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Research Article

A Partial Solution To An Open Problem

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Abstract

Batkunde et al. [Acta Univ. M. Belii Ser. Math., 2013] have defined a multilinear n -functional on l^p . Regarding the n -functional F_Y on $\left(l^p, \|.,\ldots,.\|_p\right)$, they want to compute the exact norm of F_Y, especially for $p \neq 2$. In this paper, we deal with a partial solution to an open problem given in their paper.

Keywords: Inner product spaces, n-normed spaces, bounded multilinear n-functional, p -summable sequences.

1. Introduction

Let $n \geq 2$ be an integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|.,\ldots,.\|$ on X^n satisfying the following four properties:

$i.$ $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

$ii.$ $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

$iii.$ $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$, for any $\alpha \in \mathbb{R}$,

$iv.$ $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n -norm on X , and the pair $\left(X, \|.,\ldots,.\|\right)$ is called an n -normed space (Gunawan & Mashadi 2001).

Recent results and related topics may be found in (Gahler 1965; Batkunde et al. 2013; Gunawan & Mashadi 2001; Gunawan et al. 2005; Gozali et al. 2010; Pangalela & Gunawan 2013; Gunawan 2002; Gozali et al. 2010; Gunawan 2001; Milicic 1993).

Throughout the paper, we have focused on the space of p-summable sequences, denoted by l^p , where $1 \leq p < \infty$. Recall that a sequence $u = (u_k)_{k=1}^{\infty}$ (of real numbers) belongs l^p space if $\|u\|_p := \left(\sum_{k=1}^{\infty} |u_k|^p\right)^{\frac{1}{p}} < \infty$. It is known that the dual space of l^p is l^q where $\frac{1}{p} + \frac{1}{q} = 1$. Let $\left(X, \|.,\ldots,.\|\right)$ be a real *n*-normed space and $f : X \to \mathbb{R}$ be a linear functional on *X*. Several n -norms on l^p , which can be seen in (Batkunde et al. 2013), are given as follows:

If $\left(X, \|.\|\right)$ is a normed space and X' is its dual (consisting of bounded linear functionals on X), the following function defines an n -norm on X :

$$\|x_1, \ldots, x_n\|^G := \sup_{f_i \in X', \|f_i\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}. \tag{1.1}$$

Using the formula (1.1), l^p may be equipped with the following n -norm:

$$\|x_1, \ldots, x_n\|_p^G := \sup_{y_i \in l^q,\ \|y_i\|_q \leq 1} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \cdots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \cdots & \sum_k x_{nk} y_{nk} \end{vmatrix}, \tag{1.2}$$

where q denotes the dual exponent of p . There is another formula of n -norm which can be defined on l^p (Batkunde et al. 2013), namely

$$\|x_1, \ldots, x_n\|_p := \left(\frac{1}{n!} \sum_{k_1} \ldots \sum_{k_n} \left\| \begin{matrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{matrix} \right\|^p \right)^{\frac{1}{p}}, \tag{1.3}$$

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where $x_i = (x_{ik})_{k=1}^{\infty}$, $i = 1, 2, ..., n$. As shown in (Gunawan 2002), the two n-norms are equivalent:

$$(n!)^{\frac{1}{p}-1} \|x_1, ..., x_n\|_p \le \|x_1, ..., x_n\|_p^G \le (n!)^{\frac{1}{p}} \|x_1, ..., x_n\|_p .$$

On l^2, both n-norms coincide with the standard n-norm given by $\|x_1, ...x_n\|_S := \sqrt{\det(\langle x_i, x_j \rangle)}$. Next observe that the determinant on the right hand side of (1.2) can be rewritten as

$$\frac{1}{n!} \sum_{j_1} ... \sum_{j_n} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix} . \qquad (1.4)$$

By Hölder's inequality, it is dominated by $\|x_1, ..., x_n\|_p \|y_1, ..., y_n\|_q$. Another n-norm on l^p defined by Batkunde et al. (2013), namely

$$\|x_1, ..., x_n\|_p^I := \sup_{y_i \in l^q,\ \|y_1, ..., y_n\|_q^H \le 1} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \cdots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \cdots & \sum_k x_{nk} y_{nk} \end{vmatrix} .$$

(1.5)

As can be seen in (Batkunde et al. 2013), the three n-norms on l^p given in (1.2), (1.3) and (1.5) are equivalent:

$$\|x_1, ..., x_n\|_p^I \le \|x_1, ..., x_n\|_p \le (n!)^{\frac{1}{p'}} \|x_1, ..., x_n\|_p^G \le n! \|x_1, ..., x_n\|_p^I .$$

(1.6)

On a normed space $(X, \|.\|)$, the functional $g : X^2 \to \mathbb{R}$ defined by the formula $g(x, y) := \frac{\|x\|}{2} (\lambda_+(x, y) + \lambda_-(x, y))$, where $\lambda_{\pm}(x, y) := \lim_{t \to \pm 0} t^{-1} (\|x + ty\| - \|x\|)$, satisfies the following properties:

i. $g(x, x) = \|x\|^2$ for all $x \in X$,

ii. $g(\alpha x, \beta y) = \alpha\beta\, g(x, y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$,

iii. $g(x, x + y) = \|x\|^2 + g(x, y)$ for all $x, y \in X$,

iv. $|g(x, y)| \le \|x\| \|y\|$ for all $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear in $y \in X$, it is called a semi-inner product on X (Milicic 1993).

The functional

$$g(x, y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \operatorname{sgn}(x_k) y_k , \quad x = (x_k),\ y = (y_k) \in l^p$$

(1.7)

defines a semi-inner product on the space l^p, for $1 \le p < \infty$, where $\|.\|_p$ is the usual norm on l^p. Using a semi-inner product g, one may define the notion of orthogonality on X. In particular, it can be defined

$$x \perp_g y \Leftrightarrow g(x, y) = 0 . \qquad (1.8)$$

Note that since g is in general not commutative, $x \perp_g y$ does not imply that $y \perp_g x$ (Milicic 1993).

2. Bounded Multilinear n-Functionals on l^p

A multilinear n-functional on a real vector space X is a mapping $F : X^n \to \mathbb{R}$ which is linear in each variable. A multilinear n-functional F is bounded on an n-normed space $(X, \|., ..., .\|)$ if and only if there exists $K > 0$ such that

$$|F(x_1, ..., x_n)| \le K \|x_1, ..., x_n\| \qquad (2.1)$$

for every $x_1, ..., x_n \in X$. Note that for a bounded multilinear n-functional F on an n-normed space $(X, \|., ..., .\|)$, we have $F(x_1, ..., x_n) = 0$ when $x_1, ..., x_n$ are linearly dependent (Batkunde et al. 2013).

If F is a bounded multilinear n-functional on an n-normed space $(X, \|., ..., .\|)$, then F is antisymmetric, that is

$$F(x_1, ..., x_n) = \operatorname{sgn}(\sigma) F(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

for any $x_1,\dots,x_n \in X$ and any permutation σ of $(1,\dots,n)$. Here $\operatorname{sgn}(\sigma)=1$ if σ is an even permutation and $\operatorname{sgn}(\sigma)=-1$ if σ is an odd permutation. These properties do not hold for bounded multilinear n-functionals on a normed space $(X,\|.\|)$ (Batkunde et al. 2013).

The set X' of all bounded multilinear n-functionals on $(X,\|.,\dots,.\|)$ forms a vector space. A bounded multilinear n-functional F is defined

$$\|F\| := \inf\{K>0 : (2.1)\text{ holds}\},$$

or equivalently

$$\|F\| := \sup\{|F(x_1,\dots,x_n)| : \|x_1,\dots,x_n\| \le 1\}.$$

This formula defines a norm on X'.

Let $Y:=\{y_1,\dots,y_n\}$ in l^q, where q is the dual exponent of p. Batkunde et al. (2013) defined the following multilinear n-functional on l^p where $1\le p<\infty$:

$$F_Y(x_1,\dots,x_n) := \frac{1}{n!}\sum_{j_1}\dots\sum_{j_n}\begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n}\end{vmatrix}\begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n}\end{vmatrix},$$

(2.2)

for $x_1,\dots,x_n \in l^p$. From the definition of the multilinear n-functional F_Y in (2.2), clearly, if Y is linearly dependent set, then $F_Y(x_1,\dots,x_n)=0$. For this purpose, we separate this case and we assume that if Y is linearly dependent set, then $F_Y(x_1,\dots,x_n):=0$ and if $Y:=\{y_1,\dots,y_n\}$ is linearly independent set in l^q, then the multilinear n-functional F_Y on l^p is defined as in (2.2).

Clearly F_Y is linear in each variable. Further, $|F_Y(x_1,\dots,x_n)| \le \|x_1,\dots,x_n\|_p \|y_1,\dots,y_n\|_q$ and so F_Y is bounded on $(l^p,\|.,\dots,.\|_p)$ with $\|F_Y\| \le \|y_1,\dots,y_n\|_q$. For $p=2$, the following fact is obtained (Batkunde et al. 2013).

Fact 2.3. (Batkunde et al. 2013) Consider the n-normed space $(l^2,\|.,\dots,.\|_2)$. For fixed linearly independent set $Y:=\{y_1,\dots,y_n\}$ in l^2, let F_Y be the multilinear n-functional defined as in (2.2). Then F_Y is bounded on $(l^2,\|.,\dots,.\|_2)$ with $\|F_Y\| = \|y_1,\dots,y_n\|_2$.

Proof. From the inequality

$$|F_Y(x_1,\dots,x_n)| \le \|x_1,\dots,x_n\|_2 \|y_1,\dots,y_n\|_2$$

we see that F_Y is bounded with

$$\|F_Y\| \le \|y_1,\dots,y_n\|_2.$$

Since $Y:=\{y_1,\dots,y_n\}$ is a linearly independent set in l^2, we can choose $x_i := \frac{y_i}{\sqrt[n]{\|y_1,\dots,y_n\|_2}}$, $i=1,\dots,n$. If $x_i := \frac{y_i}{\sqrt[n]{\|y_1,\dots,y_n\|_2}}$, $i=1,\dots,n$, then $\|x_1,\dots,x_n\|_2 = 1$ and $F_Y(x_1,\dots,x_n) = \|y_1,\dots,y_n\|_2$. Hence, this conclude that $\|F_Y\| = \|y_1,\dots,y_n\|_2$.

3. Main Results

Regarding the n-functional F_Y on $(l^p,\|.,\dots,.\|_p)$, an open problem was given in (Batkunde et al. 2013). In this paper, we give a partial solution for this open problem.

Open Problem. Compute the exact norm of F_Y in (2.2), especially for $p \ne 2$.

The proof is not easy. If an exact solution can not be found, then it may be possible to obtain equivalence of norms such that

$$\frac{1}{n!}\|y_1,\dots,y_n\|_p \le \|F_Y\| \le \|y_1,\dots,y_n\|_p .$$

Proof. Recall that (1.4) can be obtained from the determinant given on the right hand side of the equation (1.2). Then the multilinear n-functional on l^p can be rewritten as:

$$F_Y(x_1,\dots,x_n) := \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{vmatrix}.$$

It is known from above that F_Y is bounded on $\left(l^p, \|\cdot,\dots,\cdot\|_p\right)$ with $|F_Y(x_1,\dots,x_n)| \le \|y_1,\dots,y_n\|_q$. To show the left part of the inequality, choose the linearly independent set $Y:=\{y_1,\dots,y_n\}$ be a left g-orthogonal in l^q such that $y_i \perp_g y_j$ with $i<j$ for $1\le i, j\le n$. Next, if we take $z_{i_{j_k}} := \left|y_{i_{j_k}}\right|^{q-1}\operatorname{sgn}\left(y_{i_{j_k}}\right)$, $1\le i, k\le n$ and $x_i := \frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\,\|y_i\|_q^q}|y_{ik}|^{q-1}\operatorname{sgn}(y_{ik})$, $i=1,\dots,n$, $k\in\mathbb{N}$ and $y_i\neq 0$ for each $i\in\mathbb{N}$, then

$$\|x_1,\dots,x_n\|_p$$

$$=\left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left\|\begin{matrix} \frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_1\|_q^q}\left|y_{1j_1}\right|^{q-1}\operatorname{sgn}\left(y_{1j_1}\right) & \cdots & \frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_1\|_q^q}\left|y_{1\,j_n}\right|^{q-1}\operatorname{sgn}\left(y_{1\,j_n}\right) \\ \vdots & \ddots & \vdots \\ \frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_n\|_q^q}\left|y_{nj_1}\right|^{q-1}\operatorname{sgn}\left(y_{nj_1}\right) & \cdots & \frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_n\|_q^q}\left|y_{n\,j_n}\right|^{q-1}\operatorname{sgn}\left(y_{n\,j_n}\right)\end{matrix}\right\|^p\right]^{\frac{1}{p}}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!\prod_{i=1}^n\|y_i\|_q^q}\left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left\|\begin{matrix}\left|y_{1j_1}\right|^{q-1}\operatorname{sgn}\left(y_{1j_1}\right) & \cdots & \left|y_{1\,j_n}\right|^{q-1}\operatorname{sgn}\left(y_{1\,j_n}\right)\\ \vdots & \ddots & \vdots \\ \left|y_{nj_1}\right|^{q-1}\operatorname{sgn}\left(y_{nj_1}\right) & \cdots & \left|y_{n\,j_n}\right|^{q-1}\operatorname{sgn}\left(y_{n\,j_n}\right)\end{matrix}\right\|^p\right]^{\frac{1}{p}}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{(n!)^1\prod_{i=1}^n\|y_i\|_q^q}\|z_1,\dots,z_n\|_p$$

$$=\frac{\|y_1,\dots,y_n\|_q\|z_1,\dots,z_n\|_p}{(n!)^{\frac{1}{p}}\prod_{i=1}^n\|y_i\|_q^1(n!)^{\frac{1}{q}}\prod_{i=1}^n\|y_i\|_q^{q-1}}$$

$$=\frac{\|y_1,\dots,y_n\|_q\|z_1,\dots,z_n\|_p}{(n!)^{\frac{1}{p}}\prod_{i=1}^n\|y_i\|_q^1(n!)^{\frac{1}{q}}\prod_{i=1}^n\left[\sum_j\left|y_{i_j}\right|^q\right]^{\frac{q-1}{q}}}$$

$$=\frac{\|y_1,\dots,y_n\|_q\|z_1,\dots,z_n\|_p}{(n!)^{\frac{1}{p}}\prod_{i=1}^n\|y_i\|_q^1(n!)^{\frac{1}{q}}\prod_{i=1}^n\left[\sum_j\left|\left|y_{i_j}\right|^{q-1}\operatorname{sgn}\left(y_{i_j}\right)\right|^p\right]^{\frac{1}{p}}}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{(n!)^{\frac{1}{p}}\prod_{i=1}^n\|y_i\|_q}\frac{\|z_1,\dots,z_n\|_p}{(n!)^{\frac{1}{q}}\prod_{i=1}^n\|z_i\|_p}$$

$$\le 1,$$

since $\|y_1,\dots,y_n\|_q \le (n!)^{\frac{1}{p}}\|y_1\|_q\dots\|y_n\|_q$ and $\|z_1,\dots,z_n\|_p \le (n!)^{\frac{1}{q}}\|z_1\|_p\dots\|z_n\|_p$ [see, (Gunawan 2001)].

Hence

$$F_Y(x_1,\dots,x_n)$$

$$=\begin{vmatrix}\sum_k\frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_1\|_q^q}|y_{1k}|^{q-1}\operatorname{sgn}(y_{1k})y_{1k} & \cdots & \sum_k\frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_1\|_q^q}|y_{1k}|^{q-1}\operatorname{sgn}(y_{1k})y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k\frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_n\|_q^q}|y_{nk}|^{q-1}\operatorname{sgn}(y_{nk})y_{1k} & \cdots & \sum_k\frac{\sqrt[n]{\|y_1,\dots,y_n\|_q}}{\sqrt[n]{n!}\|y_1\|_q^q}|y_{nk}|^{q-1}\operatorname{sgn}(y_{nk})y_{nk}\end{vmatrix}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!}\begin{vmatrix}\sum_k\frac{1}{\|y_1\|_q^q}|y_{1k}|^{q-1}\operatorname{sgn}(y_{1k})y_{1k} & \cdots & \sum_k\frac{1}{\|y_1\|_q^q}|y_{1k}|^{q-1}\operatorname{sgn}(y_{1k})y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k\frac{1}{\|y_n\|_q^q}|y_{nk}|^{q-1}\operatorname{sgn}(y_{nk})y_{1k} & \cdots & \sum_k\frac{1}{\|y_n\|_q^q}|y_{nk}|^{q-1}\operatorname{sgn}(y_{nk})y_{nk}\end{vmatrix}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!}\begin{vmatrix}\frac{g(y_1,y_1)}{\|y_1\|_p^2} & \cdots & \frac{g(y_1,y_n)}{\|y_1\|_p^2} \\ \vdots & \ddots & \vdots \\ \frac{g(y_n,y_1)}{\|y_n\|_p^2} & \cdots & \frac{g(y_n,y_n)}{\|y_n\|_p^2}\end{vmatrix}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!}\begin{vmatrix}\frac{g(y_1,y_1)}{\|y_1\|_p^2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{g(y_n,y_1)}{\|y_n\|_p^2} & \cdots & \cdots & \frac{g(y_n,y_n)}{\|y_n\|_p^2}\end{vmatrix}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!}\frac{g(y_1,y_1)}{\|y_1\|_p^2}\frac{g(y_2,y_2)}{\|y_2\|_p^2}\cdots\frac{g(y_n,y_n)}{\|y_n\|_p^2}$$

$$=\frac{\|y_1,\dots,y_n\|_q}{n!}.$$

Thus

$$\|F_Y\| \ge \frac{\|y_1,\dots,y_n\|_q}{n!}.$$

Hence

$$\frac{1}{n!}\|y_1,\dots,y_n\|_q \le \|F_Y\| \le \|y_1,\dots,y_n\|_q .$$

4. Concluding Remarks

In this paper, we have found a partial solution to this open problem given in (Batkunde et al. 2013) since we obtained $\|F_Y\| \cong \|y_1,\dots,y_n\|_q$. But an exact solution still remains an open problem.

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