuous multi-utility representation if there exists a set $\mathcal{F}$ of continuous isotone functions

$$f : (X, \tau, \preceq) \to (\mathbb{R}, \tau_u, \leq)$$

(τ_u denotes the natural topology and "$\leq$"denotes usual order on $\mathbb{R}$) such that, for any two points $x, y \in X$

$$x \preceq y \Leftrightarrow f(x) \leq f(y) \text{ for all } f \in \mathcal{F}$$

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[18]. Looking for the topological conditions for the utility representation is crucial for economists and utility representation is an important application of topological ordered spaces. Therefore, Čech closure ordered spaces which are more general structures then topological ordered spaces can also find applications in economics, game theory and computer sciences.

The concept of an ordered Hyperspace was introduced in [23] and relations between an ordered topological space and its ordered hyperspace was investigated in [25]. According to [25], if $(X, \tau, \preceq)$ is an ordered topological space and $D(X)$ denote the set of all closed, decreasing subsets of X, then, the sets

$$B(G; G_1, ..., G_n) = \{F \in D(X) : F \subseteq G \text{ and } F \cap G_i \neq \emptyset \text{ for all i=1,2,...,n } \}$$

,where G is an open decreasing and $G_1, G_2, ..., G_n$ are open increasing subsets of X, form an open base of $D(X)$ and $(D(X), \tau, \preceq)$ is an ordered topological space.

In the setting of an ordered Čech closure space $(X, c, \preceq)$, we construct hyperspace topologies of subsets of X by using both Čech closure and the underlying order. Motivated by [3] and [25], we introduce the ordered Čech based Vietoris topology and investigate some of its properties.

2. Preliminaries

A partially ordered set (poset) is a set X with a binary relation "$\preceq$" which is reflexive, antisymmetric and transitive. If the relation is only reflexive and transitive then it is called preorder. In a preordered set $(X, \preceq)$, a subset A of X is called decreasing if $a \in A$, $b \in X$ and $b \preceq a$ implies $b \in A$ and called increasing if $a \in A$, $b \in X$ and $a \preceq b$ implies $b \in A$. The smallest decreasing set containing A is denoted by $\downarrow A$ and the smallest increasing set containing A is denoted by $\uparrow A$. If A is a decreasing (increasing) set, then, the complement of A which will be denoted by A^c is an increasing (decreasing) set.

An ordered topological space is a nonempty set X endowed with a topology τ and a partial order which will be denoted by $(X, \tau, \preceq)$. If we endow X with a Čech closure operator c, which reduces the idempotency property when compared with a topological closure operator, and a preorder "$\preceq$", then $(X, c, \preceq)$ is called an ordered Čech closure space. If the preorder on X is the discrete order defined as

$$a \preceq b \Leftrightarrow a = b,$$

then every ordered Čech closure space is an ordinary Čech closure space. An ordered Čech closure space $(X, c, \preceq)$ is called

(i) upper T_1-$ordered$ if for each pair of elements $a \npreceq b$ in X, there exists a decreasing neighbourhood U of b such that $a \notin U$

(ii) lower T_1-$ordered$ if for each pair of elements $a \npreceq b$ in X, there exists an increasing neighbourhood U of a such that $b \notin U$.

(iii) T_2-$ordered$ if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing neighbourhood U of a and a decreasing neighbourhood V of b such that $U \cap V = \emptyset$.

(iv) lower regular ordered if for each decreasing set $A \subseteq X$ and each element $x \notin c(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is increasing and V is decreasing.

(v) upper regular ordered if for each increasing set $A \subseteq X$ and each $x \notin c(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is decreasing and V is increasing.

(vi) regular ordered if both (iv) and (v) are satisfied.

An open set in a hyperspace X is a family of closed subsets of the underlying topological (τ) or metric structure (τ_d) on X. As usual we will denote the nonempty closed subsets of X by $CL(X)$. Most of the hyperspace topologies on closed subsets of a Hausdorff space X are usually defined by subbases consisting of members in the following sense:

$$A^+ := \{B \in CL(X) \mid B \subset A\} \text{ and } A^- := \{B \in CL(X) \mid A \cap B \neq \emptyset\}$$

for an arbitrary subset A in X. The Vietoris topology τ_V is defined as the topology having as a subbase of all sets of the form V^- where $V \in \tau$ and all sets of the form W^+ where $W \in \tau$ (see, [8]). In [3], the authors starts with a Čech closure space and defines a modification of the Vietoris topology as follows: if

$$\begin{aligned} \mathcal{H} &= \{c(A) \mid A \subseteq X\} \backslash \{\emptyset\}, \\ \mathcal{J} &= \{int_c(A) \mid A \subseteq X\} \end{aligned}$$

and for an arbitrary $A \subseteq X$

$$A^+ = \{H \in \mathcal{H} \mid H \subseteq A\} \text{ and } A^- = \{H \in \mathcal{H} \mid H \cap A \neq \emptyset\},$$

then, for an arbitrary $n \in \mathbb{N}$ and $G, G_1, ..., G_n \in \mathcal{J}$ satisfying that $G_i \subseteq G$ for $i \in \{1, 2, ..., n\}$, the collections

$$\begin{aligned} G^+ \cap (\bigcap_{i=1}^{n} G_i^-) &= \langle G; G_1, ..., G_n \rangle \\ &= \{H \in \mathcal{H} \mid H \subseteq G, (\forall i)(1 \leq i \leq n) : H \cap G_i \neq \emptyset\} \end{aligned}$$

forms the basis elements of the Vietoris topology on $\mathcal{H}$.

3. On Ordered Čech Based Hyperspace Topologies

In the setting of an ordered Čech closure space $(X, c, \preceq)$, we shall construct a hypertopology on

$$\mathcal{H}^{\downarrow} = \{c(A) \mid A \subset X, A = \downarrow A\} \backslash \{\emptyset\}$$

by using the following families of subsets of X;

$$\begin{aligned} \mathcal{J}^+ &= \{int_c(A) \mid A \subset X, A = \uparrow A\} \\ \mathcal{J}^- &= \{int_c(A) \mid A \subset X, A = \downarrow A\} \end{aligned}$$

For all $G \in \mathcal{J}^-$, $n \in \mathbb{N}$ and $G_1, ..., G_n \in \mathcal{J}^+$ satisfying that $G_i \subseteq G$ for $i \in \{1, 2, ..., n\}$, let

$$\begin{aligned} \langle G; G_1, ..., G_n \rangle &= G^+ \cap (\bigcap_{i=1}^{n} G_i^-) \\ &= \{H \in \mathcal{H}^{\downarrow} \mid H \subseteq G, (\forall i)\, (1 \leq i \leq n) : H \cap G_i \neq \emptyset\}, \end{aligned}$$

then the collection;

$$\mathcal{S} = \{\langle G; G_1, ..., G_n \rangle\}_{\substack{G \in \mathcal{J}^- \\ G_1, ..., G_n \in \mathcal{J}^+}}$$

forms a base for a topology on $\mathcal{H}^{\downarrow}$. This hypertopology is called ordered Čech based Vietoris topology and denoted by $\mathcal{W}^{\downarrow}$. In addition, with the ordinary inclusion relation "$\subseteq$" on subsets of X, $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$ is an ordered hypertopological space. Similarly, one may define a coarser topology $\mathcal{W}$ on $\mathcal{H}^{\downarrow}$ by replacing $\mathcal{J}^+$ and $\mathcal{J}^-$ with the family of the open decreasing sets and open increasing sets of $(X, c, \preceq)$, respectively.

Dually, if we define $\mathcal{H}^{\uparrow} = \{c(A) \mid A \subset X, A = \uparrow A\} \backslash \{\emptyset\}$, then the collection

$$\mathcal{S}' = \{\langle G; G_1, ..., G_n \rangle\}_{\substack{G \in \mathcal{J}^+ \\ G_1, ..., G_n \in \mathcal{J}^-}}$$

also forms a base for a topology $\mathcal{W}^{\uparrow}$ on $\mathcal{H}^{\uparrow}$. Hence $(\mathcal{H}^{\uparrow}, \mathcal{W}^{\uparrow}, \subseteq)$ is an ordered hypertopological space.

Most of our investigation will be considered for down case, but also one may obtain similar versions of our results for up case.

Example 3.1. Consider $\mathbb{N}$ endowed with the usual order $\leq$ and the closure operator c defined by

$$c(A) = A \cup \{y \in \mathbb{N} : \exists x \in A \text{ and } x \leq y\}$$

for $A \subseteq X$. Then, increasing and decreasing sets are $\{\emptyset, \mathbb{N}\} \cup \{\{n, n+1, ...\} : n \in \mathbb{N}\}$ and $\{\emptyset, \mathbb{N}\} \cup \{\{0, 1, ..., n\} : n \in \mathbb{N}\}$, respectively. Therefore, we obtain $\mathcal{H}^{\downarrow} = \{\mathbb{N}\}$,

$$\mathcal{J}^+ = \{\emptyset, \mathbb{N}\} \text{ and } \mathcal{J}^- = \{\emptyset, \mathbb{N}\} \cup \{\{0, 1, ..., n\} : n \in \mathbb{N}\}$$

Morever, the basis elements are

$$\langle \{0, 1, ..., n\}; \mathbb{N} \rangle = \{\mathbb{N}\} = \langle \mathbb{N}; \mathbb{N} \rangle \text{ and } \langle \emptyset; \mathbb{N} \rangle = \{\emptyset\}$$

Then, $\mathcal{W}^{\downarrow} = \{\emptyset, \mathcal{H}^{\downarrow}\}$. Hence, $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$ is an ordered hypertopological space.

Example 3.2. Let $X = \{x, y, z\}$, $\preceq= \{(x, x), (y, y), (z, z), (x, y)\}$ and c be a closure operator defined as:

$$c(\emptyset) = \emptyset,\ \ c(\{x\}) = \{x, y\},$$
$$c(\{y\}) = \{y, z\},\ \ c(\{z\}) = \{x, z\},$$
$$c(\{x, y\}) = c(\{x, z\}) = c(\{y, z\}) = c(X) = X.$$

Then we obtain

$$int_c\{\emptyset\} = int_c\{x\} = int_c\{y\} = int_c\{z\} = \emptyset$$
$$int_c\{x, y\} = \{y\}, int_c\{y, z\} = \{z\}$$
$$int_c\{x, z\} = \{x\}, int_c\{X\} = X.$$

Therefore, $\mathcal{H}^{\uparrow} = \{\{z, y\}, \{x, z\}, X\}$ and the basis elements

$$\langle X; \{y\}\rangle = \{\{y, z\}, X\},\ \langle X; \{x\}\rangle = \{\{x, z\}, X\},$$
$$\langle X; \{y\}, \{x\}\rangle = \{X\},\ \langle \{y\}; \{y\}\rangle = \emptyset$$

construct the topology $\mathcal{W}^{\uparrow} = \{\mathcal{H}^{\uparrow}, \emptyset, \{X\}, \{\{x, z\}, X\}, \{\{y, z\}, X\}\}$ on $\mathcal{H}^{\uparrow}$.

Lemma 3.1. *If $(X, c, \preceq)$ is an ordered Čech closure space, then*
i) $\forall A \in \mathcal{H}^{\downarrow} : A^+$ is decreasing and closed in $\mathcal{W}^{\downarrow}$,
ii) $\forall A \subseteq X \ni A =\uparrow A : (c(A))^-$ is increasing and closed in $\mathcal{W}^{\downarrow}$,
iii) $\forall G \subseteq X$ and $G_1, G_2, ..., G_n \subseteq X$:

$$cl_{\mathcal{W}^{\downarrow}}\langle G; G_1, ..., G_n\rangle \subseteq \langle c(\downarrow G); c(\uparrow G_1), ..., c(\uparrow G_n)\rangle.$$

Proof. i) Let $A \in \mathcal{H}^{\downarrow}$. For an arbitrary $K \in\downarrow (A^+)$, there exists $H \in A^+$ such that $K \subseteq H$. This implies that $K \in A^+$. For the second claim, we know that there exists a decreasing set K such that $A = c(K)$. Thus K^c is increasing and $A^c = (c(K))^c = int_c(K^c)$. Therefore $(A^+)^c = (A^c)^- = (int_c(K^c))^- = \langle X; int_c(K^c)\rangle \in \mathcal{W}^{\downarrow}$. Hence A^+ is closed.

ii) Let A be an increasing subset of X. To show that $(c(A))^-$ is increasing, let $K \in\uparrow ((c(A))^-)$. Then there exists $H \in (c(A))^-$ such that $H \subseteq K$. It follows that $K \cap c(A) \neq \emptyset$. Hence $K \in (c(A))^-$. For the second claim, it is clear that $((c(A))^-)^c = ((c(A))^c)^+ = (int_c(A^c))^+ = \langle int_c(A^c); X\rangle \in \mathcal{W}^{\downarrow}$. Hence $(c(A))^-$ is closed.

iii) Let $G \in \mathcal{J}^-$ and $(G_i)_{1\leq i\leq n} \subset \mathcal{J}^+$. Since

$$\langle c(\downarrow G); c(\uparrow G_1), ..., c(\uparrow G_n)\rangle = (c(\downarrow G))^+ \cap (c(\uparrow G_1))^- \cap ... \cap (c(\uparrow G_n))^-$$

and $(c(\downarrow G))^+$, $(c(\uparrow G_i))^-$ are closed for each $i \in \{1, ..., n\}$, then $\langle c(\downarrow G); c(\uparrow G_1), ..., c(\uparrow G_n)\rangle$ is closed in $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$. It follows that

$$cl_{\mathcal{W}^{\downarrow}}\langle G; G_1, ..., G_n\rangle \subseteq \langle c(\downarrow G); c(\uparrow G_1), ..., c(\uparrow G_n)\rangle.$$

□

Theorem 3.1. *If $(X, c, \preceq)$ is a lower T_1-ordered Čech closure space, then for each $G \subseteq X$ and $G_1, G_2, ..., G_n \subseteq X$ satisfying $G_i \subseteq G$ for each $i \in \{1, 2, ..., n\}$*

$$\langle c(G); c(G_1), ..., c(G_n) \subseteq cl^{\mathcal{H}^{\downarrow}}_{\mathcal{W}}\langle G; G_1, ..., G_n\rangle.$$

Proof. Let $G \subseteq X$ and $G_1, G_2, ..., G_n \subseteq X$ such that $G_i \subseteq G$ for each $i \in \{1, 2, ..., n\}$. To show the inclusion, let $H \in \langle c(G); c(G_1), ..., c(G_n)\rangle$. Then $H \subseteq c(G)$ and $H \cap c(G_i) \neq \emptyset$ for each $i \in \{1, ..., n\}$. Therefore

$$\forall i \in \{1, 2, \ldots, n\} : \exists x_i \in H \cap c(G_i).$$

On the other hand, for an open neighbourhood $\mathcal{V}$ of H in $\mathcal{W}$, there exists open decreasing set U and family of open increasing sets $(U_j)_{1\leq j\leq m}$ in $(X, c, \preceq)$ such that $H \in \langle U; U_1, ..., U_m\rangle \subset \mathcal{V}$. It follows that $H \subseteq U$ and $H \cap U_j \neq \emptyset$ for each $j \in \{1, 2, \ldots, m\}$. Then

$$\forall j \in \{1, 2, \ldots, m\} : \exists\ z_j \in H \cap U_j.$$

Since $x_i \in U \cap c(G_i)$ for each $i \in \{1, ..., n\}$, there exists $x'_i \in U \cap G_i$ for each $i \in \{1, ..., n\}$. Similarly, since $z_j \in c(G) \cap U_j$ for each $j \in \{1, 2, \ldots, m\}$, there exists $z'_j \in G \cap U_j$ for each $j \in \{1, 2, \ldots, m\}$ .Now let $A = \{x'_1, \ldots, x'_n, z'_1, \ldots, z'_n\}$, then to complete the proof we have to show that

$$\downarrow A \in \langle G; G_1, ..., G_n\rangle \cap \mathcal{V}.$$

Since X is lower T_1-$ordered$, $\downarrow A$ is closed and decreasing. Thus $\downarrow A \in \mathcal{H}^{\downarrow}$. $\downarrow A \subseteq\downarrow G = G$ and $\downarrow A \subseteq\downarrow U = U$ implies that $\downarrow A \subseteq G \cap U$. Therefore $A \cap G_i \neq \emptyset$ for each $i \in \{1, ..., n\}$ and $A \cap U_j \neq \emptyset$ for each $j \in \{1, ..., m\}$. Consequently, $A \in \langle G; G_1, ..., G_n\rangle \cap \langle U; U_1, ..., U_m\rangle \subset \langle G; G_1, ..., G_n\rangle \cap \mathcal{V}$. □

In the sequal, the decreasing closed sets of the Čech closure space $(X, c, \preceq)$ will be denoted by $\mathcal{D}(X)$. Also we will consider the Čech based Vietoris topologies $\mathcal{W}^{\downarrow}$ on $\mathcal{D}(X)$ which is defined similar to the one defined on $\mathcal{H}^{\downarrow}$.

Theorem 3.2. *If $(X, c, \preceq)$ is a lower T_1-$ordered$ Čech closure space , then $\{\downarrow F \mid F$ is a finite subset of $X\}$ is dense in $(\mathcal{H}^{\downarrow}, \mathcal{W}, \subseteq)$.*

Proof. If $\mathcal{P}$ is a nonempty open set in $\mathcal{W}$, then for each $A \in \mathcal{P}$ there exists an open decreasing set G and a family of open increasing sets $(G_i)_{1 \leq i \leq n}$ in $(X, c, \preceq)$ such that $A \in \langle G; G_1, ..., G_n \rangle \subset \mathcal{P}$. Then, $A \subseteq G$ and $A \cap G_i \neq \emptyset$ for each $i \in \{1, ..., n\}$. Therefore there exists $x_i \in A \cap G_i$ for each $i \in \{1, ..., n\}$ and this implies that

$$\downarrow \{x_1, ..., x_n\} \in \langle G; G_1, ..., G_n \rangle.$$

Since $(X, c, \preceq)$ is lower T_1-$ordered$, $\downarrow \{x_1, ..., x_n\}$ is closed and decreasing in $(X, c, \preceq)$. Thus we obtain that

$$\langle G; G_1, ..., G_n \rangle \cap \{\downarrow F \mid F \text{ is a finite subset of } X\} \neq \emptyset$$

which completes the proof. □

Remark 3.1. If we consider Theorem 2, then $\{\downarrow F \mid F$ is a finite subset of $X\}$ is also dense in $(\mathcal{D}(X), W, \subseteq)$. So we obtain Proposition 2 of [25] as a result.

Proposition 3.1. *If $(X, c, \preceq)$ is a regular ordered Čech closure space, then $(\mathcal{D}(X), \mathcal{W}^{\downarrow}, \subseteq)$ is T_2-$ordered$ space.*

Proof. Let $A, B \in \mathcal{D}(X)$ and $A \subsetneq B$. Then there exists $a \in A$ such that $a \notin B = c(B)$. Therefore regularity of X implies the existence of an increasing neighbourhood U of a and decreasing neighbourhood V of B such that $U \cap V = \emptyset$. Thus $a \in int_c U$ and $B \subseteq int_c V$. Therefore $\langle X; int_c U \rangle$ is an increasing neighbourhood of A and $\langle int_c V; X \rangle$ is a decreasing neighbourhood of B. Now claim that

$$\langle X; int_c U \rangle \cap \langle int_c V; X \rangle = \emptyset.$$

If $\langle X; int_c U \rangle \cap \langle int_c V; X \rangle \neq \emptyset$, then there exists $K \in \langle X; int_c U \rangle \cap \langle int_c V; X \rangle$, so $K \cap int_c U \neq \emptyset$ and $K \subseteq int_c V$. Therefore $U \cap V \neq \emptyset$ which is a contradiction. □

A collection $(U_\alpha)_{\alpha \in I}$ is called an interior cover of a set A in a Čech closure space (X, c), if the collection $\{int_c (U_\alpha)\}_{\alpha \in I}$ covers A [6].

Definition 3.1. An ordered Čech closure space $(X, c, \preceq)$ is called strongly down(up)-compact if every interior cover of X which consists of decreasing sets (increasing sets) has a finite sub-interior cover. $(X, c, \preceq)$ is called down(up)-compact if every interior cover of X which consist of decreasing sets (increasing sets) has a finite subcover not necessarily an interior cover.

Remark 3.2. If we assume that X is endowed with the discrete order, then the notions strongly down(up)-compactness and down(up)-compactness coincide with strongly compactness and compactness, respectively.

Next example shows that there exists an ordered closure space which is not compact and strongly compact but it is up compact and also strongly up compact.

Example 3.3. On the set of integers $\mathbb{Z}$ the topology τ which has a basis

$$\mathcal{B} = \{\{2m-1\} \mid m \in \mathbb{Z}\} \cup \{\{2m-1, 2m, 2m+1\} \mid m \in \mathbb{Z}\}$$

is called the digital topology. For an arbitrary $A \subset \mathbb{Z}$, define $c_\theta(A)$ as the set of points whose each closed neighbourhood intersects A. Then, $(\mathbb{Z}, c_\theta)$ (given as Example 4 in [15]) is called the $\theta-$closure space of the digital topology. $(\mathbb{Z}, c_\theta)$ is a Čech closure space and with the ordinary order $\leq$ on integers $(\mathbb{Z}, c_\theta, \leq)$ is an ordered closure space. $(\mathbb{Z}, c_\theta, \leq)$ is neither compact nor strongly compact, but it is up and strongly up compact space since the only interior cover of $\mathbb{Z}$ which consists of increasing sets is $\{\mathbb{Z}\}$.

Proposition 3.2. *Let $(X, c, \preceq)$ be an ordered Čech closure space. If $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$ is a compact space, then $(X, c, \preceq)$ is strongly up-compact.*

Proof. Let $(U_i)_{i\in I}$ be an interior cover of X which consists of increasing sets. Then $X = \bigcup\limits_{i\in I} int_c U_i$ and clearly $\{\langle X; int_c U_i\rangle\}_{i\in I}$ is an open cover of $\mathcal{H}^{\downarrow}$. It follows from the hypothesis that there exists a finite subcover of $\mathcal{H}^{\downarrow}$. Let $J \subseteq I$ and $\{\langle X; int_c U_i\rangle\}_{i\in J}$ be the finite subcover of $\mathcal{H}^{\downarrow}$. Then $\{U_i\}_{i\in J}$ is a finite sub-interior cover of X. Therefore, $(X, c, \preceq)$ is strongly up-compact space. □

Proposition 3.3. *Let $(X, c, \preceq)$ be an ordered Čech closure space and $\mathfrak{C} = (C_i)_{i\in I}$ be a collection which consists of the subsets of X. If C_i is strongly down-compact for each $i \in I$ and $\mathfrak{C}$ is compact in $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$, then $\bigcup\limits_{i\in I} C_i$ is down-compact.*

Proof. If $(U_\lambda)_{\lambda\in\Gamma}$ is an interior cover of $\bigcup\limits_{i\in I} C_i$ which consists of decreasing sets, then for each $i \in I$ there exists a finite set $\Gamma_i \subseteq \Gamma$ such that $C_i \subseteq \bigcup\limits_{\lambda\in\Gamma_i} int_c(U_\lambda) \subseteq \bigcup\limits_{\lambda\in\Gamma_i} U_\lambda = U_{\Gamma_i}$. For each $i \in I$, U_{Γ_i} is a decreasing set and

$$\mathcal{S} = (S_i)_{i\in I} = \left(\langle int_c U_{\Gamma_i}; \{int_c(\uparrow U_\lambda)\}_{\lambda\in\Gamma_i}\rangle\right)_{i\in I}$$

is an open cover for $\mathfrak{C}$ in $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$. It follows that there exists a finite set $J \subset I$ such that $\mathfrak{C} \subseteq \bigcup\limits_{j\in J} S_j$. Then for each $i \in I$, there exists $j_i \in J$ such that $C_i \in S_j$, i.e. $C_i \subset int_c\left(U_{\Gamma_{j_i}}\right)$. Hence

$$\bigcup_{i\in I} C_i \subset \bigcup_{j_i\in J} int_c\left(U_{\Gamma_{j_i}}\right)$$

which completes the proof. □

Proposition 3.4. *Let $(X, c, \preceq)$ be a regular ordered Čech closure space with a neighbourhood base consisting of increasing sets. If $\mathfrak{C}$ is a compact collection in $(\mathcal{H}^{\downarrow}, \mathcal{W}^{\downarrow}, \subseteq)$ consisting of closed and decreasing sets, then $\bigcup\limits_{A\in\mathfrak{C}} A$ is closed and decreasing.*

Proof. Let $x \in c(\bigcup\limits_{A\in\mathfrak{C}} A)$. Then for each neighbourhood W of x there is an increasing neighbourhood U of x such that $U \subseteq W$. Since $(X, c, \preceq)$ is a regular ordered space, there exists an increasing neighbourhood V of x such that $V \subseteq c(V) \subseteq U \subseteq W$. Since $V \cap (\bigcup\limits_{A\in\mathfrak{C}} A) \neq \emptyset$, there exists $A \in \mathfrak{C}$ such that $c(V) \cap A \neq \emptyset$ which implies $A \in (c(V))^-$. Then it follows from Lemma 1 that $\mathfrak{C}\cap(c(V))^-$ is closed in $\mathfrak{C}$ and

$$\{\mathfrak{C}\cap(c(V))^- \mid V \text{ is an increasing neighbourhood of } x\}$$

has the finite intersection property since $\{V_i\}_{1\leq i\leq n}$ is a family of increasing neighbourhoods of x implies $\bigcap\limits_{i=1}^{n} V_i$ is an increasing neighbourhood of x and since

$$\mathfrak{C}\cap(c(\bigcap_{i=1}^{n} V_i))^- \subseteq \mathfrak{C}\cap((\bigcap_{i=1}^{n} c(V_i)^-),$$

$\mathfrak{C}\cap((\bigcap\limits_{i=1}^{n} c(V_i)^-) \neq \emptyset$. Compactness of $(\mathfrak{C}, \mathcal{W}^{\uparrow}_{\mathfrak{C}}, \subseteq)$ implies the existence of a nonempty subcollection $\mathfrak{C}' \subseteq \mathfrak{C}$ such that, for each $A' \in \mathfrak{C}'$, $A' \in \{\mathfrak{C}\cap(c(V))^- \mid V$ is an increasing neighbourhood of $x\}$. Thus $A' \cap c(V) \neq \emptyset$ and since $c(V) \subseteq U \subseteq W$, $A' \cap W \neq \emptyset$, for each neighbourhood W of x and so $x \in c(A') = A' \subseteq \bigcup\limits_{A\in\mathfrak{C}} A$ and $x \in \bigcup\limits_{A\in\mathfrak{C}} A$. Consequently, $\bigcup\limits_{A\in\mathfrak{C}} A$ is a closed and decreasing set in $(X, c, \preceq)$. □

Corollary 3.1. *When the preorder on X is the discrete order, then we present Proposition 6, Proposition 7 and Proposition 8 of [3] as results of Proposition 2, Proposition 3, Proposition 4, respectively.*

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