Two-Point Iterative Methods for Solving Quadratic Equations and its Applications

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Abstract
Kung-Traub conjecture states that an iterative method without memory for locating the zero of a scalar equation could achieve convergence order 2^{d-1}, where d is the total number of function evaluations, but proposed algorithm produces convergence order of $r + 2$, where r is a positive integer with three function evaluations for solving quadratic equations, which is better than expected maximum convergence order. Therefore, we show that the conjecture fails for quadratic equations. Also, we extend proposed algorithm to solving systems which involving quadratic equations. We test our methods with some numerical experiments including application to one dimensional and two dimensional Bratu problems.

Keywords: Quadratic equation; Two-step iterative methods; Kung-Traub's conjecture; Efficiency Index; Systems of equations.

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1. Introduction

It is accepted that a boundary value problems among theory of gases, elasticity and different applied area are for the most part reduced in one variable nonlinear equations. The zeros of nonlinear equations is also real or complex. Generally, there are two eccentric of technique acting offered to seek out the zeros of scalar nonlinear equations. First, direct technique, that don't seem to be continually applicable to seek out the roots and second, iterative technique acting supported the idea of consecutive approximations. In this case, the ultimate procedure is to begin out with one or additional initial approximations to the zero and attain a sequence of iterates whose the limit converges to the reality solution. Multipoint iterative methods for locating nonlinear equations are of nice sensible importance since they overcome theoretical limits of one-point methods concerning the order of convergence and computational potency. Here, we have a tendency to specializing in finding the simple root of nonlinear equations by an iterative method.

One of the simplest root-finding strategies is Newton's iteration methodology for finding a nonlinear equation. The order of convergence of Newton's methodology 2 and it's optimal with two function evaluations per iterative step. In recent years, varied higher order iterative methods are developed and analyzed for finding nonlinear equations that improve classical strategies similar to Newton's, Euler's, Chebyshev-Halley's methods, etc. Recently, the order of convergence of the many variants of Newton's methodology has been improved victimisation constant range of functional evaluations by suggests that of weight functions [2, 4, 5, 12, 13, 15, 16, 19, 20]. The aim of such analysis is to develop best methods that satisfy Kung-Traub's conjecture [13]. Conjecture says that the order of convergence of any multi-point method without memory with d evaluations cannot exceed the bound 2^{d-1}, the optimal order.

Let $x^{(k+1)} = \psi(x^{(k)})$ define an Iterative Function (I.F.).

Definition 1.1. [22] If the sequence $\{x^{(k)}\}$ tends to a limit x^* in such a way that

$$\lim_{n \to \infty} \frac{x^{(k+1)} - x^*}{(x^{(k)} - x^*)^p} = C$$

for $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant. If $p = 1$, $p = 2$ or $p = 3$, the convergence is said to be linear, quadratic or cubic, respectively. Let $\varepsilon^{(k)} = x^{(k)} - x^*$, then the relation

$$\varepsilon^{(k+1)} = C\,(\varepsilon^{(k)})^p + O\left(\varepsilon^{(k)}\right)^{p+1} = O\left(\varepsilon^{(k)}\right)^p. \tag{1.1}$$

is called the error equation. The value of p is called the order of convergence of the method.

Definition 1.2. [18] The Efficiency Index is given by

$$EI = p^{\frac{1}{d}}, \tag{1.2}$$

where d is the total number of new function evaluations per iteration.

Let $x^{(k+1)}$ be determined by new information at $x^{(k)}, \phi_1(x^{(k)}), ..., \phi_i(x^{(k)}), i \geq 1$. No old information is reused. Thus,

$$x^{(k+1)} = \psi(x^{(k)}, \phi_1(x^{(k)}), ..., \phi_i(x^{(k)})). \tag{1.3}$$

Then ψ is called a multi-point I.F without memory.
Kung-Traub's Conjecture [13]
Let ψ be an I.F. without memory with d evaluations. Then

$$p(\psi) \leq p_{opt} = 2^{d-1}, \tag{1.4}$$

where p_{opt} is the maximum order.
The second order Newton I.F. ($2^{nd}NR$) is given by

$$\psi_{2^{nd}NR}(x) = x - n(x), \ \ n(x) = \frac{f(x)}{f'(x)}. \tag{1.5}$$

The $2^{nd}NR$ I.F. is satisfies the Kung-Traub conjecture with $d = 2$. Thus, $EI_{2^{nd}NR} = 1.414$. Let us consider two-step third order I.F. ($3^{rd}CM$) [7] is given by

$$\psi_{3^{rd}CM}(x) = x - \frac{1}{2}(3 - \eta(x))n(x), \ \eta(x) = \frac{f'(x - \frac{2}{3}n(x))}{f'(x)}. \tag{1.6}$$

The $3^{rd}CM$ I.F. with three function evaluations and it is not satisfies the conjecture. According to conjecture, not possible to derive an I.F. without memory more than order four with three function evaluations. Recently, Ahmad [1] and Babajee [3] developed two-step eighth order methods using weight functions, also both are showed Kung-Traub's conjecture is fails for solving quadratic equations. They termed as $8^{th}BQIM$ and it is given below

$$\psi_{(8)^{th}BQIM}(x) = x - H(\eta(x), 6)n(x), \tag{1.7}$$

where $H(\eta(x), 6) = 1 - \frac{3}{4}(\eta(x)-1) + \frac{9}{8}(\eta(x)-1)^2 - \frac{135}{64}(\eta(x)-1)^3 + \frac{567}{128}(\eta(x)-1)^4 - \frac{5103}{512}(\eta(x)-1)^5 + \frac{24057}{1024}(\eta(x)-1)^6$. Thukral [21], demonstrated that the Kung and Traub conjecture fails for a particular case, that is when the simple root of a nonlinear equation is equal to zero. Madhu [14], author developed two-step iterative methods with three function evaluations reaching convergence order $r + 3$, where r is a positive integer. In this paper, we develop two-step higher order iterative methods for solving quadratic equations and study Kung-Traub's conjecture for proposed methods.

The rest of the paper is organized as follows. Section 2 presents the development of the new methods and its convergence analysis for scalar equation. In section 3, we extend these methods to systems of quadratic equations and derived its convergence analysis. Section 4 carries out the tests on the numerical examples and compare the present methods with Newton's method and $8^{th}BQIM$. Section 5 gives application on 1-D and 2-D bratu problems. Finally, section 6 gives conclusions on our work.

2. Developments of the methods

Let us consider the proposed scheme as,

$$\psi_{(r+2)^{th}KM}(x) = x - \frac{1}{2}(3 - \eta(x))H(\eta(x), r)n(x), \qquad (2.1)$$

where

$$H(\eta(x), r) = 1 + \sum_{i=1}^{r} a_i(\eta(x) - 1)^i,$$

where a_i's are constants. If $r = 6$, then the error equation of the I.F. (2.1) is as follows

$$
\begin{aligned}
\psi(x) - x^* &= \frac{1}{3}(c_2 + 4c_2a_1)(\varepsilon^{(k)})^2 - \frac{2}{9}(c_2^2(-3 + 20a_1 + 8a_2))(\varepsilon^{(k)})^3 \\
&+ \frac{2}{27}c_2^3(-63 + 150a_1 + 152a_2 + 32a_3)(\varepsilon^{(k)})^4 \\
&- \frac{4}{81}(c_2^4(-351 + 432a_1 + 948a_2 + 448a_3 + 64a_4))(\varepsilon^{(k)})^5 \\
&+ \frac{8}{243}c_2^5(729a_1 + 2(-810 + 2367a_2 + 1920a_3 + 592a_4 + 64a_5))(\varepsilon^{(k)})^6 \\
&+ \frac{16}{729}c_2^6(1701a_1 - 2(10017a_2 + 2(-1701 + 6408a_3 + 3216a_4 + 736a_5 + 64a_6)))(\varepsilon^{(k)})^7 \\
&- \frac{32}{2187}(c_2^7(26973 + 25515a_1 - 72009a_2 - 145098a_3 - 106272a_4 - 38688a_5 - 7040a_6))(\varepsilon^{(k)})^8 \\
&+ O(\varepsilon^{(k)})^9,
\end{aligned}
$$

where $c_2 = f''(x^*)/2f'(x^*)$.

Eliminating the terms in $(\varepsilon^{(k)})^j$, $j = 2, 3, 4, 5, 6, 7$ and we obtain the following system of linear equations

$$
\begin{aligned}
&\frac{4}{3}a_1 = -\frac{1}{3}, \\
&-\frac{40}{9}a_1 - \frac{16}{9}a_2 = -\frac{2}{3}, \\
&\frac{300}{27}a_1 + \frac{304}{27}a_2 + \frac{64}{27}a_3 = \frac{126}{27}, \\
&\frac{300}{27}a_1 + \frac{304}{27}a_2 + \frac{64}{27}a_3 = \frac{126}{27}, \\
&-\frac{1728}{81}a_1 - \frac{3792}{81}a_2 - \frac{1792}{81}a_3 - \frac{256}{81}a_4 = -\frac{1404}{81}, \\
&\frac{5832}{243}a_1 + \frac{37872}{243}a_2 + \frac{30720}{243}a_3 + \frac{9472}{243}a_4 + \frac{1024}{243}a_5 = \frac{12960}{243}, \\
&\frac{27216}{729}a_1 - \frac{320544}{729}a_2 - \frac{410112}{729}a_3 - \frac{205824}{729}a_4 - \frac{47104}{729}a_5 - \frac{4096}{729}a_6 = -\frac{108864}{729}.
\end{aligned}
$$

The solutions of system of linear equation can obtained easily from first equation. The solutions are $a_1 = -\frac{1}{4}$, $a_2 = 1$, $a_3 = -\frac{103}{64}$, $a_4 = \frac{29}{8}$, $a_5 = -\frac{4175}{512}$ and $a_6 = \frac{9941}{512}$.

In this approach, we tend to get a family of higher order I.F.s $((r + 2)^{nd}KM)$ for solving quadratic equations. Consider first six members of proposed family (2.1) with their error equation,

1. $r = 1$: two-step $3^{rd}KM$ I.F.

$$H(\eta(x), 1) = 1 - \frac{1}{4}(\eta(x) - 1).$$

$$\psi_{3^{rd}KM}(x) - x^* = \left(\frac{16}{9}c_2^2\right)(\varepsilon^{(k)})^3 + O\left((\varepsilon^{(k)})^4\right).$$

2. $r = 2$: two-step $4^{th}KM$ I.F.

$$H(\eta(x), 2) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2.$$

$$\psi_{4^{th}KM}(x) - x^* = \left(\frac{103}{27}c_2^3\right)(\varepsilon^{(k)})^4 + O\left((\varepsilon^{(k)})^5\right).$$

3. $r = 3$: two-step $5^{th} KM$ I.F.

$$H(\eta(x), 3) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2 - \frac{103}{64}(\eta(x) - 1)^3.$$

$$\psi_{5^{th} KM}(x) - x^* = \left(\frac{928}{81} c_2{}^4 \right) (\varepsilon^{(k)})^5 + O\left((\varepsilon^{(k)})^6 \right).$$

4. $r = 4$: two-step $6^{th} KM$ I.F.

$$H(\eta(x), 4) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2 - \frac{103}{64}(\eta(x) - 1)^3 + \frac{29}{8}(\eta(x) - 1)^4.$$

$$\psi_{6^{th} KM}(x) - x^* = \left(\frac{8350}{243} c_2{}^5 \right) (\varepsilon^{(k)})^6 + O\left((\varepsilon^{(k)})^7 \right).$$

5. $r = 5$: two-step $7^{th} KM$ I.F.

$$H(\eta(x), 5) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2 - \frac{103}{64}(\eta(x) - 1)^3 + \frac{29}{8}(\eta(x) - 1)^4 - \frac{4175}{512}(\eta(x) - 1)^5.$$

$$\psi_{7^{th} KM}(x) - x^* = \left(\frac{79528}{729} c_2{}^6 \right) (\varepsilon^{(k)})^7 + O\left((\varepsilon^{(k)})^8 \right).$$

6. $r = 6$: two-step $8^{th} KM$ I.F.

$$H(\eta(x), 6) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2 - \frac{103}{64}(\eta(x) - 1)^3 + \frac{29}{8}(\eta(x) - 1)^4 - \frac{4175}{512}(\eta(x) - 1)^5 + \frac{9941}{512}(\eta(x) - 1)^6.$$

$$\psi_{8^{th} KM}(x) - x^* = \left(\frac{779167}{2187} c_2{}^7 \right) (\varepsilon^{(k)})^8 + O\left((\varepsilon^{(k)})^9 \right).$$

Note that we've got obtained associate eighth order with only three function evaluations for solving quadratic equations.

2.1 Convergence Analysis
Theorem 2.1. *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root x^* in the open interval D. Then the six members of two-step $(r + 2)^{rd} KM's$ family in (2.1) ($r = 1, 2, 3, 4, 5, 6$) are of local third to eighth order convergence, respectively.*

Proof. We will prove the third and eighth order convergence, fourth to seventh order I.F.s follow on similar lines. It's straightforward to examine that for a quadratic function,

$$f(x) = f'(x^*)\left[\varepsilon^{(k)} + c_2(\varepsilon^{(k)})^2 \right],$$

and

$$f'(x) = f'(x^*)\left[1 + 2c_2 \, \varepsilon^{(k)} \right].$$

By Taylor expansion and using computer algebra software MATHEMATICA

$$n(x) = \varepsilon^{(k)} - c_2(\varepsilon^{(k)})^2 + 2 c_2{}^2(\varepsilon^{(k)})^3 - 4 c_2{}^3(\varepsilon^{(k)})^4 + 8 c_2{}^4(\varepsilon^{(k)})^5 - 16 c_2{}^5(\varepsilon^{(k)})^6$$
$$+ 32 c_2{}^6(\varepsilon^{(k)})^7 - 64 c_2{}^7(\varepsilon^{(k)})^8 + 128 c_2{}^8(\varepsilon^{(k)})^9 + \dots ,$$

(2.2)

so that

$$\eta(x) = 1 - \frac{4}{3} c_2 \, \varepsilon^{(k)} + 4 c_2{}^2(\varepsilon^{(k)})^2 - \frac{32}{3} c_2{}^3(\varepsilon^{(k)})^3 + \frac{80}{3} c_2{}^4(\varepsilon^{(k)})^4 - 64 c_2{}^5(\varepsilon^{(k)})^5 + \frac{448}{3} c_2{}^6(\varepsilon^{(k)})^6$$
$$- \frac{1024}{3} c_2{}^7(\varepsilon^{(k)})^7 + 768 c_2{}^8(\varepsilon^{(k)})^8 + \dots .$$

(2.3)

Now,

$$H(\eta(x), 1) = 1 + c_2 \, \varepsilon^{(k)} - 3 \, c_2{}^2 (\varepsilon^{(k)})^2 + 8 \, c_2{}^3 (\varepsilon^{(k)})^3 + \dots . \tag{2.4}$$

Using eqs. (2.2), (2.3) and (2.4), we have

$$\frac{1}{2}(3 - \eta(x)) \, H(\eta(x), 1) \, n(x) = \varepsilon^{(k)} - \left(\frac{16}{9} \, c_2{}^2 \right) (\varepsilon^{(k)})^3 + O\left((\varepsilon^{(k)})^4 \right),$$

which leads to the error equation for the two-step $3^{rd}KM$ I.F.
Similarly,

$$H(\eta(x), 6) = 1 + c_2 \, \varepsilon^{(k)} - c_2{}^2 (\varepsilon^{(k)})^2 + c_2{}^3 (\varepsilon^{(k)})^3 - c_2{}^4 (\varepsilon^{(k)})^4 + c_2{}^5 (\varepsilon^{(k)})^5 - c_2{}^6 (\varepsilon^{(k)})^6 - 428 c_2{}^7 (\varepsilon^{(k)})^7 + \dots . \tag{2.5}$$

Using eqs. (2.2) and (2.5), we have

$$\frac{1}{2}(3 - \eta(x)) \, H(\eta(x), 6) \, n(x) = \varepsilon^{(k)} - \left(\frac{779167}{2187} \, c_2{}^7 \right) (\varepsilon^{(k)})^8 + O\left((\varepsilon^{(k)})^9 \right),$$

which leads to the error equation for the two-step $8^{th}KM$ I.F. □

We next prove the local convergence of the two-step $(r + 2)^{nd}KM's$ family for any r.

Theorem 2.2. *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root x^* in the open interval D. Then the members of two-step $(r + 2)^{nd}KM's$ family in (2.1) are of local $r + 2$ order convergence.*

Proof. We prove this result by induction. The case $r = 1$ corresponds to the $3^{rd}KM$ I.F. Assume the two-step $(r + 2)^{th}KM$ family has order of convergence of $(r + 2)$. Then it satisfies the error equation

$$\psi_{(r+2)^{th}KM}(x) - x^* = C_r \, c_2{}^{r+1} (\varepsilon^{(k)})^{r+2} + O\left((\varepsilon^{(k)})^{r+3} \right), \tag{2.6}$$

where C_r is the asymptotic error constant. Assume that eq. (2.6) holds for $r = m$. Now from eq (2.3), we have

$$\eta(x) - 1 = -\frac{4}{3} c_2 \, \varepsilon^{(k)} \left(1 - 3c_2 \, \varepsilon^{(k)} + 8c_2{}^2 (\varepsilon^{(k)})^2 + \dots \right),$$

so that

$$\begin{aligned}
(\eta(x) - 1)^{m+1} &= \left(-\frac{4}{3} \right)^{m+1} c_2{}^{m+1} (\varepsilon^{(k)})^{m+1} \left(1 - 3c_2 \, \varepsilon^{(k)} + 8c_2{}^2 (\varepsilon^{(k)})^2 + \dots \right)^{m+1}, \\
&= \left(-\frac{4}{3} \right)^{m+1} c_2{}^{m+1} (\varepsilon^{(k)})^{m+1} \left(1 + O(\varepsilon^{(k)}) \right).
\end{aligned} \tag{2.7}$$

For the case $r = m + 1$,

$$\psi_{(m+3)^{rd}KM}(x) - x^*,$$

$$= x - \frac{1}{2}(3 - \eta(x)) \, n(x) \, H(\eta(x), m + 1) - x^*,$$

$$= x - \frac{1}{2}(3 - \eta(x)) n(x) \, H(\eta(x), m) - x^* - a_{m+1} \frac{1}{2}(3 - \eta(x)) n(x) \, (\eta(x) - 1)^{m+1},$$

$$= \psi_{(m+2)^{th}KM}(x) - x^* - a_{m+1} \frac{1}{2}(3 - \eta(x)) \, n(x) \, (\eta(x) - 1)^{m+1},$$

$$= C_m \, c_2{}^{m+1} (\varepsilon^{(k)})^{m+2} - a_{m+1} \left(-\frac{4}{3} \right)^{m+1} c_2{}^{m+1} (\varepsilon^{(k)})^{m+2} + O\left((\varepsilon^{(k)})^{m+3} \right) \text{ using eqs. (2.2), (2.3), (2.6) and (2.7),}$$

$$= \left(C_m - a_{m+1} \left(-\frac{4}{3} \right)^{m+1} \right) c_2{}^{m+1} (\varepsilon^{(k)})^{m+2} + O\left((\varepsilon^{(k)})^{m+3} \right),$$

which shows that the two-step $(m + 3)^{rd}KM$ family has $(m + 3)$rd order of convergence if we choose

$$a_{m+1} = C_m \left(-\frac{3}{4} \right)^{m+1}. \tag{2.8}$$

□

From eq. (2.8), we can obtain higher order I.F. if we know the asymptotic error constant of the previous I.F. For example, for the two-step $3^{rd}KM$ I.F., $C_1 = \frac{16}{9}$ and from eq. (2.8),

$$a_2 = C_1 \left(-\frac{3}{4}\right)^2 = 1,$$

and we can obtain the $4^{th}KM$ I.F. Similarly, for the two-step $8^{th}KM$ I.F., $C_6 = \frac{779167}{2184}$ and from eq. (2.8),

$$a_7 = C_6 \left(-\frac{3}{4}\right)^7 = -\frac{779167}{16384},$$

and we can obtain the two-step $9^{th}KM$ I.F. with

$$H(\eta(x), 7) = 1 - \frac{1}{4}(\eta(x) - 1) + (\eta(x) - 1)^2 - \frac{103}{64}(\eta(x) - 1)^3 + \frac{29}{8}(\eta(x) - 1)^4 - \frac{4175}{512}(\eta(x) - 1)^5$$
$$+ \frac{9941}{512}(\eta(x) - 1)^6 - \frac{779167}{16384}(\eta(x) - 1)^7.$$

From Theorem 2.2, we conclude that we can have a family of order $r+2$, $r = 1, 2, ...$ with only 3 function evaluations.

Remark: The Efficiency Index of the two-step $(r + 2)^{th}KM$ family is

$$EI = (r + 2)^{\frac{1}{3}}, \ r \geq 1. \tag{2.9}$$

Hence, we conclude that this is the one having highest efficiency index. Next section, we extend proposed methods to systems.

3. Extension to systems of equations

Let us consider, system of nonlinear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x}))^T$, $\mathbf{x} = (x_1, x_2, ..., x_n)^T$, $f_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, 2, \ldots, n$ defined as

$$f_i(\mathbf{x}) = b_i + \sum_{l=1}^{n} \sum_{m=1}^{n} b_{l,m} \, x_l x_m, \quad b_i, \, b_{l,m}, \, i, l, m = 1, ..n, \text{ are constants.}$$

and $\mathbf{f} : \mathbf{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and $\mathbf{D}$ is an open and convex set, where we assume that $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)^T$ is a root of the system and $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right)^T$ is an initial point sufficiently close to $\mathbf{x}^*$. Let us define the two-step $(r + 2)^{th}KM's$ family for systems of quadratic equations as:

$$\psi_{(r+2)^{th}KM}(\mathbf{x}) = \mathbf{x} - \frac{1}{2}(3 - \eta(\mathbf{x}))\mathbf{H}(\eta(\mathbf{x}), r) \, \mathbf{n}(\mathbf{x}), \tag{3.1}$$

where

$$\mathbf{n}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}),$$
$$\mathbf{y}(\mathbf{x}) = \mathbf{x} - \frac{2}{3}\mathbf{n}(\mathbf{x}),$$
$$\eta(\mathbf{x}) = \mathbf{f}'(\mathbf{x})^{-1}\mathbf{f}'\left(\mathbf{y}(\mathbf{x})\right),$$
$$\mathbf{H}(\eta(\mathbf{x}), r) = \mathbf{I} + \sum_{i=1}^{r} a_i(\eta(\mathbf{x}) - \mathbf{I})^i, \ \mathbf{I} \text{ is the identity matrix.}$$

Let us define

$$\mathbf{c}_2 = \frac{1}{2}[\mathbf{f}'(\mathbf{x}^*)]^{-1}\mathbf{f}^{(2)}(\mathbf{x}^*), \ \epsilon^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^*.$$

Using the notations in [9], it is noted that $\mathbf{c}_2\epsilon^{(k)} \in \mathcal{L}(\mathbb{R}^n)$. The error at the $(k + 1)$th iteration is $\epsilon^{(k+1)} = \mathbf{L}(\epsilon^{(k)})^p + O\left((\epsilon^{(k)})^{p+1}\right)$, where $\mathbf{L}$ is a p-linear function $\mathbf{L} \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$, is called the *error equation* and p is the *order of convergence*. Observe that $(\epsilon^{(k)})^p$ is $(\epsilon^{(k)}, \epsilon^{(k)}, \cdots, \epsilon^{(k)})$. The first 6 members of $(r + 2)^{th}KM's$ family in (3.1) with their error equation are

1. $r = 1$: two-step $3^{rd}KM$ I.F.

$$H(\eta(\mathbf{x}), 1) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}).$$

$$\psi_{3^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{16}{9}\mathbf{c}_2{}^2\right)(\epsilon^{(k)})^3 + O\left((\epsilon^{(k)})^4\right).$$

2. $r = 2$: two-step $4^{th}KM$ I.F.

$$H(\eta(\mathbf{x}), 2) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}) + (\eta(\mathbf{x}) - \mathbf{I})^2.$$

$$\psi_{4^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{103}{27}\mathbf{c}_2{}^3\right)(\epsilon^{(k)})^4 + O\left((\epsilon^{(k)})^5\right).$$

3. $r = 3$: two-step $5^{th}KM$ I.F.

$$H(\eta(\mathbf{x}), 3) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}) + (\eta(\mathbf{x}) - \mathbf{I})^2 - \frac{103}{64}(\eta(\mathbf{x}) - \mathbf{I})^3.$$

$$\psi_{5^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{928}{81}\mathbf{c}_2{}^4\right)(\epsilon^{(k)})^5 + O\left((\epsilon^{(k)})^6\right).$$

4. $r = 4$: two-step $6^{th}KM$ I.F.

$$H(\eta(\mathbf{x}), 4) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}) + (\eta(\mathbf{x}) - \mathbf{I})^2 - \frac{103}{64}(\eta(\mathbf{x}) - \mathbf{I})^3 + \frac{29}{8}(\eta(\mathbf{x}) - \mathbf{I})^4.$$

$$\psi_{6^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{8350}{243}\mathbf{c}_2{}^5\right)(\epsilon^{(k)})^6 + O\left((\epsilon^{(k)})^7\right).$$

5. $r = 5$: two-step $7^{th}KM$ I.F.

$$H(\eta(\mathbf{x}), 5) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}) + (\eta(\mathbf{x}) - \mathbf{I})^2 - \frac{103}{64}(\eta(\mathbf{x}) - \mathbf{I})^3 + \frac{29}{8}(\eta(\mathbf{x}) - \mathbf{I})^4 - \frac{4175}{512}(\eta(\mathbf{x}) - \mathbf{I})^5.$$

$$\psi_{7^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{79528}{729}\mathbf{c}_2{}^6\right)(\epsilon^{(k)})^7 + O\left((\epsilon^{(k)})^8\right).$$

6. $r = 6$: two-step $8^{th}KM$ I.F.

$$\mathbf{H}(\eta(\mathbf{x}), 6) = \mathbf{I} - \frac{1}{4}(\eta(\mathbf{x}) - \mathbf{I}) + (\eta(\mathbf{x}) - \mathbf{I})^2 - \frac{103}{64}(\eta(\mathbf{x}) - \mathbf{I})^3 + \frac{29}{8}(\eta(\mathbf{x}) - \mathbf{I})^4 - \frac{4175}{512}(\eta(\mathbf{x}) - \mathbf{I})^5$$
$$+ \frac{9941}{512}(\eta(\mathbf{x}) - \mathbf{I})^6.$$

$$\psi_{8^{th}KM}(\mathbf{x}) - \mathbf{x}^* = \left(\frac{779167}{2187}\mathbf{c}_2{}^7\right)(\epsilon^{(k)})^8 + O\left((\epsilon^{(k)})^9\right).$$

3.1 Convergence Analysis

Theorem 3.1. *Let $\mathbf{f} : \mathbf{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be twice Frechet differentiable at each point of an open convex neighborhood $\mathbf{D}$ of $\mathbf{x}^* \in \mathbb{R}^n$, that is a solution of the quadratic system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Let us suppose that $\mathbf{f}'(\mathbf{x})$ is continuous and nonsingular in $\mathbf{x}^*$, and $\mathbf{x}^{(0)}$ is close enough to $\mathbf{x}^*$. Then the sequence $\{\mathbf{x}^{(k)}\}_{k \geq 0}$ obtained using the iterative expressions (3.1), $r = 1, 2, ..., 6$ converge to $\mathbf{x}^*$ with order 3 to 8, respectively.*

   Proof:
We prove now for $r = 6$ case. The other cases follows similar lines. Since $\mathbf{f}$ is a quadratic function of several variables, we have

$$\mathbf{f}(\mathbf{x}^{(k)}) = \mathbf{f}'(\mathbf{x}^*)\left[\epsilon^{(k)} + \mathbf{c}_2(\epsilon^{(k)})^2\right], \tag{3.2}$$

and

$$\mathbf{f}'(\mathbf{x}^{(k)}) = \mathbf{f}'(\mathbf{x}^*)\left[\mathbf{I} + 2\mathbf{c}_2\epsilon^{(k)}\right]. \tag{3.3}$$

$$\mathbf{f}'(\mathbf{x}^{(k)})^{-1} = \left[\mathbf{I} - 2\mathbf{c_2}\epsilon^{(k)} + 4\mathbf{c_2}^2(\epsilon^{(k)})^2 - 8\mathbf{c_2}^3(\epsilon^{(k)})^3 + 16\mathbf{c_2}^4(\epsilon^{(k)})^4 - 32\mathbf{c_2}^5(\epsilon^{(k)})^5 \right.$$
$$\left. + 64\mathbf{c_2}^6(\epsilon^{(k)})^6 - 128\mathbf{c_2}^7(\epsilon^{(k)})^7 + 256\mathbf{c_2}^8(\epsilon^{(k)})^8... \right][\mathbf{f}'(\mathbf{x}^*)]^{-1}. \tag{3.4}$$

Using eq. (3.2) and (3.4), we have

$$\mathbf{u}(\mathbf{x}^{(k)}) = \epsilon^{(k)} - \mathbf{c_2}(\epsilon^{(k)})^2 + 2\mathbf{c_2}^2(\epsilon^{(k)})^3 - 4\mathbf{c_2}^3(\epsilon^{(k)})^4 + 8\mathbf{c_2}^4(\epsilon^{(k)})^5 - 16\mathbf{c_2}^5(\epsilon^{(k)})^6$$
$$+ 32\mathbf{c_2}^6(\epsilon^{(k)})^7 - 64\mathbf{c_2}^7(\epsilon^{(k)})^8 + ... , \tag{3.5}$$

and the expression for $\mathbf{y}(\mathbf{x}^{(k)})$ is given by

$$\mathbf{y}(\mathbf{x}^{(k)}) = \mathbf{x}^* + \frac{1}{3}\epsilon^{(k)} + \frac{2}{3}\mathbf{c_2}(\epsilon^{(k)})^2 - \frac{4}{3}\mathbf{c_2}^2(\epsilon^{(k)})^3 + \frac{8}{3}\mathbf{c_2}^3(\epsilon^{(k)})^4 - \frac{16}{3}\mathbf{c_2}^4(\epsilon^{(k)})^5 + \frac{32}{3}\mathbf{c_2}^5(\epsilon^{(k)})^6$$
$$- \frac{64}{3}\mathbf{c_2}^6(\epsilon^{(k)})^7 + \frac{128}{3}\mathbf{c_2}^7(\epsilon^{(k)})^8 + ...$$

The Taylor expansion of Jacobian matrix $\mathbf{f}'(\mathbf{y}(\mathbf{x}^{(k)}))$ is then given by

$$\mathbf{f}'(\mathbf{y}(\mathbf{x}^{(k)})) = \mathbf{f}'(\mathbf{x}^*)\left[\mathbf{I} + 2\mathbf{c_2}(\mathbf{y}(\mathbf{x}^{(k)}) - \mathbf{x}^*)\right]$$
$$= \mathbf{f}'(\mathbf{x}^*)\left[\mathbf{I} + \frac{2}{3}\mathbf{c_2}(\epsilon^{(k)}) + \frac{4}{3}\mathbf{c_2}^2(\epsilon^{(k)})^2 - \frac{8}{3}\mathbf{c_2}^3(\epsilon^{(k)})^3 + \frac{16}{3}\mathbf{c_2}^4(\epsilon^{(k)})^4 - \frac{32}{3}\mathbf{c_2}^5(\epsilon^{(k)})^5 \right.$$
$$\left. + \frac{64}{3}\mathbf{c_2}^6(\epsilon^{(k)})^6 - \frac{128}{3}\mathbf{c_2}^7(\epsilon^{(k)})^7 + \frac{256}{3}\mathbf{c_2}^8(\epsilon^{(k)})^8 + ... \right].$$

Therefore, using eq. (3.4), we obtain

$$\eta(\mathbf{x}^{(k)}) = [\mathbf{f}'(\mathbf{x}^{(k)})]^{-1}\mathbf{f}'(\mathbf{y}(\mathbf{x}^{(k)}))$$
$$= \mathbf{I} - \frac{4}{3}\mathbf{c_2}(\epsilon^{(k)}) + 4\mathbf{c_2}^2(\epsilon^{(k)})^2 - \frac{32}{3}\mathbf{c_2}^3(\epsilon^{(k)})^3 + \frac{80}{3}\mathbf{c_2}^4(\epsilon^{(k)})^4 - 64\mathbf{c_2}^5(\epsilon^{(k)})^5 \tag{3.6}$$
$$+ \frac{448}{3}\mathbf{c_2}^6(\epsilon^{(k)})^6 - \frac{1024}{3}\mathbf{c_2}^7(\epsilon^{(k)})^7 + 768\mathbf{c_2}^8(\epsilon^{(k)})^8 + ...$$

so that

$$\mathbf{H}(\eta(\mathbf{x}), 6) = \mathbf{I} + \mathbf{c_2}(\epsilon^{(k)}) - \mathbf{c_2}^2(\epsilon^{(k)})^2 + \mathbf{c_2}^3(\epsilon^{(k)})^3 - \mathbf{c_2}^4(\epsilon^{(k)})^4 + \mathbf{c_2}^5(\epsilon^{(k)})^5$$
$$- \mathbf{c_2}^6(\epsilon^{(k)})^6 - 428\mathbf{c_2}^7(\epsilon^{(k)})^7 + ... \tag{3.7}$$

Using eq. (3.5), (3.6) and (3.7), we have, after simplifications,

$$\mathbf{x} - \frac{1}{2}(3 - \eta(\mathbf{x}))\mathbf{H}(\eta(\mathbf{x}), 6)\,\mathbf{u}(\mathbf{x}^{(k)}) = \mathbf{x}^* + \frac{779167}{2187}\mathbf{c_2}^7(\epsilon^{(k)})^8 + ...$$

Theorem 3.2. *Let* $\mathbf{f} : \mathbf{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be twice Frechet differentiable at each point of an open convex neighborhood* $\mathbf{D}$ *of* $\mathbf{x}^* \in \mathbb{R}^n$, *that is a solution of the quadratic system* $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. *Let us suppose that* $\mathbf{f}'(\mathbf{x})$ *is continuous and nonsingular in* $\mathbf{x}^*$, *and* $\mathbf{x}^{(0)}$ *is close enough to* $\mathbf{x}^*$. *Then the sequence* $\{\mathbf{x}^{(k)}\}_{k \geq 0}$ *obtained using the iterative expressions (3.1),* $r = 1, 2, ...$ *converges to* $\mathbf{x}^*$ *with order* $r + 2$ *with the error equation*

$$\psi_{(r+2)^{th}KM}(\mathbf{x}) - \mathbf{x}^* = C_r\,\mathbf{c_2}^{r+1}(\epsilon^{(k)})^{r+2} + ... \tag{3.8}$$

The will prove by induction and follows on similar lines. Similarly as in the case of scalar, we are able to get higher order I.F. for systems if we know the asymptotic error constant of the previous I.F. using

$$a_{r+1} = C_r\left(-\frac{3}{4}\right)^{r+1}, r = 1, 2, ...$$

4. Numerical Experiments

4.1 Scalar equation

This section deals with numerical comparisons in the MATLAB computer code rounding to 1000 significant digits. The criteria to stopping used for the iterative process $|x^{(k+1)} - x^{(k)}| < 10^{-50}$. The number of iterations for convergence based on criteria is denoted by N. The computational order of convergence (ρ) for successive zero is given by

$$\rho = \frac{\ln |(x^{(N)} - x^{(N-1)}/(x^{(N-1)} - x^{(N-2)})|}{\ln |(x^{(N-1)} - x^{(N-2)})/(x^{(N-2)} - x^{(N-3)})|}.$$

We consider the **Test problem 1** (TP1) of finding the real zero of the quadratic function $f(x) = x^2 + 4x - 1$ to compare the proposed methods. We choose $x^{(0)} = 1$. The results in Table 1 shows that, the order of the $(r + 2)^{nd}KM$ I.F. (r=1,2,3,4,5,6), methods converge in less number of iterations. The computational order of convergence (ρ) confirming that the Kung-Traub's conjecture fails for quadratic equations. By analysis of TP1, our method $8^{th}KM$ is found to be superior to the methods of $2^{nd}NR$ and $8^{th}BQIM$.

4.2 Systems of Nonlinear equations

This section deals with numerical comparisons in the MATLAB computer code rounding to 1000 significant digits. The criteria to stopping used for the iterative process

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2 < 10^{-100}. \tag{4.1}$$

The approximated computational order of convergence p_c given by (see [10])

$$p_c \approx \frac{\log \left(\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2 / \|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_2\right)}{\log \left(\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_2 / \|\mathbf{x}^{(k-1)} - \mathbf{x}^{(k-2)}\|_2\right)}. \tag{4.2}$$

We consider the **Test Problem 2** (TP2)

$$\begin{aligned} x_1^2 + x_2^2 - 7 &= 0, \\ x_1 - x_2 + 1 &= 0. \end{aligned} \tag{4.3}$$

Using the substitution method, eq. (4.3) reduces to the quadratic equation $x_2^2 - x_2 - 3 = 0$ whose positive root is given by $x_2^* = \frac{1+\sqrt{13}}{2} = 2.302775638$. Therefore $x_1^* = x_2^* - 1 = \frac{\sqrt{13}}{2} = 1.302775638$. We use $\mathbf{x}^{(0)} = (1, 2)^T$ as initial vector and apply proposed methods (3.1), $r = 1, 2, ..., 6$ to locate the approximate solutions of eq. (4.3). Table 2 shows that our method $8^{th}KM$ is converge less number of iterations with least error than $2^{nd}NR$ and $8^{th}BQIM$. Hence, Proposed methods are found to be superior to the methods of $2^{nd}NR$ and $8^{th}BQIM$.
We next consider the **Test Problem 3** (TP3) [8]

$$\begin{aligned} x_1^2 + x_2^2 - 1 &= 0, \\ x_1^2 - x_2^2 - 0.5 &= 0. \end{aligned} \tag{4.4}$$

Equation (4.4) reduces to the simple quadratic equation $2x_2^2 - 1.5 = 0$ by the elimination method, whose root is given by $x_2^* = \frac{\sqrt{3}}{2} = 0.866025403$ and therefore $x_1^* = \frac{1}{2}$. We choose initial point far from the root, $\mathbf{x}^{(0)} = (2, 3)^T$ and apply our methods (3.1), $r = 1, 2, ..., 6$ to locate the approximate solutions of eq. (4.4). In Table 3, we observe that the methods take more iterations to converge when initial point is far from the root. By analysis TP3 problem, our method $8^{th}KM$ is converge with least error than $2^{nd}NR$ and $8^{th}BQIM$. Therefore, proposed methods are found to be superior to the methods of $2^{nd}NR$ and $8^{th}BQIM$.
We next consider the **Test Problem 4** (TP4)[11].

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) &= 0, \\ x_1x_3 + x_4(x_1 + x_3) &= 0, \\ x_1x_2 + x_4(x_1 + x_2) &= 0, \\ x_1x_2 + x_1x_3 + x_2x_3 &= 1. \end{aligned} \tag{4.5}$$

Apply the substitution method in eq. (4.5) reduces to the scalar nonlinear equation $3x_1^2 - 1 = 0$ whose real root is given by $x_1^* = \frac{1}{\sqrt{3}} = 0.577350269$. Therefore $x_2^* = x_3^* = x_1^* = \frac{1}{\sqrt{3}} = 0.577350269$.
and $x_4^* = -\frac{x_1^*}{2} = -\frac{1}{2\sqrt{3}} = -0.288675134$. Using $\mathbf{x}^{(0)} = (0.5, 0.5, 0.5, -0.25)^T$ as initial point and we apply our

error	Existing Methods		Members of Proposed Methods					
	2^{nd}NR	8^{th}BQIM	3^{rd}KM	4^{th}KM	5^{th}KM	6^{th}KM	7^{th}KM	8^{th}KM
$|x_1 - x_0|$	0.6667	0.7638	0.7426	0.7583	0.7620	0.7633	0.7637	0.7638
$|x_2 - x_1|$	0.0952	9.5e-005	0.0214	0.0056	0.0019	6.4e-004	2.3e-004	8.5e-005
$|x_3 - x_2|$	0.0020	8.3e-035	8.4e-007	4.3e-011	7.0e-016	1.4e-021	4.9e-028	2.8e-035
$|x_4 - x_3|$	9.1e-007	2.8e-275	5.4e-020	1.4e-043	5.0e-078	1.6e-127	1.0e-193	4.0e-279
$|x_5 - x_4|$	1.8e-013	-	1.4e-059	1.9e-173	-	-	-	-
$|x_6 - x_5|$	7.9e-027	-	-	-	-	-	-	-
$|x_7 - x_6|$	1.4e-053	-	-	-	-	-	-	-
ρ	1.99	7.99	2.99	3.99	4.99	5.99	6.99	7.99

Table 1. Results of TP1 for the 2^{nd}NR, 8^{th}BQIM with new families of I.F.s (r = 1,2,3,4,5,6)

error	Existing Methods		Members of Proposed Methods					
	2^{nd}NR	8^{th}BQIM	3^{rd}KM	4^{th}KM	5^{th}KM	6^{th}KM	7^{th}KM	8^{th}KM
$\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2$	0.4714	0.4282	0.4203	0.4299	0.4276	0.4284	0.4281	0.4282
$\|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\|_2$	0.0429	3.0e-005	0.0079	0.0017	5.7e-004	1.8e-004	6.5e-005	2.3e-005
$\|\mathbf{x}^{(3)} - \mathbf{x}^{(2)}\|_2$	3.6e-004	3.9e-039	3.3e-008	2.4e-013	1.0e-018	4.4e-025	3.3e-032	4.0e-040
$\|\mathbf{x}^{(4)} - \mathbf{x}^{(3)}\|_2$	2.5e-008	3.0e-310	2.5e-024	1.0e-052	2.2e-092	7.4e-149	2.8e-223	2.6e-318
$\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_2$	1.2e-016	-	1.1e-072	3.4e-210	0	-	-	-
$\|\mathbf{x}^{(6)} - \mathbf{x}^{(5)}\|_2$	3.1e-033	-	1.1e-217	-	-	-	-	-
$\|\mathbf{x}^{(7)} - \mathbf{x}^{(6)}\|_2$	1.9e-066	-	-	-	-	-	-	-
$\|\mathbf{x}^{(8)} - \mathbf{x}^{(7)}\|_2$	7.4e-133	-	-	-	-	-	-	-
p_c	2.00	8.12	3	3.99	5	6.14	7.16	8.16

Table 2. Results of TP2 for the 2^{nd}NR, 8^{th}BQIM with new families of I.F.s (r = 1,2,3,4,5,6)

error	Existing Methods		Members of Proposed Methods					
	2^{nd}NR	8^{th}BQIM	3^{rd}KM	4^{th}KM	5^{th}KM	6^{th}KM	7^{th}KM	8^{th}KM
$\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2$	1.6694	2.5146	2.0799	2.2689	2.3649	2.4334	2.4824	2.5197
$\|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\|_2$	0.7437	0.2477	0.5974	0.4729	0.3931	0.3291	0.2805	0.2424
$\|\mathbf{x}^{(3)} - \mathbf{x}^{(2)}\|_2$	0.2824	1.5e-004	0.0883	0.0242	0.0076	0.0021	5.5e-004	1.2e-004
$\|\mathbf{x}^{(4)} - \mathbf{x}^{(3)}\|_2$	0.0681	1.7e-028	8.9e-004	1.1e-006	2.7e-010	3.1e-015	1.7e-021	2.7e-029
$\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_2$	0.0046	3.9e-220	1.2e-009	6.1e-024	1.8e-047	3.3e-086	4.4e-144	1.3e-226
$\|\mathbf{x}^{(6)} - \mathbf{x}^{(5)}\|_2$	2.1e-005	-	3.6e-027	5.4e-093	2.4e-233	0	-	-
$\|\mathbf{x}^{(7)} - \mathbf{x}^{(6)}\|_2$	4.4e-010	-	8.4e-080	0	-	-	-	-
$\|\mathbf{x}^{(8)} - \mathbf{x}^{(7)}\|_2$	1.9e-019	-	1.0e-237	-	-	-	-	-
$\|\mathbf{x}^{(9)} - \mathbf{x}^{(8)}\|_2$	3.9e-038	-	-	-	-	-	-	-
$\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_2$	1.5e-075	-	-	-	-	-	-	-
$\|\mathbf{x}^{(11)} - \mathbf{x}^{(10)}\|_2$	2.3e-150	-	-	-	-	-	-	-
p_c	2.00	7.50	3.00	4.00	4.99	6.00	6.98	7.60

Table 3. Results of TP3 for the 2^{nd}NR, 8^{th}BQIM with new families of I.F.s (r = 1,2,3,4,5,6)

	Existing Methods		Members of Proposed Methods					
error	2^{nd}NR	8^{th}BQIM	3^{rd}KM	4^{th}KM	5^{th}KM	6^{th}KM	7^{th}KM	8^{th}KM
$\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2$	0.1502	0.1394	0.1379	0.1397	0.1394	0.1395	0.1394	0.1394
$\|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\|_2$	0.0107	1.4e-006	0.0015	2.5e-004	6.2e-005	1.5e-005	4.0e-006	1.1e-006
$\|\mathbf{x}^{(3)} - \mathbf{x}^{(2)}\|_2$	5.5e-005	3.9e-047	1.3e-009	1.6e-015	5.9e-022	1.2e-029	2.5e-038	4.6e-048
$\|\mathbf{x}^{(4)} - \mathbf{x}^{(3)}\|_2$	1.4e-009	0	1.1e-027	3.2e-060	4.7e-107	2.9e-174	8.6e-264	0
$\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_2$	1.0e-018	-	5.5e-082	4.4e-239	-	-	-	-
$\|\mathbf{x}^{(6)} - \mathbf{x}^{(5)}\|_2$	5.1e-037	-	7.1e-245	-	-	-	-	-
$\|\mathbf{x}^{(7)} - \mathbf{x}^{(6)}\|_2$	1.2e-073	-	-	-	-	-	-	-
$\|\mathbf{x}^{(8)} - \mathbf{x}^{(7)}\|_2$	7.9e-147	-	-	-	-	-	-	-
p_c	2.00	8.11	2.99	4.01	4.99	6.14	7.10	8.11

Table 4. Results of TP4 for the 2^{nd}NR, 8^{th}BQIM with new families of I.F.s (r = 1,2,3,4,5,6)

methods (3.1), $r = 1, 2, ..., 6$ to locate the approximate solutions of eq. (4.5). In Table 4, shows that as the convergence order of the algorithm increase the methods converge in less number of iterations with least error. By analysis TP4 problem, our method 8^{th}KM is converge with least error than 2^{nd}NR and 8^{th}BQIM. Hence, Proposed methods are found to be superior to the methods of 2^{nd}NR and 8^{th}BQIM.

5. Application

5.1 1-D Bratu Problem
The one dimensional Bratu problem [6] is given by

$$\frac{d^2U}{dx^2} + \lambda \exp U(x) = 0, \ \lambda > 0, \ 0 < x < 1, \tag{5.1}$$

with boundary conditions $U(0) = U(1) = 0$. The 1-D Bratu problem has two known, bifurcated, actual solutions for values of $\lambda < \lambda_c$, one solution for $\lambda = \lambda_c$ and no solution for $\lambda > \lambda_c$.

The critical value of λ_c is simply $8(\eta^2 - 1)$, where η is the fixed point of the hyperbolic cot function $\coth(x)$. The exact solution to equation (5.1) is known and can be presented here as

$$U(x) = -2 \ln \left[\frac{\cosh \left(x - \frac{1}{2}\right) \frac{\theta}{2}}{\cosh \left(\frac{\theta}{4}\right)} \right], \tag{5.2}$$

where θ may be a constant to be determined, that satisfies the boundary conditions and is fastidiously chosen and assumed to be the answer of the equation (5.1). Employing a similar procedure as in [17], we tend to show the way to acquire the crucial worth of λ. Substitute equation (5.2) in equation (5.1), simplify and collocate at the point $x = \frac{1}{2}$ because its the centre of the interval. Selected another point, however low order of approximations are probably to being higher suppose the collocation points are distributed somewhat equally throughout the region. Then, we have

$$\theta^2 = 2\lambda \cosh^2 \left(\frac{\theta}{4}\right). \tag{5.3}$$

Differentiating equation (5.3) with respect to θ and setting $\frac{d\lambda}{d\theta} = 0$, the critical value λ_c satisfies

$$\theta = \frac{1}{2} \lambda_c \cosh \left(\frac{\theta}{4}\right) \sinh \left(\frac{\theta}{4}\right). \tag{5.4}$$

By eliminating λ from equations (5.3) and (5.4), we have the value of θ_c for the critical λ_c satisfying

$$\frac{\theta_c}{4} = \coth \left(\frac{\theta_c}{4}\right) \tag{5.5}$$

for which $\theta_c = 4.798714560$ can be obtained using an iterative method. We then get $\lambda_c = 3.513830720$ from Equation (5.3). Figure 1 illustrates this critical value of λ. The finite dimensional problem using standard finite difference
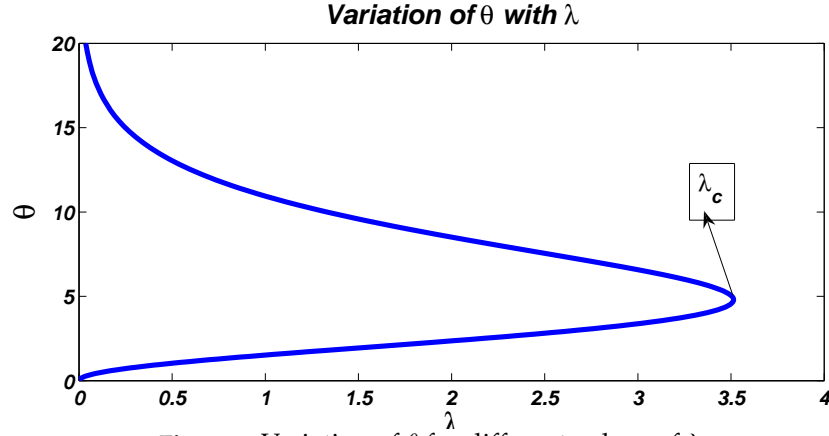
Figure 1. Variation of θ for different values of λ.

scheme is given by

$$F_j(U_j) = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \lambda \exp U_j = 0, \; j = 1..N-1 \tag{5.6}$$

with distinct boundary conditions $U_0 = U_N = 0$ and also the step size $h = 1/N$. There are square measure $N-1$ unknowns ($n = N - 1$). The Jacobian may be a sparse matrix and its typical range of nonzero per row is 3. Its noted that the finite difference scheme converges to the lower solution of the 1-D Bratu using the initial vector $U^{(0)} = (0, 0, .., 0)^T$.

We use $N = 101$ ($n = 100$) and check for 350 λ's in the interval $(0, 3.5]$ (interval breadth = 0.01). For every λ, we let M_λ be the minimum number of iterations for which $\|U_j^{(k+1)} - U_j^{(k)}\|_2 < 1e - 13$, where the approximation $U_j^{(k)}$ is calculated correct to fourteen decimal places. Let $\overline{M_\lambda}$ be the mean of iteration number for the 350 λ's.

Table 5. Comparison of number of λ's (out of 350 λ's) converging for 1-D Bratu problem

Method	$M=2$	$M=3$	$M=4$	$M=5$	$M>5$	$\overline{M_\lambda}$
2^{nd}NR	0	12	114	143	81	4.92
$8^{th}BQIM$	4	257	86	2	1	3.26
$3^{rd}KM$	1	139	170	35	5	3.73
$4^{th}KM$	4	231	104	10	1	3.35
$5^{th}KM$	4	250	90	6	0	3.28
$6^{th}KM$	4	254	88	1	3	3.27
$7^{th}KM$	4	256	87	1	2	3.26
$8^{th}KM$	4	256	87	1	2	3.27

Table 5 give the results for the 1-D Bratu problem, where M denoting number of iterations for convergence. It can be observed from the methods considered in Table 5, proposed methods are most efficient methods than 2^{nd}NR method, because it has the lowest mean iteration number.

5.2 2-D Bratu problem

We consider the solution of the Bratu problem in two-dimensions

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \lambda exp(U) = 0, \; x, y \in D = [0,1] \times [0,1] \tag{5.7}$$

subject to the boundary conditions

$$U(x, y) = 0, \; x, y \in D. \tag{5.8}$$

The 2-D Planar Bratu problem has 2 known, bifurcated, actual solutions for values of $\lambda < \lambda_c$, one solution for $\lambda = \lambda_c$ and no solutions for $\lambda > \lambda_c$. The exact solution to eq. (5.7) is known and can be presented here as

$$U(x, y) = 2\ln\left[\frac{\cosh\left(\frac{\theta}{4}\right)\cosh\left((x - \frac{1}{2})(y - \frac{1}{2})\theta\right)}{\cosh\left((x - \frac{1}{2})(\frac{\theta}{2})\right)\cosh\left((y - \frac{1}{2})(\frac{\theta}{2})\right)}\right], \tag{5.9}$$
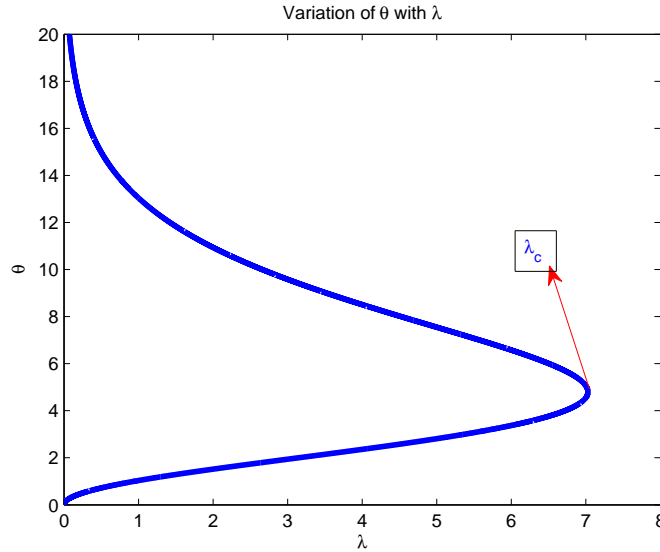
Figure 2. Variation of θ for different values of λ.

where θ may be a constant and to be determined, which satisfies the boundary conditions and is carefully chosen and assumed to be the solution of the equation (5.7). The following procedure found in [17], for how to obtain the critical value of λ. Substituting eq. (5.9) in (5.7), simplifying and collocating at the point $x = \frac{1}{2}$ and $y = \frac{1}{2}$ because it is the midpoint of the interval. Another point could be chosen, but low-order approximations are likely to be better if the collocation points are distributed somewhat evenly throughout the region. Then, we have

$$\theta^2 = \lambda \cosh^2\left(\frac{\theta}{4}\right). \tag{5.10}$$

Differentiating eq. (5.10) with respect to θ and setting $\frac{d\lambda}{d\theta} = 0$, the critical value λ_c satisfies

$$\theta = \frac{1}{4}\lambda_c \cosh\left(\frac{\theta}{4}\right)\sinh\left(\frac{\theta}{4}\right). \tag{5.11}$$

By eliminating λ from eqs.(5.10) and (5.11), we have the value of θ_c for the critical λ_c satisfying

$$\frac{\theta_c}{4} = \coth\left(\frac{\theta_c}{4}\right) \tag{5.12}$$

and $\theta_c = 4.798714561$. We then get $\lambda_c = 7.027661438$ from eq. (5.11). Fig. (2) illustrates this critical value of λ_c. The differential equation (5.7) is usually discretized by using the finite-difference five-point formula with the step size h, the resulting nonlinear equations are

$$F(U_{i,j}) = -(4U_{i,j} - \lambda h^2 exp(U_{i,j})) + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \tag{5.13}$$

where $U_{i,j}$ is U at (x_i, y_j), $x_i = ih$, $y_j = jh$, $i, j = 1, 2, ...N$. Equation (5.13) represents a set of $N \times N$ nonlinear equations in $U_{i,j}$ which are then solved by using iterative methods. We use $N = 10$ and $N = 20$ for test 700 λ's in the interval $(0, 7]$ (interval width = 0.01). For each λ, we let M_λ be the minimum number of iterations for which $\|U_{i,j}^{(k+1)} - U_{i,j}^{(k)}\|_2 < 1e - 11$, where the approximation $U_{i,j}^{(k)}$ is calculated correct to 14 decimal places. Let $\overline{M_\lambda}$ be the mean of iteration number for the 700 λ's.

   Tables 6, 7 gives the results for 2-D Bratu problem. It can be observed from the proposed methods are considered in Table 6, 7, all the grid points are convergent in two and three iterations only. The proposed methods are most efficient methods than $2^{nd}NR$, in both the case $N = 10$ and $N = 20$ because it has the lowest mean iteration number.

Table 6. Comparison of number of λ's for 2-D Bratu problem for N=10

Method	$M=2$	$M=3$	$M=4$	$M=5$	$\overline{M_\lambda}$
$2^{nd} NR$	0	101	520	79	3.96
$8^{th} BQIM$	43	657	0	0	2.93
$3^{rd} KM$	18	606	76	0	3.08
$4^{th} KM$	43	657	0	0	2.93
$5^{th} KM$	43	657	0	0	2.93
$6^{th} KM$	43	657	0	0	2.93
$7^{th} KM$	43	657	0	0	2.93
$8^{th} KM$	43	657	0	0	2.93

Table 7. Comparison of number of λ's for 2-D Bratu problem for N=20

Method	$M=2$	$M=3$	$M=4$	$M=5$	$\overline{M_\lambda}$
$2^{nd} NR$	1	212	487	0	3.69
$8^{th} BQIM$	89	611	0	0	2.87
$3^{rd} KM$	35	665	0	0	2.95
$4^{th} KM$	89	611	0	0	2.87
$5^{th} KM$	89	611	0	0	2.87
$6^{th} KM$	89	611	0	0	2.87
$7^{th} KM$	89	611	0	0	2.87
$8^{th} KM$	89	611	0	0	2.87

6. Conclusion

In this work, we have developed a family of iterative methods with three functions evaluations reaching convergence order more than 4 for solving scalar quadratic equations. Furthermore, we have a tendency to show that it is possible to develop methods with same function evaluations of order $r + 2$. Also, we have extended these methods to systems for solving quadratic equations. We have showed that Kung-Traub's conjecture fails for quadratic functions. The most advantages of the proposed schemes are the following to solve systems: (i) they are doing not use second order Frechet derivative, (ii) measure only one inverse of first order Frechet derivative and (iii) measure less number of linear systems per iteration. The proposed new methods and their theoretical results are validated through examples whose results are tabulated which gives better results among compared existing methods. For practical applications, the new methods are verified on 1-D, 2-D Bratu problems which gives encouraging results compared to existing methods.

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